Electroweak corrections to the muon anomalous magnetic moment

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The two-loop electroweak radiative corrections to the muon's anomalous magnetic moment, $a_\mu \equiv (g_\mu - 2)/2$, are presented. We obtain an overall 22.6% reduction in the electroweak contribution $a_\mu^{\rm EW}$ from 195×10^{-11} to $151(4) \times 10^{-11}$. Implications for the full standard model prediction and an upcoming high precision measurement of a_μ are briefly discussed. Some aspects of the calculations are discussed in detail.

The anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$, provides both a sensitive quantum loop test of the standard $SU(3)_C \times SU(2)_L \times U(1)$ model and a window to potential "new physics" effects. The current experimental average [1]

$$a_\mu^{\rm exp} = 116592300(840) \times 10^{-11} \tag{1}$$

is in good agreement with theoretical expectations and already constrains physics beyond the standard model such as supersymmetry and supergravity [2, 3], dynamical or loop muon mass generation [4], compositeness [5], leptoquarks [6] etc.

An upcoming experiment E821 [7] at Brookhaven National Laboratory is expected to start taking data in 1997. With one month of dedicated running, it is expected to reduce the uncertainty in $a_\mu^{\rm exp}$ to roughly $\pm 40 \times 10^{-11}$, more than a factor of 20 improvement. With subsequent longer dedicated runs it could statistically approach the anticipated systematic uncertainty of about $\pm 10 - 20 \times 10^{-11}$ [8]. At those levels, both electroweak one and two loop effects become important and "new physics" at the multi-TeV scale is probed. Indeed, generic muon mass generating mechanisms (via perturbative or dynamical loops [4]) lead to $\Delta a_\mu \approx m_\mu^2/\Lambda^2$, where Λ is the scale of "new physics". At $\pm 40 \times 10^{-11}$ sensitivity, $\Lambda \approx 5$ TeV is being explored.

To fully exploit the anticipated experimental improvement, the standard model prediction for a_μ must be known with comparable precision. That requires detailed studies of very high order QED loops, hadronic effects, and electroweak contributions through two loop order. The contributions to a_μ are traditionally divided into

$$a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm Hadronic} + a_\mu^{\rm EW} \tag{2}$$

QED loops have been computed to very high order [9, 13, 14]

$$a_\mu^{\rm QED} \quad = \quad 116584706(2) \times 10^{-11}. \tag{3}$$

The uncertainty of the QED result is well within the $\pm 20 - 40 \times 10^{-11}$ goal.

Hadronic vacuum polarization corrections to a_μ enter at $\mathcal{O}(\alpha/\pi)^2$. They can be evaluated via a dispersion relation using $e^+e^- \to hadrons$ data and perturbative QCD (for the very high energy regime). Employing a recent analysis of e^+e^- data [15, 16] along with an estimate of the leading $\mathcal{O}(\alpha/\pi)^3$ effects, we find [17]

$$a_\mu^{\rm Hadronic}({\rm vac.\,pol.}) = 6934(153) \times 10^{-11} \tag{4}$$

Unfortunately, the error has not yet reached the desired level of precision. Ongoing improvements in $e^+e^- \to hadrons$ measurements at low energies along with additional theoretical input should significantly lower the uncertainty in (4). Nevertheless, reducing the hadronic error below $\pm 20 \times 10^{-11}$ or even $\pm 40 \times 10^{-11}$ remains a formidable challenge.

The result in (4) must be supplemented by hadronic light by light amplitudes (which are of three loop origin) [10, 11, 12]. Here, we employ a recently updated study by Hayakawa, Kinoshita, and Sanda [11] which gives

$$a_\mu^{\rm Hadronic}({\rm light\,by\,light}) = -52(18) \times 10^{-11} \tag{5}$$

However, we note that the result is somewhat dependent on the low energy model of hadronic physics employed and continues to be scrutinized. Combining (4) and (5) leads to the total hadronic contribution

$$a_\mu^{\rm Hadronic} = 6882(154) \times 10^{-11} \tag{6}$$

Now we come to the electroweak contributions to a_μ, the main focus of our work and the impetus for forthcoming experimental effort. At the one loop level, the standard model predicts [19, 20, 21, 22, 23]

$$a_\mu^{\rm EW}(1\,{\rm loop}) = \frac{5}{3}\frac{{\rm G}_\mu {\rm m}_\mu^2}{8\sqrt{2}\pi^2} \times \left[1+\frac{1}{5}(1-4s_W^2)^2+\mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)\right] \approx 195\times 10^{-11} \quad (7)$$

where $G_\mu = 1.16639(1)\times 10^{-5}\ {\rm GeV}^{-2}$, $M = M_W$ or $M_{\rm Higgs}$, and the weak mixing angle $\sin^2\theta_W \equiv s_W^2 = 1-M_W^2/M_Z^2 = 0.224$. We can safely neglect the $\mathcal{O}\left(m_\mu^2/M^2\right)$ terms in (7).

The one loop result in (7) is about five to ten times the anticipated experimental error. Naively, one might expect higher order (2 loop) electroweak contributions to be of relative $\mathcal{O}(\alpha/\pi)$ and hence negligible; however, that is not the case. Kukhto, Kuraev, Schiller, and Silagadze (KKSS) [25] have shown that some two loop electroweak contributions can be quite large and must be included in any serious theoretical estimate of $a_\mu^{\rm EW}$ or future confrontation with experiment. Given the KKSS observation, a detailed evaluation of the two loop electroweak contributions to a_μ is clearly warranted. We have reported the complete results of such an analysis in two recent papers [17, 18]. The types of diagrams contributing to two-loop electroweak corrections are shown in 1. These fall into a number of different categories. Diagrams with a closed fermion loop, together with a class of bosonic diagrams, represent corrections to vertices and propagators of the one-loop electroweak diagrams (shown in the first line in Fig.1).

In addition we have to calculate new types of diagrams which appear at the two-loop level, such as γZ mixing, an induced $\gamma\gamma H$ vertex etc. (lines two and three of Fig.1). Finally, there are non-planar diagrams and diagrams with quartic couplings; some examples are shown in Fig.2.

The final result of the total electroweak contributions to the g-2 of leptons – including one- and two-loop corrections – is

$$\begin{aligned} a_e^{\rm EW} &= 3.0\cdot 10^{-14} \\ a_\mu^{\rm EW} &= 151\cdot 10^{-11} \\ a_\tau^{\rm EW} &= 4.7\cdot 10^{-7}\,. \end{aligned} \quad (8)$$

In this paper we discuss some details of the procedures we have employed in that project. In particular we discuss the method for projecting out the Pauli formfactor in d dimensions, and some aspects of the fermionic contributions.

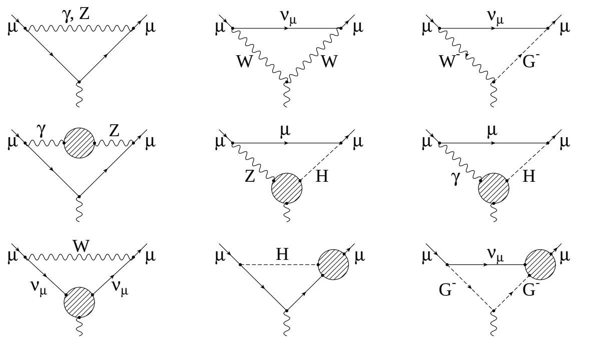


Figure 1. Bosonic diagrams contributing to g-2 of the muon

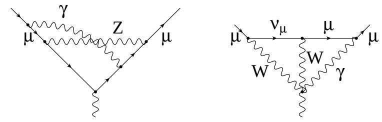


Figure 2. Examples of non-planar diagrams and diagrams with quartic couplings.

Diagrams with charged bosons

Contributions of diagrams with a fermion loop connected to the muon line via two charged bosons are shown in Fig. 3. Fermions with isospin $\pm 1/2$ are denoted by u and d, respectively. Of course, in the case of leptons diagrams (a) and (f) vanish.

We first consider the case when the fermions in the loop belong to the first two generations. Here the masses of the fermions in the loop do not influence the result very much and we neglect them. The ratio of the neglected terms to the result is at most of the order of

$$\frac{m_c^2}{M_W^2}\ln\frac{M_W^2}{m_c^2} < 0.3\% \quad (9)$$

with m_c denoting the mass of the charm quark $\approx$ 1.5GeV. In the massless approximation we need to consider only the first four diagrams in Fig. 3. In ref.[26] it has been argued that the diagrams 1(a) and 1(b) vanish by virtue of Furry's theorem. We find that this is not true even after adding contributions of all fermions in a generation. Furry's theorem consists in the observation that the sum of contributions of diagrams with two different orientations of the fermion loop vanishes. This is not the case for diagrams 1(a) and 1(b) because for every fermion flavor interacting with the external photon there is only one possible orientation of the fermion line. Only those parts of the expressions which contain a single γ_5 cancel out after adding contributions from the up-type quark, down-type quark and from the lepton.

Let us consider diagrams 1(a) and 1(b) in more detail. The contributions of these diagrams can be divided up into those with an odd number of vector couplings to the fermion loop (VVV or VAA) and with two vector, one axial vector coupling (AVV). The dominant contribution is obtained by taking massless fermions in the loop. For diagrams of type 1(a) and 1(b) we find (the subscript indicates the fermion to which the photon couples)

$$\begin{aligned}\Delta a_f^{VVV} &= \frac{CN_C^f}{s_W^2} Q_f I_f \left(\frac{19}{36} + \frac{7}{18\varepsilon} - \frac{7}{9}\ln M_W^2\right) \\ \Delta a_f^{AVV} &= \frac{CN_C^f}{s_W^2} Q_f \left(+\frac{1}{4}\right) \end{aligned} \tag{10}$$

with

$$C = \frac{G_\mu \alpha m_\mu^2}{8\sqrt{2}\pi^3} \,. \tag{11}$$

Here we used the shorthand notation VVV for the sum of VVV+VAA. $N_C^{quarks} = 3$ is the number of colors (with $N_C^{lepton} = 1$ understood) and Q_f denotes the fermion charge. Only the parity conserving part contains divergences; they are proportional to the lowest order γWW vertex.

Summing the AVV part over one light generation (l.g.) we obtain

$$\Delta a_{l.g.}^{AVV} = 0 \tag{12}$$

due to the fact that in the standard model

$$\sum_f N_C^f Q_f = 0 \,. \tag{13}$$

The VVV contribution of one light generation is

$$\begin{aligned}\Delta a_{l.g.}^{VVV} &= \frac{C}{s_W^2}\sum_f N_C^f Q_f I_f \times \\ &\quad \times \left(\frac{19}{36} + \frac{7}{18\varepsilon} - \frac{7}{9}\ln M_W^2\right) \\ &= \frac{C}{s_W^2}\cdot 2 \cdot \left(\frac{19}{36} + \frac{7}{18\varepsilon} - \frac{7}{9}\ln M_W^2\right)\end{aligned} \tag{14}$$

This example demonstrates explicitly that the contributions from three vector couplings to the fermion loop do not cancel. We would like to emphasize that this has nothing to do with the masses inside the fermion loop. We can put them equal (even equal to zero as in our example) just like in the QED case. The Furry theorem is not applicable because, unlike in QED, the charge flow through the fermion loop destroys the symmetry between the fermion loop diagram and the diagram with reversed fermion direction.

We summarize now the total contributions from a single light generation (where appropriate, we always include a mirror counterpart of the diagrams)

$$\begin{aligned}\Delta a_{1a} &= \frac{C}{s_W^2} Q_u \left(\frac{37}{72} + \frac{7}{36\varepsilon} - \frac{7}{18}\ln M_W^2\right) \\ \Delta a_{1b} &= \frac{C}{s_W^2} Q_d \left(-\frac{1}{72} - \frac{7}{36\varepsilon} + \frac{7}{18}\ln M_W^2\right) \\ \Delta a_{1c} &= \frac{C}{s_W^2} \left(\frac{1}{54} - \frac{1}{6\varepsilon} + \frac{1}{3}\ln M_W^2\right) \\ \Delta a_{1d} &= \frac{C}{s_W^2} \left(-\frac{1}{216} - \frac{1}{36\varepsilon} + \frac{1}{18}\ln M_W^2\right)\end{aligned} \tag{15}$$

These expressions have to be multiplied by a color factor where necessary. We put the relevant Kobayashi-Maskawa matrix elements equal to 1. Adding contributions of all fermions we obtain for a single light generation

$$\Delta a_{light} = \frac{C}{s_W^2}\frac{10}{9} \tag{16}$$

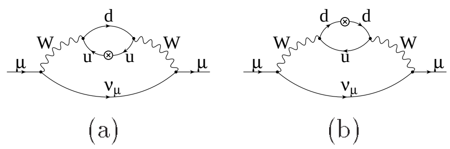
We now proceed to the contribution of the third generation. For the τ lepton loop we can still use eq. (15). We obtain

$$\Delta a_\tau = \frac{C}{s_W^2}\frac{1}{36} \tag{17}$$

In diagrams with quarks we neglect the mass of the b. To leading order in $\frac{M_W^2}{m_t^2}$ we obtain for the sum of all diagrams containing top and bottom quark loops

$$\begin{aligned}\Delta a_{tb} = \frac{CN_C}{s_W^2}\Bigg[&-\frac{1}{36} - \frac{1}{6}\ln\frac{m_t^2}{M_W^2} \\ &+\frac{m_t^2}{M_W^2}\left(-\frac{8}{9} - \frac{5}{12\varepsilon} + \frac{5}{6}\ln m_t M_W\right)\Bigg]\end{aligned} \tag{18}$$

The singular terms $\frac{m_t^2}{\varepsilon}$ are cancelled by renormalization of the W boson mass present in the one loop electroweak contributions to muon g-2.



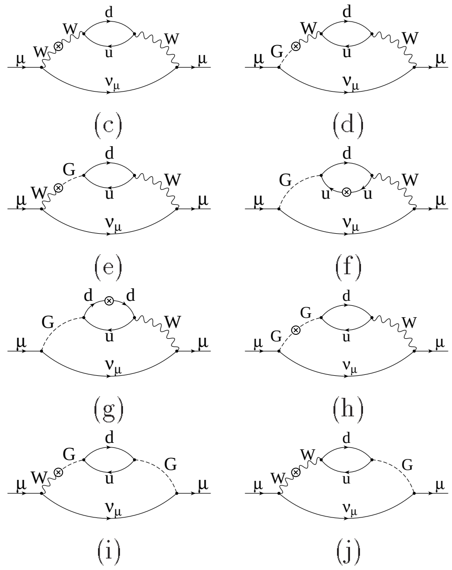


Figure 3: Diagrams with a charged boson and a fermion loop contributing to the muon g-2. Crossed circles denote interactions with an external photon.

Projector for g-2 form factor
An efficient projector for g-2 calculations has been obtained in ref.[24]. Here we extend their formulae to $d = 4 - 2\varepsilon$ dimensions.

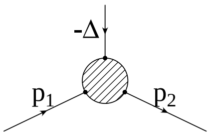


Figure 4: Definition of momentum flow

We consider the most general matrix element of a current between spin 1/2 fermions

$$\begin{aligned}\langle\alpha_f|J_\mu(x)|\alpha_i\rangle &= \qquad (19)\\ \bar{u}_f(p_2)&\left[F_1(t)\gamma_\mu - \frac{i}{2m}F_2(t)\sigma_{\mu\nu}\Delta^\nu + \frac{1}{m}F_3(t)\Delta_\mu\right.\\ &\left.+\gamma_5\left(G_1(t)\gamma_\mu - \frac{i}{2m}G_2(t)\sigma_{\mu\nu}\Delta^\nu + \frac{1}{m}G_3(t)\Delta_\mu\right)\right]\\ &\cdot u_i(p_1)e^{i\Delta x}\end{aligned}$$

with $\Delta = p_1 - p_2$ and $\Delta^2 = t$. For on-shell external fermions we have

$$p_1^2 = p_2^2 = m^2 \qquad (20)$$

Introducing $p = \frac{1}{2}(p_1 + p_2)$ we obtain in addition:

$$p^2 = \frac{1}{4}(4m^2 - t)\,, \qquad p\cdot\Delta = 0\,. \qquad (21)$$

Conservation of the electromagnetic current requires $F_3(t) = 0$. $F_1(t)$ is the charge form factor, $F_2(t)$ the magnetic moment form factor, $G_2(t)$ the electric moment form factor. The anomalous magnetic moment of the fermion a is given by

$$a \equiv \frac{1}{2}(g-2) = F_2(0) \qquad (22)$$

In order to extract the magnetic moment form factor, one can introduce a projection operator

$$\begin{aligned}N_\mu &= \qquad (23)\\ &(\not{p}_1 + m)\left[g_1\gamma_\mu - \frac{1}{m}g_2p_\mu - \frac{1}{m}g_3\Delta_\mu\right](\not{p}_2 + m)\end{aligned}$$

In order to determine the coefficients g_i we take the trace of $N_\mu M^\mu$:

$$\begin{aligned}\mathrm{Tr}(N_\mu M^\mu) &= \left\{\left[8m^2 + 2t(d-2)\right]g_1\right.\\ &\qquad\left.+2(t-4m^2)g_2\right\}F_1(t)\\ &+\left[2t(d-1)g_1 + t\frac{(t-4m^2)}{2m^2}g_2\right]F_2(t)\\ &+\left[2t\frac{(t-4m^2)}{m^2}g_3\right]F_3(t) \qquad (24)\end{aligned}$$

This leads us to the following system of equations for $F_2(t)$:

$$\begin{aligned}0 &= \left[8m^2 + 2t(d-2)\right]g_1 + 2(t-4m^2)g_2\\ 1 &= 2t(d-1)g_1 + t\frac{(t-4m^2)}{2m^2}g_2 \qquad (25)\end{aligned}$$

and we can put $g_3 = 0$ already at this step. The solution is

$$g_1 = \frac{-2m^2}{(d-2)\,t(t-4m^2)} \qquad (26)$$

$$g_2 = \frac{2m^2\left[4m^2 + (d-2)t\right]}{(d-2)\,t(t-4m^2)^2} \qquad (27)$$

and we find

$$\begin{aligned}F_2(t) &= \frac{-2m^2}{(d-2)\,t(t-4m^2)}\mathrm{Tr}\left[(\not{p}_1 + m)\,\times\right.\\ &\left.\left(\gamma_\mu + \frac{4m^2 + (d-2)t}{m(t-4m^2)}p_\mu\right)(\not{p}_2 + m)M^\mu\right] \qquad (28)\end{aligned}$$

It is also possible to extract directly the anomaly $F_2(0)$. As a first step the general amplitude M_μ can then be expanded to first order in Δ_μ:

$$\begin{aligned}M_\mu(p,\Delta) &\approx M_\mu(p,0) + \Delta_\nu\frac{\partial}{\partial\Delta_\nu}M_\mu(p,\Delta)\Big|_{\Delta=0}\\ &\equiv V_\mu(p) + \Delta_\nu T_\mu^\nu(p)\,. \qquad (29)\end{aligned}$$

The next step is to average over the spatial directions of Δ with the formulas

$$\begin{aligned}\langle\Delta_\mu\Delta_\nu\rangle &= \frac{1}{d-1}\Delta^2\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\,,\\ \langle\Delta_\mu\rangle &= 0\,; \qquad (30)\end{aligned}$$

after that, the limit $t \to 0$ can be taken. The result is

$$a = \frac{1}{2(d-1)(d-2)m^2}\times \quad (31)$$
$$\mathrm{Tr}\left\{\frac{d-2}{2}\left[m^2\gamma_\mu - dp_\mu \not{p} - (d-1)mp_\mu\right]V^\mu + \frac{m}{4}(\not{p}+m)[\gamma_\nu,\gamma_\mu](\not{p}+m)T^{\mu\nu}\right\}$$

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