TTP96-04*
MPI/PhT/96-10
DTP/96/12
hep-ph/9603313
March 1996

DOUBLE BUBBLE CORRECTIONS TO HEAVY QUARK PRODUCTION‡

K.G. Chetyrkin[a], A.H. Hoang[b†], J.H. Kühn[b], M. Steinhauser[b†], T. Teubner[c]

[a] Max-Planck-Institut für Physik, Werner-Heisenberg-Institut,
Föhringer Ring 6, D-80805 Munich, Germany

[b] Institut für Theoretische Teilchenphysik,
Universität Karlsruhe, D-76128 Karlsruhe, Germany

[c] Department of Physics,
University of Durham, Durham, DH1 3LE, UK

Abstract

Second order $\mathcal{O}(\alpha_s^2)$ corrections to the heavy quark production cross-section due to massless quarks and coloured scalars are calculated for all energies above threshold. Based on the method introduced in this letter also the gauge non-invariant second order corrections due to the pure gluonic self-energy insertion and a certain class of $\mathcal{O}(\alpha_s^3)$ and $\mathcal{O}(\alpha_s^4)$ corrections are determined. For the special choice of the gauge parameter, $\xi = 4$, the leading threshold and high energy behaviour of the pure second order gluonic corrections to the cross-section are governed by the gluonic self-energy insertion.

1 Introduction

As a consequence of the high precision measurements at LEP the total cross-section for quark anti-quark production in e^+e^--annihilation has been subject to extensive studies during the past years. Whereas the $\mathcal{O}(\alpha_s)$ corrections are known for all energy and mass values [1], the complete $\mathcal{O}(\alpha_s^2)$ corrections are only known in the high energy expansion including terms up to $\mathcal{O}(M^4/s^2)$, where s denotes the c.m. energy and M the mass of the produced quarks, see [2]. However, in view of future experiments (τ-charm-, B-factory, NLC) where quark anti-quark pairs will be produced

*The postscript file of this preprint, including figures, is available via anonymous ftp at ftp://www-ttp.physik.uni-karlsruhe.de (129.13.102.139) as /ttp96-04/ttp96-04.ps or via www at http://www-ttp.physik.uni-karlsruhe.de/cgi-bin/preprints/.

‡Work supported by BMFT under Contract 056KA93P6, DFG under Contract Ku502/6-1 and INTAS under Contract INTAS-93-0744.

†Supported by Graduiertenkolleg Elementarteilchenphysik, Karlsruhe.

near their production threshold, the knowledge of the complete $\mathcal{O}(\alpha_s^2)$ corrections for all mass and energy assignments is desirable. Analytical formulae are of particular importance because they provide important cross-checks for approximation methods which can be applied even where an analytical evaluation seems to be impossible. A closed analytical expression for the vector-current induced $\mathcal{O}(\alpha_s^2)$ corrections due to a massless fermion pair has been published in [3]. The complete vector-current induced gluonic contribution were obtained recently in [4] using Padé approximation methods.

In this letter a refinement and an extension of the results obtained in [3] are presented. In section 2 the second order corrections to the total cross-section of massive fermion pair production due to radiation of a light secondary fermion pair from the final state will be presented. The framework of on-shell renormalized QED will be employed with the fine structure constant as the expansion parameter of the perturbation series. The result will be parametrized in terms of moments in which all information on the vacuum polarization due to the light fermion pair is encoded. As a consequence the corresponding corrections due to a pair of light scalar particles can be easily determined. In the framework of a supersymmetric model this result can be applied to light sfermion radiation in $t\bar{t}$ production. The concept of the moments even allows for a determination of those fermionic $\mathcal{O}(\alpha^3)$ corrections to the total inclusive massive fermion pair production cross-section which originate from the insertion of the two-loop vacuum polarization into the photon line. The remaining logarithm of the square of the small mass devided by the c.m. energy can be absorbed by employing the running coupling. In section 3 the transition to QCD will be performed. Based on the results of section 2 the gauge non-invariant $\mathcal{O}(\alpha_s^2)$ contributions due to the gluonic self-energy will be presented. The result will be examined for the special gauge $\xi = 4$, where the leading contributions in the threshold region as well as for high energies are determined by the self-energy insertion. In section 4, finally, we apply the concept of moments to determine a certain class of fermionic $\mathcal{O}(\alpha_s^4)$ corrections without referring back to the corresponding QED result. This calculation constitutes an example how the method of moments can be used to fix some even higher order QCD corrections.

2 Light Fermionic and Scalar Second Order Corrections

We consider the vector-current induced inclusive cross-section for the production of a fermion anti-fermion pair, $F\bar{F}$ (with mass M), normalized to the point cross-section,

$$R_{F\bar{F}} = \frac{\sigma(e^+e^- \to \gamma^* \to F\bar{F}\ldots)}{\sigma_{pt}}, \qquad \sigma_{pt} = \frac{4\,\pi\,\alpha^2}{3\,s} \tag{1}$$

for arbitrary c.m. energy, $\sqrt{s} \geq 2\,M$. We only discuss final state corrections and restrict ourselves in this section to on-shell renormalized QED. The perturbative series of $R_{F\bar{F}}$ reads

$$R_{F\bar{F}} = r^{(0)} + \left(\frac{\alpha}{\pi}\right) r^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 r^{(2)} + \ldots, \tag{2}$$

where $\alpha = 1/137$ is the fine structure constant defined in the Thomson limit. The Born and first order [1] contributions are well known:

$$\begin{aligned}
r^{(0)} &= \frac{\beta}{2}\,(3-\beta^2)\,,\\
r^{(1)} &= \frac{(3-\beta^2)\,(1+\beta^2)}{2}\left[2\,\mathrm{Li}_2(p) + \mathrm{Li}_2(p^2) + \ln p\,\Big(2\,\ln(1-p) + \ln(1+p)\Big)\right]\\
&\quad - \beta\,(3-\beta^2)\,\Big(2\,\ln(1-p) + \ln(1+p)\Big) - \frac{(1-\beta)\,(33 - 39\,\beta - 17\,\beta^2 + 7\,\beta^3)}{16}\,\ln p\\
&\quad + \frac{3\,\beta\,(5-3\,\beta^2)}{8}\,,
\end{aligned} \tag{3}$$

where

$$p = \frac{1-\beta}{1+\beta}\,, \qquad \beta = \sqrt{1-4\,x}\,, \qquad x = \frac{M^2}{s}\,.$$

For the convenience of the reader and for later reference we also present the corresponding expansions near threshold, $\beta \equiv \sqrt{1-4\,x} \to 0$, and at high energies, $x \to 0$.

$$\begin{aligned}
r^{(0)} &\overset{\beta\to 0}{\longrightarrow} \frac{3}{2}\beta + \mathcal{O}(\beta^3)\,,\\
r^{(1)} &\overset{\beta\to 0}{\longrightarrow} \frac{9}{2}\zeta(2) - 6\,\beta + 3\,\zeta(2)\,\beta^2 + \mathcal{O}(\beta^3)\,,\\
r^{(0)} &\overset{x\to 0}{\longrightarrow} 1 - 6\,x^2 - 8\,x^3 + \mathcal{O}(x^4)\,,\\
r^{(1)} &\overset{x\to 0}{\longrightarrow} \frac{3}{4} + 9\,x + \Big(\frac{15}{2} - 18\,\ln x\Big)\,x^2 - \frac{4}{9}\Big(47 + 87\,\ln x\Big)\,x^3 + \mathcal{O}(x^4)\,.
\end{aligned} \tag{4}$$

The contribution to $r^{(2)}$ arising from a light fermion anti-fermion pair with mass m which corresponds to the sum of all possible cuts of the current-current correlator diagrams depicted in Fig. 1 is denoted by $r_f^{(2)}$.

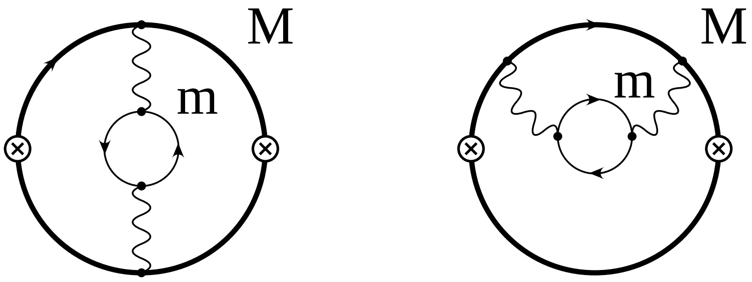

Figure 1: Fermionic double bubble diagrams.

The contribution where the light fermion pair is primarily produced is treated in [5] and will not be discussed in the following. Due to Furry's theorem the corresponding interference terms (singlet contributions) vanish identically, if vector-currents are considered. The result for $r_f^{(2)}$ can be written in the form ($x = f$)

$$r_x^{(2)} = -\frac{1}{3}\left[R_\infty^x\,\ln\frac{m^2}{s} - R_0^x\right] r^{(1)} + R_\infty^x\,\delta^{(2)}\,, \tag{5}$$

where the moments R_∞^f and R_0^f are constants which are uniquely determined by the vacuum polarization due to the light fermion pair

$$\Pi_{light}^x(q^2) \overset{m^2\to 0}{\longrightarrow} -\frac{\alpha}{3\,\pi}\left[R_\infty^x\,\ln\frac{-q^2}{4\,m^2} + R_0^x\right]. \tag{6}$$

The combination $\alpha^2 R_\infty^f/(3\,\pi)$ is the second order contribution to the QED running coupling β-function, whereas R_0^f is the zeroth momentum of the vacuum polarization due to the light fermion pair, defined in [6]

$$\begin{aligned}
R_\infty^f &= R_{f\bar f}(\infty) = 1,\\
R_0^f &= \int\limits_{4\,m^2}^{\infty} \frac{ds}{s}\left[R_{f\bar f}(s) - R_\infty^f\right] = -\frac{5}{3} + \ln 4.
\end{aligned} \tag{7}$$

$R_{f\bar{f}}$ is the normalized Born cross-section for pair production of the light fermion anti-fermion pair with mass m. The function $\delta^{(2)}$ has already been presented in [3] in a somewhat different form,

$$
\begin{aligned}
\delta^{(2)} \;=\; & -\frac{(3-\beta^2)\;(1+\beta^2)}{6}\times \\
& \Big\{ \mathrm{Li}_3(p) - 2\,\mathrm{Li}_3(1-p) - 3\,\mathrm{Li}_3(p^2) - 4\,\mathrm{Li}_3\Big(\frac{p}{1+p}\Big) - 5\,\mathrm{Li}_3(1-p^2) + \frac{11}{2}\,\zeta(3) \\
& + \mathrm{Li}_2(p)\,\ln\Big(\frac{4\;(1-\beta^2)}{\beta^4}\Big) + 2\,\mathrm{Li}_2(p^2)\,\ln\Big(\frac{1-\beta^2}{2\,\beta^2}\Big) + 2\,\zeta(2)\,\Big[\,\ln p - \ln\Big(\frac{1-\beta^2}{4\,\beta}\Big)\Big] \\
& - \frac{1}{6}\,\ln\Big(\frac{1+\beta}{2}\Big)\,\Big[\,36\,\ln 2\,\ln p - 44\,\ln^2 p + 49\,\ln p\,\ln\Big(\frac{1-\beta^2}{4}\Big) + \ln^2\Big(\frac{1-\beta^2}{4}\Big)\Big] \\
& - \frac{1}{2}\,\ln p\,\ln\beta\,\Big[\,36\,\ln 2 + 21\,\ln p + 16\,\ln\beta - 22\,\ln(1-\beta^2)\Big]\Big\} \\
& + \frac{1}{24}\Big\{(15 - 6\,\beta^2 - \beta^4)\,\Big(\mathrm{Li}_2(p) + \mathrm{Li}_2(p^2)\Big) + 3\,(7 - 22\,\beta^2 + 7\,\beta^4)\,\mathrm{Li}_2(p) \\
& - (1-\beta)\,(51 - 45\,\beta - 27\,\beta^2 + 5\,\beta^3)\,\zeta(2) \\
& + \frac{(1+\beta)\;(-9 + 33\,\beta - 9\,\beta^2 - 15\,\beta^3 + 4\,\beta^4)}{\beta}\,\ln^2 p \\
& + \Big[(33 + 22\,\beta^2 - 7\,\beta^4)\,\ln 2 - 10\,(3-\beta^2)\,(1+\beta^2)\,\ln\beta \\
& \qquad - (15 - 22\,\beta^2 + 3\,\beta^4)\,\ln\Big(\frac{1-\beta^2}{4\,\beta^2}\Big)\Big]\,\ln p \\
& + 2\,\beta\,(3-\beta^2)\,\ln\Big(\frac{4\;(1-\beta^2)}{\beta^4}\Big)\,\Big[\,\ln\beta - 3\,\ln\Big(\frac{1-\beta^2}{4\,\beta}\Big)\Big] \\
& + \frac{237 - 96\,\beta + 62\,\beta^2 + 32\,\beta^3 - 59\,\beta^4}{4}\,\ln p - 16\,\beta\,(3-\beta^2)\,\ln\Big(\frac{1+\beta}{4}\Big) \\
& - 2\,\beta\,(39 - 17\,\beta^2)\,\ln\Big(\frac{1-\beta^2}{2\,\beta^2}\Big) - \frac{\beta\;(75 - 29\,\beta^2)}{2}\Big\}\,.
\end{aligned}
\tag{8}
$$

The threshold and high energy behaviour is given by

$$
\delta^{(2)} \;\stackrel{\beta\to 0}{\longrightarrow}\; 3\,\zeta(2)\,\ln\frac{\beta}{2} + \Big(-\frac{3}{2} + 8\,\ln 2\Big)\,\beta + 2\,\zeta(2)\,\Big(\ln\frac{\beta}{2} - 2\Big)\,\beta^2 \;+\; \mathcal{O}(\beta^3)\,,
\tag{9}
$$

$$
\begin{aligned}
\delta^{(2)} \;\stackrel{x\to 0}{\longrightarrow}\; & \zeta(3) - \frac{1}{2}\,\ln 2 - \frac{23}{24} - 3\,\Big(2\,\ln 2 + \frac{1}{2}\Big)\,x \\
& + \Big(-3\,\ln^2 x + \ln x\,\Big(12\,\ln 2 + \frac{7}{2}\Big) - 4\,\zeta(3) - 18\,\zeta(2) - 5\,\ln 2 - \frac{5}{3}\Big)\,x^2 \\
& + \frac{2}{27}\,\Big(-108\,\ln^2 x + 2\,\ln x\,(174\,\ln 2 + 43) - 456\,\zeta(2) + 188\,\ln 2 + 57\Big)\,x^3 \;+\; \mathcal{O}(x^4)\,.
\end{aligned}
\tag{10}
$$

An explicit derivation of eq. (5) can be found in [7], where the two- and four-body cuts are calculated separately.

The complete information on the vacuum polarization relevant for the internal photon line is encoded in the moments R^x_∞ and R^x_0. The crucial ingredient which allows to arrive at this simple and compact formulation is the following condition on the high energy behaviour of $R_{f\bar{f}}(s)$: it has to approach a constant value R^x_∞ in the limit of large $\sqrt{s}$ fast enough, which is equivalent to the occurrence of at most one single logarithm $\ln(-q^2/4m^2)$ in the vacuum polarization function. Thus we can determine without any effort the second order correction due to a light pair of unit-charged scalar particles with mass m. The corresponding moments for the vacuum polarization, Π^s_{light}, as defined in (6) read

$$
R^s_\infty \;=\; R_{ss^*}(\infty) \;=\; \frac{1}{4},
$$

$$R_0^s \;=\; \int\limits_{4m^2}^{\infty} \frac{ds}{s}\left[R_{ss^*}(s) - R_\infty^s\right] \;=\; -\frac{2}{3}+\frac{1}{4}\ln 4\,. \tag{11}$$

Because the $\mathcal{O}(\alpha)$ corrections to the cross-section $R_{f\bar{f}}$ also approach a constant value for high energies even $\mathcal{O}(\alpha^3)$ corrections to $R_{F\bar{F}}$ due to a light fermion pair with additional real and virtual radiation of a photon off the light fermions can be calculated. These $\mathcal{O}(\alpha^3)$ contributions can be cast into the form of eq. (5) with the following two moments from the vacuum polarization function $\Pi_{light}^{f\gamma}$,

$$\begin{aligned} R_\infty^{f\gamma} &= \frac{\alpha}{\pi}\frac{3}{4}\,, \\ R_0^{f\gamma} &= \frac{\alpha}{\pi}\left(-\frac{5}{8}+3\zeta(3)+\frac{3}{4}\ln 4\right)\,. \end{aligned} \tag{12}$$

Similar arguments hold for the $\mathcal{O}(\alpha)$ corrections to R_{ss^*}. Here the moments read

$$\begin{aligned} R_\infty^{s\gamma} &= \frac{\alpha}{\pi}\frac{3}{4}\,, \\ R_0^{s\gamma} &= \frac{\alpha}{\pi}\left(-\frac{49}{16}+\frac{3}{4}\zeta(3)+\frac{3}{4}\ln 4\right)\,, \end{aligned} \tag{13}$$

where the vacuum polarization diagrams depicted in Fig. 2 have to be taken into account.

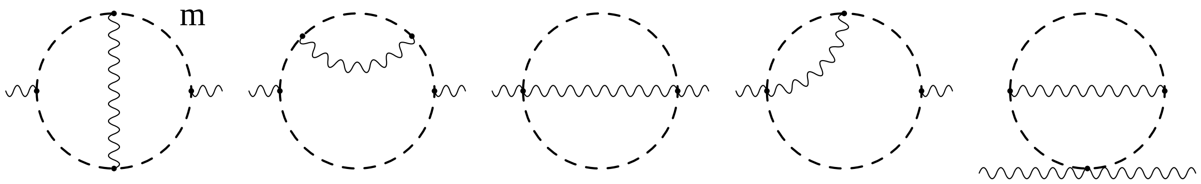

Figure 2: Scalar two-loop diagrams to the vacuum polarization. The dashed line represents a unit-charged scalar particle with mass m.

The mass singularity in the limit $m \to 0$, evident from eqs. (5) and (6) is a consequence of the renormalization scheme with the fine structure constant α defined at momentum transfer zero. For the vacuum polarization due to massless particles a mass independent renormalization scheme such as $\overline{\mathrm{MS}}$ for the coupling is more appropriate. The $\mathcal{O}(\alpha^2)$ relation between the fine structure constant and the running $\overline{\mathrm{MS}}$ coupling at the scale μ for the case of either a light fermion or a light scalar reads

$$\alpha \;=\; \alpha_{\overline{\mathrm{MS}}}(\mu^2)\left(1+\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi}\,\frac{1}{3}\,R_\infty^x\,\ln\frac{m^2}{\mu^2}\right)+\mathcal{O}(\alpha_{\overline{\mathrm{MS}}}^3)\,. \tag{14}$$

Replacing the fine structure constant in eq. (2) by $\alpha_{\overline{\mathrm{MS}}}$ results in

$$R_{F\bar{F}} \;=\; r^{(0)}+\left(\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi}\right)r^{(1)}+\left(\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi}\right)^2 r^{(2)}_{\overline{\mathrm{MS}}}+\ldots\,, \tag{15}$$

where

$$r^{(2)}_{x,\overline{\mathrm{MS}}} \;=\; -\frac{1}{3}\left[R_\infty^{x,\overline{\mathrm{MS}}}\ln\frac{\mu^2}{s}-R_0^{x,\overline{\mathrm{MS}}}\right]r^{(1)}+R_\infty^{x,\overline{\mathrm{MS}}}\,\delta^{(2)}\,,\qquad x\;=\;f,s. \tag{16}$$

It is evident that eq. (16) closely resembles eq. (5). Also in the $\overline{\mathrm{MS}}$ scheme the moments are uniquely determined via the vacuum polarization function

$$\Pi_{massless}^{x,\overline{\mathrm{MS}}}(q^2) \;=\; -\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{3\,\pi}\left[R_\infty^{x,\overline{\mathrm{MS}}}\ln\frac{-q^2}{4\,\mu^2}+R_0^{x,\overline{\mathrm{MS}}}\right]\,, \tag{17}$$

where

$$R_\infty^{x,\overline{\rm MS}} = R_\infty^x\,,\ R_0^{x,\overline{\rm MS}} = R_0^x \quad \text{for} \quad x = f,s. \tag{18}$$

It should be noted that the equality of the zero-moments, R_0^x, in the $\overline{\rm MS}$ and on-shell scheme holds because there are no non-logarithmic terms present in eq. (14). For the three-loop corrections which were mentioned above again the r.h.s. of eq. (16) can be employed in the $\overline{\rm MS}$ scheme. The corresponding $\overline{\rm MS}$ moments read

$$\begin{aligned}
R_\infty^{f\gamma,\overline{\rm MS}} &= \frac{\alpha_{\overline{\rm MS}}}{\pi}\frac{3}{4}\,,\\
R_0^{f\gamma,\overline{\rm MS}} &= \frac{\alpha_{\overline{\rm MS}}}{\pi}\left(-\frac{55}{16}+3\zeta(3)+\frac{3}{4}\ln 4\right)\,,\\
R_\infty^{s\gamma,\overline{\rm MS}} &= \frac{\alpha_{\overline{\rm MS}}}{\pi}\frac{3}{4}\,,\\
R_0^{s\gamma,\overline{\rm MS}} &= \frac{\alpha_{\overline{\rm MS}}}{\pi}\left(-\frac{43}{16}+\frac{3}{4}\zeta(3)+\frac{3}{4}\ln 4\right)\,.
\end{aligned} \tag{19}$$

Here, the zero-moments in the $\overline{\rm MS}$ scheme are different from the corresponding ones in the on-shell scheme, eqs. (12, 13), due to a non-logarithmic contribution in the $\mathcal{O}(\alpha^3)$ relation between the fine structure constant and $\alpha_{\overline{\rm MS}}$ ($x = f\gamma, s\gamma$):

$$\alpha \,=\, \alpha_{\overline{\rm MS}}(m^2)\left[1+\frac{1}{3}\left(\frac{\alpha_{\overline{\rm MS}}(m^2)}{\pi}\right)^2\left(R_0^{x,\overline{\rm MS}} - R_0^x\right)\right] + \mathcal{O}(\alpha_{\overline{\rm MS}}^4)\,. \tag{20}$$

Eqs. (16) and (17) provide a simple and unambiguous method to determine second order corrections to $R_{F\bar{F}}$ due to the vacuum polarization from arbitrary massless particles. For that one has to compute the $\overline{\rm MS}$-renormalized one-loop vacuum polarization function, $\Pi^{x,\overline{\rm MS}}_{massless}$, where the index x denotes the types of massless particles considered, and to identify the moments $R_\infty^{x,\overline{\rm MS}}$ and $R_0^{x,\overline{\rm MS}}$, as defined in eq. (17). Inserting the moments into eq. (16) gives the desired result.

3 Transition to QCD and Gluon Bubble Contribution

To obtain the $\mathcal{O}(\alpha_s^2)$ corrections to massive quark production due to massless quarks (or squarks) and the $\mathcal{O}(\alpha_s^3)$ contributions corresponding to the $\mathcal{O}(\alpha^3)$ corrections presented in the previous section we have to multiply the QED results by the corresponding SU(3) group theoretical factors $T = 1/2$ (the moments $R_\infty^{x,\overline{\rm MS}}$, $R_0^{x,\overline{\rm MS}}$, $R_\infty^{x\gamma,\overline{\rm MS}}$, and $R_0^{x\gamma,\overline{\rm MS}}$ ($x = f,s$)) and $C_F = 4/3$ (the $r^{(1)}$, $\delta^{(2)}$, $R_\infty^{x\gamma,\overline{\rm MS}}$, and $R_0^{x\gamma,\overline{\rm MS}}$ ($x = f,s$)). For the $\mathcal{O}(\alpha_s^3)$ contributions this leads to the colour factor $T\,C_F^2$. Furthermore an additional global colour factor $N_c = 3$ has to be taken into account. $\alpha_{\overline{\rm MS}}$ now represents the $\overline{\rm MS}$ renormalized QCD coupling constant.

Using the method described in section 2 we are now in a position to determine the gluonic self-energy contribution to $r^{(2)}$, as illustrated in Fig. 3, by determining the corresponding moments for the gluonic contributions to the $\mathcal{O}(\alpha_s)$ gluon propagator. The moments which correspond to eq. (17) read

$$\begin{aligned}
R_\infty^{g,\overline{\rm MS}} &= C_A\left(-\frac{5}{4}-\frac{3}{8}\xi\right)\,,\\
R_0^{g,\overline{\rm MS}} &= C_A\left(\frac{31}{12}-\frac{3}{4}\xi+\frac{3}{16}\xi^2+\left(-\frac{5}{4}-\frac{3}{8}\xi\right)\ln 4\right)\,,
\end{aligned} \tag{21}$$

where the gauge parameter ξ is defined via the gluon propagator in lowest order

$$\frac{i}{q^2+i\,\epsilon}\left(-\,g^{\mu\nu}+\xi\,\frac{q^\mu\,q^\nu}{q^2}\right)\,. \tag{22}$$

Of course, the moments and $r^{(2)}_{g,\overline{\rm MS}}$ are not gauge invariant. However, it is an interesting fact that

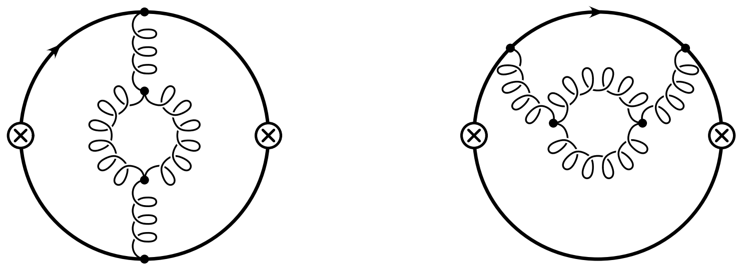

Figure 3: Gluonic double bubble diagrams. The ghost diagrams are not depicted explicitly.

for the special choice $\xi = 4$ the combination $\alpha_s^2\, R_\infty^{g,\overline{\rm MS}}/(3\,\pi)$ coincides with the gluonic contribution to the QCD β-function of $\mathcal{O}(\alpha_s^2)$. Thus for $\xi = 4$ the term $r^{(2)}_{g,\overline{\rm MS}}$ accounts for the leading logarithmic behaviour of the sum of all gluonic $\mathcal{O}(\alpha_s^2)$ diagrams in the high energy limit. It is quite obvious that such a ξ can be found. Remarkably enough, the complete gluonic contributions of $\mathcal{O}(\alpha_s^2)$ to the QCD potential are described by $r^{(2)}_{g,\overline{\rm MS}}$ for the choice $\xi = 4$. To be specific,

$$V_{\rm QCD}(Q^2) \;=\; -4\pi\, C_F\, \frac{\alpha_V(Q^2)}{Q^2} \tag{23}$$

with [8]

$$\begin{aligned} \alpha_V(Q^2) &= \alpha_s(\mu^2)\left(1 - \Pi^{g,\overline{\rm MS}}(-Q^2)|_{\xi=4}\right) \\ &= \alpha_s(\mu^2)\left[1 + \frac{\alpha_s(\mu^2)}{3\,\pi}\, C_A\left(-\frac{11}{4}\ln\frac{Q^2}{\mu^2} + \frac{31}{12}\right)\right]. \end{aligned} \tag{24}$$

For this choice of gauge the leading threshold behaviour of order α_s^2 with the colour structure proportional to $C_A C_F$ is incorporated in the gluonic double bubble diagrams. This can be seen as follows: The leading term to $R_{F\bar{F}}$ of $\mathcal{O}(\alpha_s)$ is given by

$$R_{F\bar{F}} \;\stackrel{\beta\to 0}{\longrightarrow}\; N_c\, C_F\, \frac{3\,\pi}{4}\, \alpha_s\,. \tag{25}$$

(The leading term of order α_s^2 which is proportional to π^2/β can be derived from Sommerfeld's rescattering formula for the Coulomb problem. It is, however, proportional to C_F^2.) We are interested in terms of order α_s^2 proportional to $C_A C_F$ which are characteristic for the non-abelian nature of the interaction. On the basis of general considerations this term is obtained from eq. (25) through the replacement of α_s by $\alpha_V(\beta^2 s)$ as given in eq. (24). This prediction of the leading logarithmic and constant $C_A C_F$ term coincides with the result of the analytic calculation of the diagrams (Fig. 3) in the $\xi = 4$ gauge. This aspect strongly resembles the behaviour of the leading threshold contribution from massless quark loops as discussed in [9] where a similar relation between the leading threshold terms and eq. (25) expressed through an effective coupling constant has been observed.

4 Application at $\mathcal{O}(\alpha_s^4)$

The concept of the moments allows for the determination of higher order QCD corrections from one gluon exchange diagrams with the insertion of massless vacuum polarization functions into the gluon line which contain at most one single logarithm $\ln(-q^2/4\mu^2)$. This condition is fulfilled by the sum

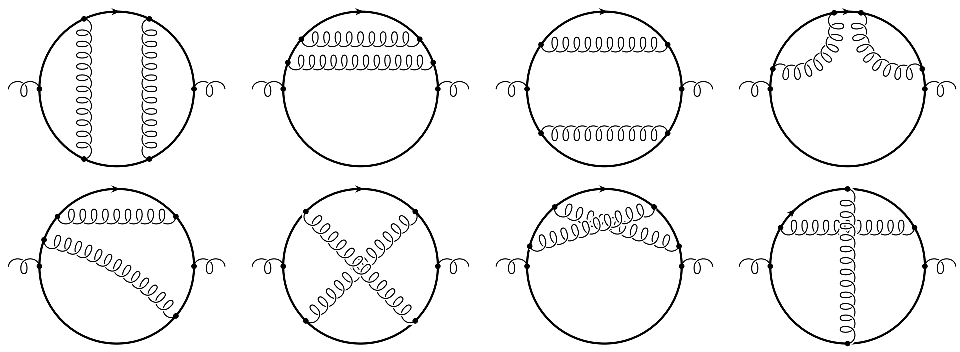

Figure 4: Three-loop diagrams contributing to $R_\infty^{fgg,\overline{\rm MS}}$ and $R_0^{fgg,\overline{\rm MS}}$.

of those terms in the third order vacuum polarization due to a massless fermion anti-fermion pair (depicted in Fig. 4) which are proportional to $T\,C_F^2$. It is the same class of contributions which would also be present in an abelian theory. The result for the corresponding vacuum polarization function in the $\overline{\rm MS}$ scheme reads [10]

$$\Pi^{fgg,\overline{\rm MS}}_{massless}(q^2) \;=\; -\frac{\alpha_s(\mu^2)}{3\,\pi}\left[R_\infty^{fgg,\overline{\rm MS}}\ln\frac{-q^2}{4\,\mu^2}+R_0^{fgg,\overline{\rm MS}}\right], \tag{26}$$

with

$$\begin{aligned} R_\infty^{fgg,\overline{\rm MS}} &= \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 C_F^2\,T\,\left(-\frac{3}{32}\right), \\ R_0^{fgg,\overline{\rm MS}} &= \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 C_F^2\,T\,\left(\frac{143}{96}+\frac{37}{8}\zeta(3)-\frac{15}{2}\zeta(5)-\frac{3}{32}\ln 4\right). \end{aligned} \tag{27}$$

The same is also true for the corresponding third order contributions to the vacuum polarization due to a massless squark anti-squark pair, $\Pi^{sgg,\overline{\rm MS}}_{massless}$ with the moments [11]

$$\begin{aligned} R_\infty^{sgg,\overline{\rm MS}} &= \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 C_F^2\,T\,\frac{87}{128}, \\ R_0^{sgg,\overline{\rm MS}} &= \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 C_F^2\,T\,\left(-\frac{251}{96}+\frac{5}{2}\zeta(3)-\frac{15}{8}\zeta(5)+\frac{87}{128}\ln 4\right). \end{aligned} \tag{28}$$

Thus the corrections to the total heavy quark production cross-section from the insertion of $\Pi^{xgg,\overline{\rm MS}}_{massless}$ into the one gluon exchange diagrams read ($x = f$ for massless fermions or $x = s$ for massless squarks)

$$r^{(4)}_{xgg,\overline{\rm MS}} = N_c\left\{-\frac{1}{3}\left[R_\infty^{xgg,\overline{\rm MS}}\ln\frac{\mu^2}{s}-R_0^{xgg,\overline{\rm MS}}\right]C_F\,r^{(1)}+R_\infty^{xgg,\overline{\rm MS}}\,C_F\,\delta^{(2)}\right\}. \tag{29}$$

This four-loop result exemplifies the power of the concept of moments used in this paper to evaluate massive higher order contributions to the production of heavy quarks.

References

[1] G. Källen and A. Sabry, *K. Dan. Vidensk. Selsk. Mat.-Fys. Medd.* **29** (1955) No. 17, see also J. Schwinger, *Particles, Sources and Fields*, Vol.II, (Addison-Wesley, New York, 1973).

[2] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, *Phys. Reports* (in print), hep-ph/9503396 and references therein.

[3] A.H. Hoang, J.H. Kühn and T. Teubner, *Nucl. Phys.* **B 452** (1995) 173.

[4] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, *Phys. Lett.* **B 371** (1996) 93.

[5] A.H. Hoang, M. Jeżabek, J.H. Kühn and T. Teubner, *Phys. Lett.* **B 338** (1994) 330.

[6] B.A. Kniehl, M. Krawczyk, J.H.Kühn and R.G. Stuart, *Phys. Lett.* **B 209** (1988) 337.

[7] A.H. Hoang and T. Teubner, in preparation.

[8] W. Fischler, *Nucl. Phys.* **B 129** (1977) 157.

[9] S.J. Brodsky, A.H. Hoang, J.H. Kühn and T. Teubner, *Phys. Lett.* **B 359** (1995) 355.

[10] S.G. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B 273** (1991) 141; erratum ibid, **B 275** (1992) 512; erratum ibid, **B 341** (1995) 448.

[11] D.J. Broadhurst, R. Delbourgo and D. Kreimer, *Phys. Lett.* **B 366** (1996) 421.