Electroweak corrections to the muon anomalous magnetic moment

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Abstract

The bosonic two-loop electroweak radiative corrections to the muon's anomalous magnetic moment, $a_\mu \equiv (g_\mu - 2)/2$, are presented. We find $\Delta a_\mu^{\rm EW}(2\,\text{loop bosonic})/a_\mu^{\rm EW}(1\,\text{loop}) \approx \frac{\alpha}{\pi}\left(-3.6\ln\left(\frac{M_W^2}{m_\mu^2}\right)+0.10\right) \approx -0.11$ for $M_{\rm Higgs} \approx 250$ GeV. Combining that result with our previous two-loop fermionic calculation, we obtain an overall 22.6% reduction in $a_\mu^{\rm EW}$ from 195×10^{-11} to $151(4)\times10^{-11}$. Implications for the full standard model prediction and an upcoming high precision measurement of a_μ are briefly discussed. We also give the two-loop electroweak corrections to the anomalous magnetic moments of electron and tau lepton; they result in a reduction of the one-loop estimates by 35% and 15%, respectively.

The anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$, provides both a sensitive quantum loop test of the standard $SU(3)_C \times SU(2)_L \times U(1)$ model and a window to potential "new physics" effects. The current experimental average [1]

$$a_\mu^{\rm exp} = 116592300(840) \times 10^{-11} \tag{1}$$

is in good agreement with theoretical expectations and already constrains physics beyond the standard model such as supersymmetry and supergravity [2,3], dynamical or loop muon mass generation [4], compositeness [5], leptoquarks [6] etc.

An upcoming experiment E821 [7] at Brookhaven National Laboratory is expected to start in 1996. With one month of dedicated running, it is expected to reduce the uncertainty in $a_\mu^{\rm exp}$ to roughly $\pm 40 \times 10^{-11}$, more than a factor of 20 improvement. With subsequent longer dedicated runs it could statistically approach the anticipated systematic uncertainty of about $\pm 10 - 20 \times 10^{-11}$ [8]. At those levels, both electroweak one and two loop effects become important and "new physics" at the multi-TeV scale is probed. Indeed, generic muon mass generating mechanisms (via perturbative or dynamical loops [4]) lead to $\Delta a_\mu \approx m_\mu^2/\Lambda^2$, where Λ is the scale of "new physics". At $\pm 40 \times 10^{-11}$ sensitivity, $\Lambda \approx 5$ TeV is being explored.

To fully exploit the anticipated experimental improvement, the standard model prediction for a_μ must be known with comparable precision. That requires detailed studies of very high order QED loops, hadronic effects, and electroweak contributions through two loop order. The contributions to a_μ are traditionally divided into

$$a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm Hadronic} + a_\mu^{\rm EW} \tag{2}$$

QED loops have been computed to very high order [9,10]

$$\begin{aligned} a_\mu^{\rm QED} = & \frac{\alpha}{2\pi} \\ & +0.765857381(51)\left(\frac{\alpha}{\pi}\right)^2 + 24.050531(40)\left(\frac{\alpha}{\pi}\right)^3 \\ & +126.02(42)\left(\frac{\alpha}{\pi}\right)^4 + 930(170)\left(\frac{\alpha}{\pi}\right)^5 \end{aligned} \tag{3}$$

Employing $\alpha = 1/137.03599944(57)$ obtained from the electron $g_e - 2$, implies [10]

$$a_\mu^{\rm QED} = 116584706(2) \times 10^{-11} \tag{4}$$

The uncertainty is well within the $\pm 20 - 40 \times 10^{-11}$ goal. Indeed, even if we take the last known term in (3) as indicative of its truncation uncertainty, the QED error remains relatively small.

Hadronic vacuum polarization corrections to a_μ enter at $\mathcal{O}(\alpha/\pi)^2$. They can be evaluated via a dispersion relation using $e^+e^- \to$ *hadrons* data and perturbative QCD (for the very high energy regime). Employing a recent analysis of e^+e^- data [11] along with an estimate of the leading $\mathcal{O}(\alpha/\pi)^3$ effects, we find [12]

$$a_\mu^{\rm Hadronic}({\rm vac.\,pol.}) = 6934(153) \times 10^{-11} \tag{5}$$

Unfortunately, the error has not yet reached the desired level of precision. Ongoing improvements in $e^+e^- \to$ *hadrons* measurements at low energies along with additional theoretical input should significantly lower the uncertainty in (5). Nevertheless, reducing the hadronic error below $\pm 20 \times 10^{-11}$ or even $\pm 40 \times 10^{-11}$ remains a formidable challenge.

The result in (5) must be supplemented by hadronic light by light amplitudes (which are of three loop origin) [13–15]. Here, we employ a recently updated study by Hayakawa, Kinoshita, and Sanda [14] which gives

$$a_\mu^{\rm Hadronic}({\rm light\,by\,light}) = -52(18) \times 10^{-11} \tag{6}$$

However, we note that the result is somewhat dependent on the low energy model of hadronic physics employed and continues to be scrutinized. Combining (5) and (6) leads to the total hadronic contribution

$$a_\mu^{\rm Hadronic} = 6882(154) \times 10^{-11} \tag{7}$$

Now we come to the electroweak contributions to a_μ, the main focus of our work and the impetus for forthcoming experimental effort. At the one loop level, the standard model predicts [16–20]

$$
\begin{aligned}
a_\mu^{\mathrm{EW}}(1\,\mathrm{loop}) &= \frac{5}{3}\frac{\mathrm{G}_\mu \mathrm{m}_\mu^2}{8\sqrt{2}\pi^2} \\
&\times \left[1+\frac{1}{5}(1-4s_W^2)^2+\mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)\right] \\
&\approx 195\times 10^{-11}
\end{aligned} \tag{8}
$$

where $G_\mu = 1.16639(1)\times 10^{-5}$ GeV^{-2}, $M = M_W$ or M_{Higgs}, and the weak mixing angle $\sin^2\theta_W \equiv s_W^2 = 1 - M_W^2/M_Z^2 = 0.224$. We can safely neglect the $\mathcal{O}\left(m_\mu^2/M^2\right)$ terms in (8).

The one loop result in (8) is about five to ten times the anticipated experimental error. Naively, one might expect higher order (2 loop) electroweak contributions to be of relative $\mathcal{O}(\alpha/\pi)$ and hence negligible; however, that is not the case. Kukhto, Kuraev, Schiller, and Silagadze (KKSS) [21] have shown that some two loop electroweak contributions can be quite large and must be included in any serious theoretical estimate of a_μ^{EW} or future confrontation with experiment. Given the KKSS observation, a detailed evaluation of the two loop electroweak contributions to a_μ is clearly warranted. Here, we report the complete results of such an analysis.

The two loop electroweak contributions to a_μ^{EW} naturally divide into so-called fermion and boson parts

$$
\begin{aligned}
a_\mu^{\mathrm{EW}} &= a_\mu^{\mathrm{EW}}(1\,\mathrm{loop}) \\
&+a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{ferm.}) + a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{bos.})
\end{aligned} \tag{9}
$$

The $a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{ferm.})$ includes all two loop electroweak corrections which contain closed fermion loops while all other contributions are lumped into $a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{bos.})$. In a previous study [12], we computed $a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{ferm.})$. For $M_{\mathrm{Higgs}} \approx 250$ GeV it reduces a_μ^{EW} by 11.8%. We have now completed that effort by computing $a_\mu^{\mathrm{EW}}(2\,\mathrm{loop};\ \mathrm{bos.})$. Our results are described below.

The one-loop diagrams which contribute to the lowest order electroweak corrections to a_μ are shown in fig. 1 (there is another diagram obtained by exchanging W and G in 1(c) but its value is the same as 1(c). This is true also for mirror reflections of two loop diagrams, and hence we do not depict them.) The diagrams of fig. 1, minus Schwinger's photon exchange diagram in 1(a), lead to the formula (8).

The two-loop diagrams fall in two general categories. The first and largest group consists of all diagrams which can be viewed as corrections to the one-loop diagrams in fig. 1. Those are one-loop insertions in the propagators and vertices, but also non-planar diagrams and diagrams with quartic couplings. The second group includes all the new types which appear at the two-loop level, as shown in fig. 2.

The complete set of all two-loop diagrams is quite large, together with fermionic loops it includes the total of 1678 diagrams [22]. However, the diagrams with two or more scalar couplings to the muon line are suppressed by an extra factor of m_μ^2/M_W^2 and can be discarded. This is true already at the one-loop level, where one neglects the diagrams with the Higgs boson loop and with two Goldstone boson couplings to the muon. Making this approximation and taking advantage of the mirror symmetry mentioned above reduces the number of relevant diagrams to about 240 in the linear 't Hooft-Feynman gauge. This number can be almost halved by choosing a non-linear gauge [23] in which the $\gamma W^\pm G^\mp$ vertex vanishes. We performed the calculation in both gauges to have a sensitive check of the accuracy of our procedures. For both gauges, two loop divergences are canceled by counterterm insertions in the one loop diagrams of fig. 1.

The smallness of the muon mass compared to the electroweak scales allows us to employ the asymptotic expansion method [24]. In the present calculation we also assume that mass of the Higgs is large compared to $M_{W,Z}$ and compute the first two terms in the expansion in $M_{W,Z}^2/M_H^2$. In diagrams where both W and Z bosons are present we also expand in their relative mass difference. This corresponds to an expansion in $\sin^2\theta_W$ and we keep the first four terms in this expansion. This number of powers is also sufficient to obtain an exact coefficient of the large logarithms $\ln M_W^2/m_\mu^2$; these terms are generated by diagrams with either Z or W boson and hence their coefficient is a polynomial, rather than a series, in $\sin^2\theta_W$.

The large logs have been considered by KKSS [21] in the approximation $\sin^2\theta_W = 1/4$. We obtain a slightly different coefficient even in this special case. The difference between KKSS and our calculation is that KKSS did not consider the diagram shown in fig. 3, where a contribution to the leading log comes from a loop with the Higgs boson. There is no Higgs mass suppression in this diagram because of the HG^+G^- coupling. Without this diagram the result is gauge dependent. For example, this diagram vanishes exactly in the non-linear gauge we adopted in the cross-check, but not in the linear gauge. It should, therefore, have been included in the linear gauge calculation as in KKSS.

Altogether, we find for the two-loop electroweak corrections

$$
\begin{aligned}
a_\mu^{\rm EW}(2\,\text{loop; bos.}) &= \frac{m_\mu^2 \alpha G_\mu}{8\sqrt{2}\pi^3} \\
&\times \left(\sum_{i=-1}^{2} \left[a_{2i} s_W^{2i} + \frac{M_W^2}{M_H^2} b_{2i} s_W^{2i} \right] + \mathcal{O}(s_W^6) \right)
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
a_{-2} &= \frac{19}{36} - \frac{99}{8} S_2 - \frac{1}{24} \ln \frac{M_H^2}{M_W^2} \\
a_0 &= -\frac{859}{18} + 11\frac{\pi}{\sqrt{3}} + \frac{20}{9}\pi^2 + \frac{393}{8} S_2 \\
&\quad -\frac{65}{9} \ln \frac{M_W^2}{m_\mu^2} + \frac{31}{72} \ln \frac{M_H^2}{M_W^2} \\
a_2 &= \frac{165169}{1080} - \frac{385}{6}\frac{\pi}{\sqrt{3}} - \frac{29}{6}\pi^2 + \frac{33}{8} S_2 \\
&\quad +\frac{92}{9} \ln \frac{M_W^2}{m_\mu^2} - \frac{133}{72} \ln \frac{M_H^2}{M_W^2} \\
a_4 &= -\frac{195965}{864} + \frac{265}{3}\frac{\pi}{\sqrt{3}} + \frac{163}{18}\pi^2 + \frac{223}{12} S_2 \\
&\quad -\frac{184}{9} \ln \frac{M_W^2}{m_\mu^2} - \frac{5}{8} \ln \frac{M_H^2}{M_W^2}
\end{aligned}
$$

$$
\begin{aligned}
b_{-2} &= \frac{155}{192} + \frac{3}{8}\pi^2 - \frac{9}{8}S_2 + \frac{3}{2}\ln^2\frac{M_H^2}{M_W^2} - \frac{21}{16}\ln\frac{M_H^2}{M_W^2} \\
b_0 &= \frac{433}{36} + \frac{5}{24}\pi^2 - \frac{51}{8}S_2 + \frac{3}{8}\ln^2\frac{M_H^2}{M_W^2} + \frac{9}{4}\ln\frac{M_H^2}{M_W^2} \\
b_2 &= -\frac{431}{144} + \frac{3}{8}\pi^2 + \frac{315}{8}S_2 + \frac{3}{2}\ln^2\frac{M_H^2}{M_W^2} \\
&\quad -\frac{11}{8}\ln\frac{M_H^2}{M_W^2} \\
b_4 &= \frac{433}{216} + \frac{13}{24}\pi^2 + \frac{349}{24}S_2 + \frac{21}{8}\ln^2\frac{M_H^2}{M_W^2} \\
&\quad -\frac{49}{12}\ln\frac{M_H^2}{M_W^2}
\end{aligned}
\tag{11}
$$

and

$$
S_2 \equiv \frac{4}{9\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = 0.2604341... \tag{12}
$$

We have used the mass shell renormalization prescription [25]. Part of the two-loop bosonic corrections have been absorbed into the lowest order result, by expressing one-loop contributions in eq. (8) in terms of the muon decay constant G_μ.

Employing $\sin^2\theta_W = 0.224$ and $M_H = 250$ GeV in eq. (11,8) gives

$$
\frac{a_\mu^{\mathrm{EW}}(2\,\mathrm{loop\ bosonic})}{a_\mu^{\mathrm{EW}}(1\,\mathrm{loop})} \approx \frac{\alpha}{\pi}\left(-3.6\ln\frac{M_W^2}{m_\mu^2} + 0.10\right) \tag{13}
$$

which corresponds to a 11.0% reduction. For comparison, the partial leading log calculation of KKSS gave $-\frac{49\alpha}{15\pi}\ln(m_Z^2/m_\mu^2)$, a 10.3% reduction.

Combining our new result and previous fermionic two-loop calculation leads to a total reduction of a_μ^{EW} by a factor $(1-97\alpha/\pi)\approx 0.77$ and the new electroweak prediction

$$
a_\mu^{\mathrm{EW}} = 151(4)\times 10^{-11} \tag{14}
$$

The assigned error of $\pm 4\times 10^{-11}$ is due to uncertainties in M_H and quark two loop effects. It also allows for possible three loop (or higher) electroweak contributions. In that regard, we note that our calculation of the $\ln(M_W/m_\mu)$ coefficients can be combined with a renormalization group analysis to sum up leading log corrections of the form $\left(\frac{\alpha}{\pi}\ln(M_W/m_\mu)\right)^n$, $n=1,2,\ldots$; that analysis will be given in a future publication.

With minor modifications, our results give also the two-loop electroweak corrections to anomalous magnetic moments of other leptons. For the electron we find for the combined fermionic and bosonic loops

$$
\frac{a_e^{\mathrm{EW}}(2\,\mathrm{loop})}{a_e^{\mathrm{EW}}(1\,\mathrm{loop})} \approx -150\frac{\alpha}{\pi} \tag{15}
$$

The two-loop corrections result in a 35% reduction of the one-loop prediction.

For the tau lepton the total result of two-loop bosonic and fermionic loops is

$$\frac{a_\tau^{\rm EW}(2\,{\rm loop})}{a_\tau^{\rm EW}(1\,{\rm loop})} \approx -65\frac{\alpha}{\pi} \tag{16}$$

The fermionic contribution has been computed assuming charm quark mass equal approximately to the tau mass. In the case of the τ-lepton the two-loop corrections amount to a 15% reduction of the one-loop result.

Our final result in (14) along with the current best estimates for $a_\mu^{\rm QED}$ and $a_\mu^{\rm hadronic}$ are illustrated in table I. For comparison, the 1990 values are also given [2]. Changes reflect the evolution and continuing scrutiny of theoretical studies. At present

$$a_\mu^{\rm theory} = 116591739(154) \times 10^{-11} \tag{17}$$

with extremely small QED and EW uncertainties. What remains is to reduce the hadronic uncertainty by a factor of 4 (or more) via improved $e^+e^- \to$ *hadrons* data and additional theoretical input. Then, one can fully exploit the anticipated improvement in $a_\mu^{\rm exp}$ from E821 at Brookhaven, a measurement we anxiously await.

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TABLES

TABLE I. Update in $a_\mu \equiv \frac{g_\mu - 2}{2}$ since the 1990 estimate. All numbers have to be multiplied by 10^{-11}.

	Current Value	1990 Estimate	Change
a_μ^{QED}	116 584 706 (2)	116 584 696 (5)	+10
$a_\mu^{hadronic}$	6 882 (154)	7 027 (175)	−145
a_μ^{EW}	151 (4)	195 (10)	−44
a_μ^{theory}	116 591 739 (154)	116 591 918 (176)	−179

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FIGURES

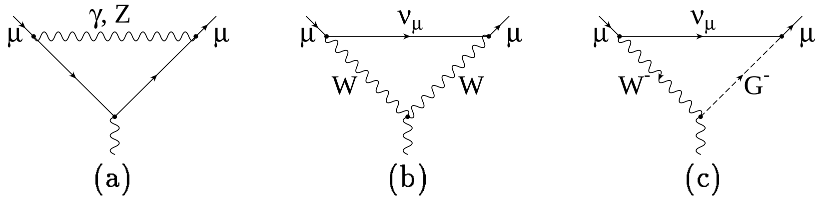

FIG. 1. One-loop electroweak corrections to a_μ (including the QED contribution)

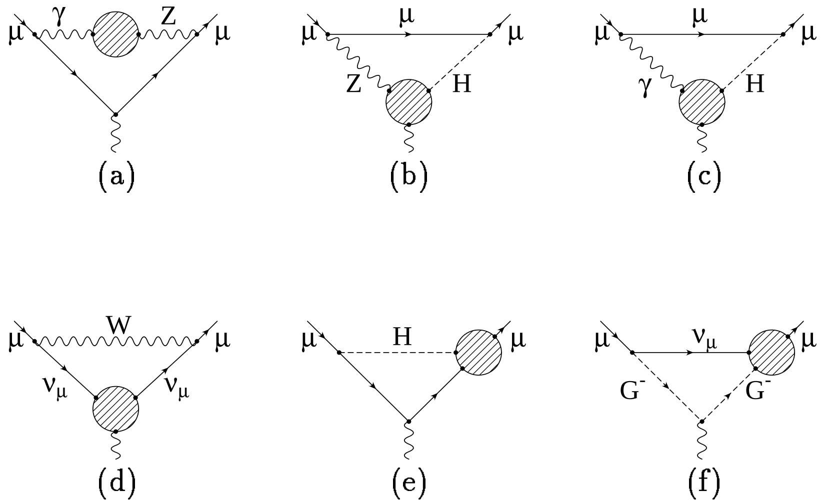

FIG. 2. New types of diagrams at the two-loop level

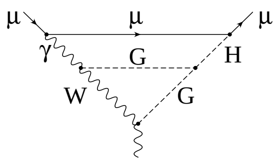

FIG. 3. An effective $\gamma\gamma H$ coupling diagram which gives a contribution to the leading logarithms in linear gauges.