FERMILAB–PUB–95/195–T
MAD/PH/894
MPI/PhT/95–64
TTP95–27*
hep-ph/9508241
July 1995

Three-Loop $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ Corrections to Higgs Production and Decay at e^+e^- Colliders

Bernd A. Kniehl†
Theoretical Physics Department, Fermi National Accelerator Laboratory,
P.O. Box 500, Batavia, IL 60510, USA
and
Department of Physics, University of Wisconsin,
1150 University Avenue, Madison, WI 53706, USA

Matthias Steinhauser
Institut für Theoretische Teilchenphysik, Universität Karlsruhe,
Kaiserstraße 12, 76128 Karlsruhe, Germany

Abstract

We evaluate the next-to-leading-order QCD corrections of $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ to the Standard-Model $\ell^+\ell^- H$, ZZH, and W^+W^-H couplings in the heavy-top-quark limit. Exploiting these results together with knowledge of $\Delta\rho$ to the same order, we analyze a variety of production and decay processes of low-mass Higgs bosons at e^+e^- colliders. Specifically, we consider $H \to \ell^+\ell^-$, $H \to \ell^+\ell^-\ell'^+\ell'^-$, $e^+e^- \to ZH$, $Z \to f\bar{f}H$, and $e^+e^- \to f\bar{f}H$, with $f = \ell, \nu$. We work in the electroweak on-shell scheme formulated with G_F and employ both the on-shell and $\overline{\mathrm{MS}}$ definitions of the top-quark mass in QCD. As expected, the scheme and scale dependences are greatly reduced when the next-to-leading-order corrections are taken into account. In the on-shell scheme of top-quark mass renormalization, the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections act in the same direction as the $\mathcal{O}(\alpha_s G_F M_t^2)$ ones and further increase the screening of the $\mathcal{O}(G_F M_t^2)$ terms. The coefficients of $(\alpha_s/\pi)^2$ range from -6.847 for the ZZH coupling to -16.201 for the $\ell^+\ell^- H$ coupling. This is in line with the value -14.594 recently found for $\Delta\rho$.

*The complete paper, including figures, is also available via anonymous ftp at ftp://ttpux2.physik.uni-karlsruhe.de/, or via www at http://ttpux2.physik.uni-karlsruhe.de/cgi-bin/preprints/.

†Permanent address: Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany.

1 Introduction

After the celebrated discovery of the top quark [1], the Higgs boson is the last missing link in the Standard Model (SM). The detection of this particle and the study of its characteristics are among the prime objectives of present and future high-energy colliding-beam experiments. Following Bjorken's proposal [2], the Higgs boson is currently being searched for with the CERN Large Electron-Positron Collider (LEP1) and the SLAC Linear Collider (SLC) via $e^+e^- \to Z \to f\bar{f}H$. At the present time, the failure of this search allows one to rule out the mass range $M_H \leq 64.3$ GeV at the 95% confidence level [3]. The quest for the Higgs boson will be continued with LEP2 by exploiting the Higgs-strahlung mechanism [4, 5], $e^+e^- \to ZH \to f\bar{f}H$. In next-generation e^+e^- linear supercolliders (NLC), also $e^+e^- \to \bar{\nu}_e\nu_e H$ via W^+W^- fusion and, to a lesser extent, $e^+e^- \to e^+e^-H$ via ZZ fusion will provide copious sources of Higgs bosons.

Once a novel scalar particle is discovered, it will be crucial to decide if it is the very Higgs boson of the SM or if it lives in some more extended Higgs sector. To that end, precise knowledge of the SM predictions will be mandatory, *i.e.*, quantum corrections must be taken into account. The status of the radiative corrections to the production and decay processes of the SM Higgs boson has recently been summarized [6]. Since the top quark is by far the heaviest established elementary particle, with a pole mass of $M_t = (180 \pm 12)$ GeV [1], the leading high-M_t terms, of $\mathcal{O}(G_F M_t^2)$, are particularly important, and it is desirable to acquire information on their quantumchromodynamical (QCD) corrections. During the last year, a number of papers have appeared in which the two-loop $\mathcal{O}(\alpha_s G_F M_t^2)$ corrections to various Higgs-boson production and decay processes are presented. These processes include $H \to f\bar{f}$, with $f \neq b$ [7] and $f = b$ [8, 9], $Z \to f\bar{f}H$ and $e^+e^- \to ZH$ [10], $e^+e^- \to \bar{\nu}_e\nu_e H$ via W^+W^- fusion [11], $gg \to H$ [11, 12], and more [11]. In this paper, we shall take the next step and tackle with three-loop $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections. To keep matters as simple as possible, we shall restrict our considerations to light Higgs bosons, with $M_H \ll M_t$, and to reactions with colourless particles in the initial and final states. Such reactions typically involve the $\ell^+\ell^-H$, W^+W^-H, and ZZH couplings together with gauge couplings of the W and Z bosons to leptons. We are thus led to incorporate the next-to-leading QCD corrections in the low-M_H effective $\ell^+\ell^-H$, W^+W^-H, and ZZH interaction Lagrangians. This will be achieved in Section 2.

Recently, the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ correction to $\Delta\rho$ has been calculated and found to be sizeable [13]. This is relevant for present and future precision tests of the standard electroweak theory. It is of great theoretical interest to find out whether the occurrence of significant $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections is specific to $\Delta\rho$ or whether this is a common feature in the class of electroweak observables with a quadratic M_t dependence at one loop. In the latter case, there must be some underlying principle which is able to explain this phenomenon. Our analysis will put us into a position where we can investigate this matter for four independent quantities. We shall return to this issue in Section 5.

The complete evaluation of the one-loop electroweak correction to a process which involves more than four external particles is enormously intricate. To our knowledge, the literature does not contain a single example of such a calculation. However, the so-called improved Born approximation (IBA) [14] allows us to conveniently extract at least the dominant fermionic loop corrections. As a by-product of our analysis, we shall illustrate the usefulness of the IBA for Higgs-boson production and decay in high-energy e^+e^- collisions. The appropriate formalism will be developed in Section 3.

This paper is organized as follows. In Section 2, we shall extend the low-M_H effective $\ell^+\ell^-H$, W^+W^-H, and ZZH interaction Lagrangians to $\mathcal{O}(\alpha_s^2 G_F M_t^2)$. In the G_F formulation of the electroweak on-shell scheme, knowledge of the QCD-corrected W^+W^-H coupling is sufficient to control the related four- and five-point Higgs-boson production and decay processes which emerge by connecting one or both of the W bosons with lepton lines, respectively. Contrariwise, the corresponding processes involving a ZZH coupling receive additional QCD corrections from the gauge sector, which we shall evaluate by invoking the IBA in Section 3. In Section 4, we shall quantitatively analyze the phenomenological consequences of our results. Section 5 contains our conclusions.

2 Effective Lagrangians

Throughout this paper, we shall work in the electroweak on-shell renormalization scheme [15], with G_F as a basic parameter, and define $c_w^2 = 1 - s_w^2 = M_W^2/M_Z^2$ [16]. In particular, this implies that the lowest-order formulae are expressed in terms of G_F, c_w, s_w, and the physical particle masses. The self-energies of the W, Z, and Higgs bosons to $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ for zero external four-momentum squared will be the basic ingredients of our analysis. While the results for the W and Z bosons are now well established [13], the Higgs-boson self-energy requires a separate analysis, which will be performed here. Our calculation will proceed along the lines of Ref. [13]. We shall employ dimensional regularization in $n = 4 - 2\epsilon$ space-time dimensions and introduce a 't Hooft mass, μ, to keep the coupling constants dimensionless. We shall suppress terms containing $\gamma_E - \ln(4\pi)$, where γ_E is Euler's constant. These terms may be retrieved by substituting $\mu^2 \to 4\pi e^{-\gamma_E}\mu^2$. In the modified minimal-subtraction ($\overline{\mathrm{MS}}$) scheme [17], these terms are subtracted along with the poles in ϵ. This is also true for the relation between the $\overline{\mathrm{MS}}$ and pole masses of the quarks, so that these terms are also absent when the quark masses are renormalized according to the on-shell scheme. Since we wish to extract the leading high-M_t terms, we may neglect the masses of all virtual particles, except for the top quark. As usual, we shall take γ_5 to be anticommuting for n arbitrary. We shall choose a covariant gauge with an arbitrary gauge parameter for the gluon propagator. This will allow us to explicitly check that our final results are gauge independent. The requirement that the expressions for physical observables be renormalization-group (RG) invariant will serve as a further check for our calculation.

Large intermediate expressions will be treated with the help of FORM 2.0 [18]. The tadpole integrals which enter the one- and two-loop calculations may be solved straightforwardly, even for arbitrary powers of propagators. The three-loop case is more involved. After evaluating the traces, the scalar integrals may be reduced by decomposing the scalar products in the numerator into appropriate combinations of the factors in the denominator. Subsequently, recurrence relations derived using the integration-by-parts method [19] may be applied to reduce any scalar Feynman integral to a small number of so-called master diagrams, which remain to be calculated by hand. More technical details may be found in Ref. [13].

Prior to listing our results, we shall introduce our notation. We take the colour gauge group to be SU(N_c); $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$ are the eigenvalues of the Casimir operators of its fundamental and adjoint representations, respectively. As is usually done for SU(N_c), we fix the trace normalization of the fundamental representation to be $T_F = 1/2$. In our numerical analysis, we set $N_c = 3$. We explicitly include five massless quark flavours plus the massive top quark in our calculation, so that we have $n_f = 6$ active quark flavours altogether, *i.e.*, we must not consider n_f as a free parameter. We denote the QCD renormalization scale by μ. We evaluate the strong coupling constant, $\alpha_s(\mu)$, at next-to-leading order (two loops) in the $\overline{\mathrm{MS}}$ scheme, from

$$\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2_{\overline{MS}})}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(\mu^2/\Lambda^2_{\overline{MS}})}{\ln(\mu^2/\Lambda^2_{\overline{MS}})}\right], \tag{1}$$

where $\Lambda_{\overline{MS}}$ is the asymptotic scale parameter appropriate for $n_f = 6$ and [20]

$$\begin{aligned}
\beta_0 &= \frac{1}{4}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) = \frac{7}{4}, \\
\beta_1 &= \frac{1}{16}\left(\frac{34}{3}C_A^2 - 2C_F n_f - \frac{10}{3}C_A n_f\right) = \frac{13}{8}
\end{aligned} \tag{2}$$

are the first two coefficients of the Callan-Symanzik beta function of QCD. We define $a = 4h = \alpha_s(\mu)/\pi$, $x_t = [G_F m_t^2(\mu)/8\pi^2\sqrt{2}]$, $X_t = (G_F M_t^2/8\pi^2\sqrt{2})$, $l = \ln[\mu^2/m_t^2(\mu)]$, and $L = \ln(\mu^2/M_t^2)$, where $m_t(\mu)$ and M_t are the $\overline{\mathrm{MS}}$ and pole masses of the top quark, respectively, and G_F is Fermi's constant. Using the two-loop relation between $m_t(M_t)$ and M_t [21] along with the RG equation for $m_t(\mu)$, we find

$$\begin{aligned}
\frac{m_t(\mu)}{M_t} &= 1 + hC_F(-3L - 4) + h^2 C_F\left\{L^2\left(\frac{9}{2}C_F - \frac{11}{2}C_A + n_f\right) + L\left(\frac{21}{2}C_F - \frac{185}{6}C_A\right.\right. \\
&\quad \left. + \frac{13}{3}n_f\right) - 12\zeta(2) + 6 + C_F\left[-12\zeta(3) + 6\zeta(2)(8\ln 2 - 5) + \frac{7}{8}\right] \\
&\quad \left. + C_A\left[6\zeta(3) + 8\zeta(2)(-3\ln 2 + 1) - \frac{1111}{24}\right] + n_f\left[4\zeta(2) + \frac{71}{12}\right]\right\} \\
&\approx 1 - a\left(L + \frac{4}{3}\right) - a^2\left(\frac{3}{8}L^2 + \frac{35}{8}L + 9.125\,451\right).
\end{aligned} \tag{3}$$

Riemann's zeta function takes on the values $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.202\,057$, and $\zeta(4) = \pi^4/90$. The numerical constants [13]

$$\begin{aligned}
S_2 &= \frac{4}{9\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \approx 0.260\,434, \\
D_3 &\approx -3.027\,009, \\
B_4 &= 16\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{13}{2}\zeta(4) - 4\zeta(2)\ln^2 2 + \frac{2}{3}\ln^4 2 \approx -1.762\,800,
\end{aligned} \tag{4}$$

where Cl_2 is Clausen's function and Li_4 is the quadrilogarithm, occur in the evaluation of the three-loop master diagrams.

In the following , we shall frequently make use of the QCD expansion of $\Delta\rho$ through $\mathcal{O}(\alpha_s^2 G_F M_t^2)$. For the reader's convenience, we shall list it here for $N_c = 3$ and $n_f = 6$. The $\overline{\mathrm{MS}}$ and on-shell results read [13]

$$\Delta\bar{\rho} \approx N_c x_t \left[1 + a(2\,l - 0.193\,245) + a^2\left(\frac{15}{4}l^2 + 2.025\,330\,l - 3.969\,560\right)\right], \tag{5}$$

$$\Delta\rho \approx N_c X_t[1 - 2.859\,912\,a - a^2(5.004\,846\,L + 14.594\,028)], \tag{6}$$

respectively.

To start with, we shall construct the low-M_H effective $\ell^+\ell^- H$ interaction Lagrangian through $\mathcal{O}(\alpha_s^2 G_F M_t^2)$. In the following, bare quantities will be labelled with the superscript 0. The bare $\ell^+\ell^- H$ Lagrangian reads

$$\mathcal{L}_{\ell\ell H} = -m_\ell^0 \bar{\ell}^0 \ell^0 \frac{H^0}{v^0}, \tag{7}$$

where v denotes the Higgs vacuum expectation value. The renormalizations of the lepton mass and wave function do not receive corrections in $\mathcal{O}(\alpha_s^n G_F M_t^2)$, where $n = 0, 1, 2$, so that we may replace m_ℓ^0 and ℓ^0 with their renormalized counterparts. In the G_F formulation of the on-shell scheme, we have [22]

$$\frac{H^0}{v^0} = 2^{1/4} G_F^{1/2} H(1 + \delta_u), \tag{8}$$

with

$$\delta_u = -\frac{1}{2}\left[\frac{\Pi_{WW}(0)}{M_W^2} + \Pi'_{HH}(0)\right]. \tag{9}$$

Here, $\Pi_{WW}(q^2)$ and $\Pi_{HH}(q^2)$ are the W- and Higgs-boson self-energies for external momentum q, respectively, and the subscript u is to remind us that this term appears as a universal building block in the radiative corrections to all production and decay processes of the Higgs boson. Consequently, the renormalized version of Eq. (7) reads

$$\mathcal{L}_{\ell\ell H} = -2^{1/4} G_F^{1/2} m_\ell \bar{\ell}\ell H(1 + \delta_u). \tag{10}$$

The one-loop expressions for $\Pi_{WW}(q^2)$ and $\Pi_{HH}(q^2)$ have been presented in Ref. [23]. The leading-order QCD corrections to $\Pi_{WW}(q^2)$ and $\Pi_{HH}(q^2)$ for arbitrary quark masses have been found in Refs. [24, 7], respectively. The $\mathcal{O}(\alpha_s G_F M_t^2)$ term of δ_u has independently been obtained in Ref. [9] by using the computational technique outlined above at the two-loop level. Here, we shall extend this analysis to $\mathcal{O}(\alpha_s^2 G_F M_t^2)$. The $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ term of $\Pi_{WW}(0)$ may be found in Ref. [13]. The Feynman diagrams pertinent to $\Pi_{HH}(q^2)$ in $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ come in twenty different topologies. Typical examples are depicted in Fig. 1. We shall renormalize the strong coupling constant and the top-quark mass according to the $\overline{\mathrm{MS}}$ scheme. The appropriate counterterms are listed in Ref. [21]. In this way, we obtain

$$\Pi'_{HH}(0) = N_c x_t \left\{ \frac{2}{\epsilon} + 2l - \frac{4}{3} + hC_F\left(-\frac{6}{\epsilon^2} + \frac{5}{\epsilon} + 6l^2 - 10l - \frac{37}{6}\right) + h^2 C_F \left[27\zeta(3) + 6 \right.\right.$$
$$+ C_F\left(\frac{12}{\epsilon^3} - \frac{12}{\epsilon^2} + \frac{1}{\epsilon}\left(24\zeta(3) - \frac{119}{6}\right) + l\left(72\zeta(3) - \frac{93}{2}\right) + 24B_4 - 108\zeta(4)\right.$$
$$\left. + 106\zeta(3) + \frac{331}{12}\right) + C_A\left(\frac{22}{3\epsilon^3} - \frac{83}{3\epsilon^2} + \frac{1}{\epsilon}\left(-12\zeta(3) + \frac{77}{3}\right) + \frac{22}{3}l^3\right.$$
$$\left. + 14l^2 + l\left(-36\zeta(3) - \frac{961}{18}\right) - 12B_4 + 54\zeta(4) - \frac{55}{3}\zeta(3) - 7\right)$$
$$\left.\left. + n_f\left(-\frac{4}{3\epsilon^3} + \frac{10}{3\epsilon^2} - \frac{8}{3\epsilon} - \frac{4}{3}l^3 + \frac{65}{9}l - \frac{32}{3}\zeta(3) - 3\right)\right]\right\}. \tag{11}$$

When we combine Eq. (11) with the corresponding expression for $\Pi_{WW}(0)$ [13], the ultraviolet divergences cancel, and we obtain

$$\bar{\delta}_u = N_c x_t \left\{ \frac{7}{6} + hC_F\left(7l - 2\zeta(2) + \frac{19}{3}\right) + h^2 C_F\left[243S_2 - \frac{449}{6}\zeta(3) - \frac{14}{3}\zeta(2) + \frac{79}{3}\right.\right.$$
$$+ C_F\left(21l^2 + l\left(-12\zeta(2) - \frac{1}{2}\right) + 4B_4 + 2D_3 - \frac{1053}{2}S_2 + 2\zeta(4) + \frac{599}{3}\zeta(3) - \frac{259}{9}\zeta(2)\right.$$
$$\left. - \frac{3043}{72}\right) + C_A\left(\frac{77}{6}l^2 + l\left(-\frac{22}{3}\zeta(2) + \frac{1097}{18}\right) - 2B_4 - D_3 + \frac{1053}{4}S_2 + 15\zeta(4)\right.$$
$$\left.\left.\left. - \frac{509}{6}\zeta(3) - \frac{73}{3}\zeta(2) + \frac{953}{24}\right) + n_f\left(-\frac{7}{3}l^2 + l\left(\frac{4}{3}\zeta(2) - \frac{73}{9}\right) - \frac{8}{3}\zeta(3) + \frac{14}{3}\zeta(2) - \frac{55}{12}\right)\right]\right\}$$
$$\approx \frac{7}{6}N_c x_t\left[1 + a(2\,l + 0.869\,561) + a^2\left(\frac{15}{4}l^2 + 6.010\,856\,l - 2.742\,226\right)\right]. \tag{12}$$

With the help of Eq. (3), we may eliminate $m_t(\mu)$ in favour of M_t, which leads to

$$\delta_u = N_c X_t \left\{ \frac{7}{6} + hC_F(-2\zeta(2) - 3) + h^2 C_F\left[243S_2 - \frac{449}{6}\zeta(3) - \frac{98}{3}\zeta(2) + \frac{121}{3}\right.\right.$$
$$+ C_F\left(4B_4 + 2D_3 - \frac{1053}{2}S_2 + 2\zeta(4) + \frac{515}{3}\zeta(3) + \zeta(2)\left(112\ln 2 - \frac{745}{9}\right) - \frac{146}{9}\right)$$
$$+ C_A\left(L\left(-\frac{22}{3}\zeta(2) - 11\right) - 2B_4 - D_3 + \frac{1053}{4}S_2 + 15\zeta(4) - \frac{425}{6}\zeta(3)\right.$$
$$\left.\left.\left. + \zeta(2)\left(-56\ln 2 - \frac{17}{3}\right) - \frac{2459}{36}\right) + n_f\left(L\left(\frac{4}{3}\zeta(2) + 2\right) - \frac{8}{3}\zeta(3) + 14\zeta(2) + \frac{83}{9}\right)\right]\right\}$$
$$\approx \frac{7}{6}N_c X_t[1 - 1.797\,105\,a - a^2(3.144\,934\,L + 16.200\,847)]. \tag{13}$$

Equation (13) reproduces the $\mathcal{O}(G_F M_t^2)$ and $\mathcal{O}(\alpha_s G_F M_t^2)$ terms found in Refs. [22, 7], respectively. We observe that the new $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ term in Eq. (13) enhances the QCD correction and thus supports the screening of the leading-order M_t dependence. The choice $\mu = M_t$ is singled out, since it eliminates the terms containing L in Eq. (13). The nonlogarithmic coefficient of $(\alpha_s/\pi)^2$ in Eq. (13) is relatively large; it exceeds the corresponding coefficient of $\Delta\rho$ in Eq. (6) by approximately 11%. If we consider the ratio of the coefficient of $(\alpha_s/\pi)^2$ to the one of α_s/π, the difference is even more pronounced; the corresponding numbers for Eqs. (13) and (6) are roughly 9 versus 5.

A phenomenologically interesting application of Eq. (10) is to study the effect of QCD corrections on $\Gamma(H \to \ell^+\ell^-)$. The corrections through $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ to this observable may be accommodated by multiplying the Born formula [22] with

$$\begin{aligned} K_{\ell\ell H} &= (1+\delta_u)^2 \\ &= 1+2\delta_u, \end{aligned} \tag{14}$$

where we have suppressed terms of $\mathcal{O}(G_F^2 M_t^4)$ in the second line. This implies that δ_u is gauge independent and RG invariant in these orders. In order to avoid double counting, the $\mathcal{O}(G_F M_t^2)$ term must once be subtracted when the full one-loop correction [22] is included. A detailed numerical analysis will be presented in Section 4.

Next, we shall derive the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ correction to the low-M_H effective W^+W^-H interaction Lagrangian. In contrast to the $\ell^+\ell^-H$ case, we are now faced with the task of computing genuine three-point amplitudes at three loops, which, at first sight, appears to be enormously hard. Fortunately, in the limit that we are interested in, this problem may be reduced to one involving just three-loop two-point diagrams by means of a low-energy theorem [4, 25]. Generally speaking, this theorem relates the amplitudes of two processes which differ by the insertion of an external Higgs-boson line carrying zero four-momentum. It allows us to compute a loop amplitude, $\mathcal{M}(A \to B+H)$, with an external Higgs boson which is light compared to the virtual particles by differentiating the respective amplitude without that Higgs boson, $\mathcal{M}(A \to B)$, with respect to the virtual-particle masses. More precisely [4, 25],

$$\lim_{p_H \to 0} \mathcal{M}(A \to B+H) = \frac{1}{v} \sum_i \frac{m_i \partial}{\partial m_i} \mathcal{M}(A \to B), \tag{15}$$

where i runs over all massive virtual particles which are involved in the transition $A \to B$. Here, it is understood that the differential operator does not act on factors of m_i appearing in coupling constants, since this would generate tree-level interactions involving the Higgs boson that do not exist in the SM. This theorem has variously been applied at leading order [4, 25] and has even made its way into standard text books [26]. Special care must be exercised if this theorem is to be applied beyond leading order. Then, it must be formulated for the bare quantities of the theory, and the renormalization must be performed after the left-hand side of Eq. (15) has been constructed [8]. The beyond-leading-order version of this theorem [8] has recently been employed to find the $\mathcal{O}(\alpha_s G_F M_t^2)$ corrections to $\Gamma\left(H \to b\bar{b}\right)$ [8], $\Gamma\left(Z \to f\bar{f}H\right)$, and $\sigma(e^+e^- \to ZH)$ [10]. A comprehensive review of higher-order applications of this and related low-energy theorems may be found in Ref. [11]. An axiomatic formulation of these soft-Higgs theorems has recently been introduced in Ref. [27].

Proceeding along the lines of Refs. [10, 11], we find the bare W^+W^-H interaction Lagrangian including its genuine vertex corrections to be

$$\mathcal{L}_{W^+W^-H} = 2(M_W^0)^2 (W_\mu^+)^0 (W^{-\mu})^0 \frac{H^0}{v^0} \left[1 - \frac{(m_t^0)^2 \partial}{\partial (m_t^0)^2} \frac{\Pi_{WW}(0)}{(M_W^0)^2}\right], \tag{16}$$

where it is understood that $\Pi_{WW}(0)$ is expressed in terms of the bare top-quark mass, m_t^0, while all other quark masses are put to zero. We renormalize the W-boson mass and wave function according to the electroweak on-shell scheme by substituting

$$\begin{aligned}(M_W^0)^2 &= M_W^2 + \delta M_W^2,\\ (W_\mu^\pm)^0 &= W_\mu^\pm(1+\delta Z_W)^{1/2},\end{aligned} \tag{17}$$

with the counterterms

$$\begin{aligned}\delta M_W^2 &= \Pi_{WW}(0),\\ \delta Z_W &= -\Pi'_{WW}(0).\end{aligned} \tag{18}$$

For dimensional reasons, δZ_W does not receive corrections in the orders that we are interested in. Using Eq. (8), we thus obtain

$$\mathcal{L}_{W^+W^-H} = 2^{5/4}G_F^{1/2}M_W^2 W_\mu^+ W^{-\mu} H(1+\delta_{WWH}), \tag{19}$$

where

$$\delta_{WWH} = \delta_u + \delta_{nu}^{WWH} \tag{20}$$

and the non-universal part herein may be calculated from

$$\delta_{nu}^{WWH} = \left[1 - \frac{(m_t^0)^2\partial}{\partial(m_t^0)^2}\right]\frac{\Pi_{WW}(0)}{(M_W^0)^2}. \tag{21}$$

In Ref. [13], $\Pi_{WW}(0)$ is expressed in terms of renormalized parameters. Thus, we have to undo the top-quark mass renormalization [21] before we can apply Eq. (21). Then, after evaluating the right-hand side of Eq. (21), we reintroduce the renormalized top-quark mass and so obtain a finite result for δ_{nu}^{WWH}, which we combine with δ_u to get δ_{WWH}. If we define the top-quark mass according to the $\overline{\mathrm{MS}}$ scheme, then the result is

$$\begin{aligned}\bar\delta_{WWH} &= N_c x_t\left\{-\frac{5}{6} + hC_F\left(-5l - 2\zeta(2) + \frac{7}{3}\right) + h^2 C_F\left[243S_2 - \frac{449}{6}\zeta(3) - \frac{14}{3}\zeta(2) + \frac{79}{3}\right.\right.\\ &\quad + C_F\left(-15l^2 + l\left(-12\zeta(2) + \frac{83}{2}\right) + 4B_4 + 2D_3 - \frac{1053}{2}S_2 + 2\zeta(4) + \frac{383}{3}\zeta(3)\right.\\ &\quad \left. - \frac{43}{9}\zeta(2) + \frac{377}{72}\right) + C_A\left(-\frac{55}{6}l^2 + l\left(-\frac{22}{3}\zeta(2) - \frac{331}{18}\right) - 2B_4 - D_3 + \frac{1053}{4}S_2\right.\\ &\quad \left. + 15\zeta(4) - \frac{293}{6}\zeta(3) - \frac{29}{3}\zeta(2) - \frac{691}{24}\right) + n_f\left(\frac{5}{3}l^2 + l\left(\frac{4}{3}\zeta(2) + \frac{11}{9}\right) - \frac{8}{3}\zeta(3)\right.\\ &\quad \left.\left.\left. + 2\zeta(2) + \frac{53}{12}\right)\right]\right\}\\ &\approx -\frac{5}{6}N_c x_t\left[1 + a(2\,l + 0.382\,614) + a^2\left(\frac{15}{4}l^2 + 4.184\,802\,l + 1.343\,710\right)\right].\end{aligned} \tag{22}$$

The corresponding formula written in terms of M_t reads

$$\begin{aligned}
\delta_{WWH} &= N_c X_t \Bigg\{ -\frac{5}{6} + h C_F \left(-2\zeta(2)+9\right) + h^2 C_F \Bigg[243 S_2 - \frac{449}{6}\zeta(3) + \frac{46}{3}\zeta(2) + \frac{49}{3} \\
&\quad + C_F \left(4B_4 + 2D_3 - \frac{1053}{2} S_2 + 2\zeta(4) + \frac{443}{3}\zeta(3) + \zeta(2)\left(-80\ln 2 + \frac{551}{9}\right) - \frac{614}{9} \right) \\
&\quad + C_A \left(L\left(-\frac{22}{3}\zeta(2) + 33\right) - 2B_4 - D_3 + \frac{1053}{4} S_2 + 15\zeta(4) - \frac{353}{6}\zeta(3) \right. \\
&\quad \left. +\ \zeta(2)(40\ln 2 - 23) + \frac{1741}{36} \right) + n_f \left(L\left(\frac{4}{3}\zeta(2) - 6\right) - \frac{8}{3}\zeta(3) - \frac{14}{3}\zeta(2) - \frac{49}{9} \right) \Bigg] \Bigg\} \\
&\approx -\frac{5}{6} N_c X_t [1 - 2.284\,053\, a - a^2(3.997\,092\, L + 10.816\,384)]. \qquad (23)
\end{aligned}$$

We recover the $\mathcal{O}(G_F M_t^2)$ and $\mathcal{O}(\alpha_s G_F M_t^2)$ terms of Refs. [28, 11], respectively. Similarly to $\Delta\rho$ and δ_u, the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ term of Eq. (23) supports the screening of the one-loop M_t dependence by the leading-order QCD correction. Here, the coefficient of $(\alpha_s/\pi)^2$ is by 26% smaller than in the case of $\Delta\rho$, but it, too, is about five times bigger than the coefficient of α_s/π.

From Eq. (19) it follows on that $\Gamma(H \to W^+W^-)$ receives the correction factor

$$K_{WWH} = 1 + 2\delta_{WWH}. \qquad (24)$$

Thus, both δ_{nu}^{WWH} and δ_{WWH} are gauge independent and RG invariant to the orders that we are working in. The tree-level formula for $\Gamma(H \to W^+W^-)$ and its full one-loop correction may be found in Ref. [28]. In order for the Higgs boson to decay into a W^+W^- pair, it must satisfy $M_H > 2M_W$. On the other hand, the high-M_t approximation is based on $M_H \ll M_t$. Since these two conditions conflict with each other [1], the application of Eq. (19) to $\Gamma(H \to W^+W^-)$ is somewhat academic. However, the first condition is relaxed to $M_H > M_W$ or removed altogether if one or both of the W bosons are allowed to leave their mass shells, respectively. In order to avoid gluon exchange between the W^+W^-H vertex and the external fermions, we restrict our considerations to leptonic currents. The resulting class of processes includes $H \to (W^+)^* W^- \to \ell^+\nu_\ell W^-$, $H \to W^+(W^-)^* \to W^+\ell^-\bar\nu_\ell$, $H \to (W^+)^*(W^-)^* \to \ell^+\nu_\ell \ell'^-\bar\nu_{\ell'}$, as well as $e^+e^- \to \bar\nu_e\nu_e(W^+)^*(W^-)^* \to \bar\nu_e\nu_e H$ via W^+W^- fusion. The Born formulae for these $1 \to 3$, $1 \to 4$, and $2 \to 3$ processes may be found in Refs. [29, 30, 31], respectively. Since G_F is defined through the radiative correction to the muon decay, which is a charged-current process, the W-boson propagator does not receive radiative corrections in the orders of interest here. Therefore, the correction factors of all these processes coincide with the one for $\Gamma(H \to W^+W^-)$.

Finally, we shall treat the ZZH interaction. The procedure is very similar to the W^+W^-H case. Application of the low-energy theorem (15) to the bare Z-boson vacuum polarization induced by the top quark yields

$$\mathcal{L}_{ZZH} = (M_Z^0)^2 Z_\mu^0 Z^{\mu 0} \frac{H^0}{v^0} \left[1 - \frac{(m_t^0)^2 \partial}{\partial (m_t^0)^2} \frac{\Pi_{ZZ}(0)}{(M_Z^0)^2} \right]. \qquad (25)$$

Again, $(M_Z^0)^2 = M_Z^2 + \delta M_Z^2$, with $\delta M_Z^2 = \Pi_{ZZ}(0)$, and $Z_\mu^0 = Z_\mu$. Together with Eq. (8), we then have

$$\mathcal{L}_{ZZH} = 2^{1/4} G_F^{1/2} M_Z^2 Z_\mu Z^\mu H (1 + \delta_{ZZH}), \tag{26}$$

where $\delta_{ZZH} = \delta_u + \delta_{nu}^{ZZH}$, with the non-universal part,

$$\delta_{nu}^{ZZH} = \left[1 - \frac{(m_t^0)^2 \partial}{\partial (m_t^0)^2}\right] \frac{\Pi_{ZZ}(0)}{(M_Z^0)^2}, \tag{27}$$

being separately finite, gauge independent, and RG invariant.

Starting from the expression for $\Pi_{ZZ}(0)$ listed in Ref. [13] and repeating the steps of the W^+W^-H analysis, we obtain

$$\begin{aligned}
\bar{\delta}_{ZZH} = N_c x_t \Bigg\{ &-\frac{5}{6} + hC_F \left(-5l - 2\zeta(2) + \frac{25}{3}\right) + h^2 C_F \Bigg[243 S_2 - \frac{449}{6}\zeta(3) - \frac{14}{3}\zeta(2) + \frac{79}{3} \\
&+ C_F \left(-15l^2 + l\left(-12\zeta(2) + \frac{155}{2}\right) + 4B_4 + 2D_3 - \frac{1053}{2} S_2 + 2\zeta(4) + \frac{383}{3}\zeta(3) \right. \\
&\left. - \frac{259}{9}\zeta(2) + \frac{593}{72}\right) + C_A \left(-\frac{55}{6}l^2 + l\left(-\frac{22}{3}\zeta(2) + \frac{65}{18}\right) - 2B_4 - D_3 + \frac{1053}{4} S_2 \right. \\
&\left. + 15\zeta(4) - \frac{293}{6}\zeta(3) - \frac{73}{3}\zeta(2) + \frac{613}{24}\right) \\
&+ n_f \left(\frac{5}{3}l^2 + l\left(\frac{4}{3}\zeta(2) - \frac{25}{9}\right) - \frac{8}{3}\zeta(3) + \frac{14}{3}\zeta(2) - \frac{35}{12}\right)\Bigg]\Bigg\} \\
\approx -\frac{5}{6} N_c x_t &\left[1 + a(2\,l - 2.017\,386) + a^2 \left(\frac{15}{4}l^2 - 4.815\,198\,l - 1.086\,685\right)\right]
\end{aligned} \tag{28}$$

in the $\overline{\mathrm{MS}}$ scheme and

$$\begin{aligned}
\delta_{ZZH} = N_c X_t \Bigg\{ &-\frac{5}{6} + hC_F \left(-2\zeta(2) + 15\right) + h^2 C_F \Bigg[243 S_2 - \frac{449}{6}\zeta(3) + \frac{46}{3}\zeta(2) + \frac{49}{3} \\
&+ C_F \left(4B_4 + 2D_3 - \frac{1053}{2} S_2 + 2\zeta(4) + \frac{443}{3}\zeta(3) + \zeta(2)\left(-80\ln 2 + \frac{335}{9}\right) - \frac{1019}{9}\right) \\
&+ C_A \left(L\left(-\frac{22}{3}\zeta(2) + 55\right) - 2B_4 - D_3 + \frac{1053}{4} S_2 + 15\zeta(4) - \frac{353}{6}\zeta(3) \right. \\
&\left. + \zeta(2)\left(40\ln 2 - \frac{113}{3}\right) + \frac{3697}{36}\right) + n_f \left(L\left(\frac{4}{3}\zeta(2) - 10\right) - \frac{8}{3}\zeta(3) - 2\zeta(2) - \frac{115}{9}\right)\Bigg]\Bigg\} \\
\approx -\frac{5}{6} N_c X_t &[1 - 4.684\,053\,a - a^2(8.197\,092\,L + 6.846\,779)]
\end{aligned} \tag{29}$$

in the on-shell scheme. The $\mathcal{O}(G_F M_t^2)$ and $\mathcal{O}(\alpha_s G_F M_t^2)$ terms of Eq. (29) agree with those found in Refs. [23, 10], respectively. Again, the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ term of Eq. (29) reinforces the potential of the QCD corrections to reduce the leading-order M_t dependence. Comparing δ_{ZZH} with $\Delta\rho$, δ_u, and δ_{WWH}, we observe that it has the largest α_s/π coefficient but the smallest $(\alpha_s/\pi)^2$ coefficient, the ratio of the latter to the former only being about 1.5. The $(\alpha_s/\pi)^2$ coefficient of δ_{ZZH} is by 53% smaller than the one of $\Delta\rho$ in Eq. (6).

From Eq. (26), we infer that $\Gamma(H \to ZZ)$ receives the correction factor

$$K_{ZZH} = 1 + 2\delta_{ZZH}. \tag{30}$$

The Born formula for $\Gamma(H \to ZZ)$ and its full one-loop correction may be found in Ref. [23]. Since the condition $2M_Z < M_H \ll M_t$ is likely to be unrealistic [1], the high-M_t approximation underlying Eq. (26) is of limited usefulness for $H \to ZZ$. We can evade this problem by allowing for one or both of the Z bosons to go off-shell. In addition to the information contained in Eq. (26), we then need to account for the corresponding corrections arising from the gauge sector. However, in order not to invoke unknown QCD corrections, we have to restrict ourselves to the inclusion of lepton lines. The form of the additional corrections depends on the considered reaction. It is useful to divide the phenomenologically relevant processes into three classes:

(1) $H \to Z^*Z \to f\bar{f}Z$, $Z \to Z^*H \to f\bar{f}H$, and $e^+e^- \to ZH$;

(2) $H \to Z^*Z^* \to f\bar{f}f'\bar{f}'$ and $e^+e^- \to Z^* \to Z^*H \to f\bar{f}H$ (via Higgs-strahlung);

(3) $e^+e^- \to e^+e^-Z^*Z^* \to e^+e^-H$ (via ZZ fusion).

Here, f and f' stand for neutrinos and charged leptons. The results for $H \to f\bar{f}Z$ at tree level [29] and at one loop [6], for $Z \to f\bar{f}H$ at tree level [32] and at one loop [33], for $e^+e^- \to ZH$ at tree level and at one loop [31], for $H \to f\bar{f}f'\bar{f}'$ at tree level [30], and for $e^+e^- \to f\bar{f}H$ at tree level [31] are in the literature. In the next section, we shall discuss the corrections to these processes in $\mathcal{O}(\alpha_s^n G_F M_t^2)$, with $n = 0, 1, 2$.

3 Corrections from the gauge sector

The IBA [14] provides a systematic and convenient method to incorporate the dominant corrections of fermionic origin to processes within the gauge sector of the SM. These are contained in $\Delta\rho$ and $\Delta\alpha = 1 - \alpha/\overline{\alpha}$, which parameterizes the running of the fine-structure constant from its value, α, defined in Thomson scattering to its value, $\overline{\alpha}$, measured at the Z-boson scale. The recipe is as follows. Starting from the Born formula expressed in terms of α, c_w, s_w, and the physical particle masses, one substitutes

$$\begin{aligned} \alpha &\to \overline{\alpha} = \frac{\alpha}{1 - \Delta\alpha}, \\ c_w^2 &\to \overline{c}_w^2 = 1 - \overline{s}_w^2 = c_w^2(1 - \Delta\rho). \end{aligned} \tag{31}$$

To eliminate $\overline{\alpha}$ in favour of G_F, one exploits the relation

$$\frac{\sqrt{2}}{\pi} G_F = \frac{\overline{\alpha}}{\overline{s}_w^2 M_W^2} = \frac{\overline{\alpha}}{\overline{c}_w^2 \overline{s}_w^2 M_Z^2}(1 - \Delta\rho), \tag{32}$$

which correctly accounts for the leading fermionic corrections.

We shall now employ the IBA to find the additional corrections through $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ to the four- and five-point processes with a ZZH coupling, which we have classified in Section 2. We shall always assume that the Born formulae are written in terms of G_F, c_w, s_w, and the physical particle masses. The generic correction factor for class (1) reads [10, 33]

$$
\begin{aligned}
K_1^{(f)} &= \frac{(1+\delta^{ZZH})^2}{1-\Delta\rho}\,\frac{\overline{v}_f^2+a_f^2}{v_f^2+a_f^2} \\
&= 1 + 2\delta^{ZZH} + \left(1 - 8c_w^2\frac{Q_f v_f}{v_f^2+a_f^2}\right)\Delta\rho,
\end{aligned}
\tag{33}
$$

where $v_f = 2I_f - 4s_w^2 Q_f$, $\overline{v}_f = 2I_f - 4\overline{s}_w^2 Q_f$, $a_f = 2I_f$, Q_f is the electric charge of f in units of the positron charge, I_f is the third component of weak isospin of the left-handed component of f, and we have omitted terms of $\mathcal{O}(G_F^2 M_t^4)$ in the second line. Similarly, the correction factor for class (2) is given by [34]

$$
\begin{aligned}
K_2^{(ff')} &= \frac{(1+\delta^{ZZH})^2}{(1-\Delta\rho)^2}\,\frac{\overline{v}_f^2+a_f^2}{v_f^2+a_f^2}\,\frac{\overline{v}_{f'}^2+a_{f'}^2}{v_{f'}^2+a_{f'}^2} \\
&= 1 + 2\delta^{ZZH} + 2\left[1 - 4c_w^2\left(\frac{Q_f v_f}{v_f^2+a_f^2} + \frac{Q_{f'} v_{f'}}{v_{f'}^2+a_{f'}^2}\right)\right]\Delta\rho.
\end{aligned}
\tag{34}
$$

Here and in the following, we neglect interference terms of five-point amplitudes with a single fermion trace, since, in the kinematic regime of interest here, these are strongly suppressed, by Γ_V/M_V, with $V = W, Z$. Such terms have recently been included in a tree-level calculation of $\Gamma(H \to 2V \to 4f)$ for $M_H \ll M_W$ [35].

The correction factor for case (3) is slightly more complicated because the electron and positron lines run from the initial state to the final state. Allowing for generic fermion flavours, f and f', the Born cross section may be evaluated from

$$
\sigma(ff' \to ff'H) = \frac{G_F^3 M_Z^4}{64\pi^3\sqrt{2}}\left[(v_f^2+a_f^2)(v_{f'}^2+a_{f'}^2)A \pm 4v_f a_f v_{f'} a_{f'} B\right],
\tag{35}
$$

where

$$
A = \int_{M_H^2/s}^1 dx \int_x^1 dy \frac{a(x,y)}{[1+s(y-x)/M_Z^2]^2},
\tag{36}
$$

and similarly for B, $\sqrt{s}$ is the centre-of-mass energy, and the plus/minus sign refers to an odd/even number of antifermions in the initial state. The process under case (3), with an e^+e^- initial state, requires the plus sign. The integrands read

$$
\begin{aligned}
a(x,y) &= \left(\frac{2x}{y^3} - \frac{1+2x}{y^2} + \frac{2+x}{2y} - \frac{1}{2}\right)\left[\frac{z}{1+z} - \ln(1+z)\right] + \frac{x}{y^2}\left(\frac{1}{y}-1\right)\frac{z^2}{1+z}, \\
b(x,y) &= \left(-\frac{x}{y^2} + \frac{2+x}{2y} - \frac{1}{2}\right)\left[\frac{z}{1+z} - \ln(1+z)\right],
\end{aligned}
\tag{37}
$$

where $z = (y/M_Z^2)(s - M_H^2/x)$. The inner integration in Eq. (36) has been carried out analytically in Appendix A of Ref. [31]. By means of the IBA, we obtain the correction factor for Eq. (35) as

$$K_3^{(ff')} = \frac{(1+\delta^{ZZH})^2}{(1-\Delta\rho)^2}\,\frac{(\overline{v}_f^2 + a_f^2)(\overline{v}_{f'}^2 + a_{f'}^2)A \pm 4\overline{v}_f a_f \overline{v}_{f'} a_{f'} B}{(v_f^2 + a_f^2)(v_{f'}^2 + a_{f'}^2)A \pm 4 v_f a_f v_{f'} a_{f'} B}$$
$$= 1 + 2\delta^{ZZH} + 2\left[1 - \frac{4c_w^2}{1+r}\left(\frac{Q_f v_f}{v_f^2 + a_f^2} + \frac{Q_{f'} v_{f'}}{v_{f'}^2 + a_{f'}^2}\right)\right.$$
$$\left. - \frac{2c_w^2}{1+1/r}\left(\frac{Q_f}{v_f} + \frac{Q_{f'}}{v_{f'}}\right)\right]\Delta\rho, \tag{38}$$

where

$$r = \frac{\pm 4 v_f a_f v_{f'} a_{f'} B}{(v_f^2 + a_f^2)(v_{f'}^2 + a_{f'}^2)A}. \tag{39}$$

We wish to point out that, in the limit $r \to 0$, K_3 coincides with K_2. Detailed analysis reveals that r is quite small in magnitude whenever $e^+e^- \to e^+e^- H$ via ZZ fusion is phenomenologically relevant. In fact, if we consider energies $\sqrt{s} > 150$ GeV and demand that the total cross section of this process be in excess of 10^{-2} fb^{-1}, then we find $|r| < 1\%$. This concludes our discussion of the additional QCD corrections to the processes under items (1)–(3) originating in the gauge sector.

4 Numerical results

We are now in a position to explore the phenomenological implications of our results. We shall take the values of our input parameters to be $M_W = 80.26$ GeV, $M_Z = 91.1887$ GeV [36], $M_t = 180$ GeV [1], and $\alpha_s^{(5)}(M_Z) = 0.118$ [37].[1] The latter corresponds to $\alpha_s^{(6)}(M_t) = 0.1071$, which entails that $\Lambda_{\overline{MS}}^{(6)} = 91$ MeV in Eq. (1). If we use the one-loop formula for $\alpha_s^{(6)}(\mu)$, *i.e.*, Eq. (1) with the second term within the square brackets discarded, $\Lambda_{\overline{MS}}^{(6)}$ comes down to 41 MeV.

Any perturbative calculation to finite order depends on the choice of renormalization scheme and, in general, also on one or more renormalization scales. It is generally believed that the scheme and scale dependences of a calculation up to a given order indicate the size of the unknown higher-order contributions, *i.e.*, they provide us with an estimate of the theoretical uncertainty. Of course, the central values and variations of the scales must be judiciously chosen in order for this estimate to be meaningful. If the perturbation series converges, then the scheme and scale dependences are expected to decrease as the respective next order is taken into account. This principle has recently been confirmed for $\Delta\rho$ [13, 38]. Here, we have the opportunity to carry out similar studies for the three additional observables δ_u, δ_{WWH}, and δ_{ZZH}. Similarly to Ref. [13], we have presented our results in the on-shell and $\overline{\mathrm{MS}}$ schemes as functions of a single renormalization scale, μ.

[1]Note that this value does not include results from lattice computations.

In the $\overline{\rm MS}$ scheme, one could, in principle, introduce individual renormalization scales for the coupling and the mass. For simplicity, we have chosen not to do so. It is natural to define the central value of μ in such a way that, at this point, the radiative correction is devoid of logarithmic terms. This leads us to set $\mu = \xi M_t$ in the on-shell scheme and $\mu = \xi\mu_t$, where $\mu_t = m_t(\mu_t)$, in the $\overline{\rm MS}$ scheme. We may obtain μ_t as a closed function of M_t by iterating Eq. (3), with the result that

$$\begin{aligned}
\frac{\mu_t}{M_t} &= 1 - 4HC_F + H^2C_F\left\{-12\zeta(2) + 6 + C_F\left[-12\zeta(3) + 6\zeta(2)(8\ln 2 - 5) + \frac{199}{8}\right]\right. \\
&\quad \left. + \, C_A\left[6\zeta(3) + 8\zeta(2)(-3\ln 2 + 1) - \frac{1111}{24}\right] + n_f\left[4\zeta(2) + \frac{71}{12}\right]\right\} \\
&\approx 1 - \frac{4}{3}A - 6.458\,784\,A^2, \qquad (40)
\end{aligned}$$

where $A = 4H = \alpha_s(M_t)/\pi$. For $M_t = 180$ GeV, Eq. (40) yields $\mu_t = 170.5$ GeV, in good agreement with the exact fix point of Eq. (3), which is $\mu_t = 170.6$ GeV.

Table 1: Relative deviations (in %) of $\Delta\rho$, δ_u, δ_{WWH}, and δ_{ZZH} from the respective one-loop results due to their corrections up to $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$. The renormalization scale dependence is investigated by choosing $\mu = \xi M_t$, with ξ variable.

ξ	$\Delta\rho/\Delta\rho^{(1)} - 1$ [%]		$\delta_u/\delta_u^{(1)} - 1$ [%]		$\delta_{WWH}/\delta_{WWH}^{(1)} - 1$ [%]		$\delta_{ZZH}/\delta_{ZZH}^{(1)} - 1$ [%]	
	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$
1/4	-11.68	-11.88	-7.34	-8.65	-9.33	-9.35	-19.13	-16.58
1/2	-10.63	-11.72	-6.68	-8.34	-8.49	-9.24	-17.40	-16.83
1	-9.75	-11.44	-6.12	-8.01	-7.78	-9.04	-15.96	-16.76
2	-9.00	-11.11	-5.66	-7.67	-7.19	-8.79	-14.74	-16.51
4	-8.36	-10.74	-5.26	-7.35	-6.68	-8.51	-13.70	-16.15

Table 2: Relative deviations (in %) of $\Delta\bar\rho$, $\bar\delta_u$, $\bar\delta_{WWH}$, and $\bar\delta_{ZZH}$ from the respective one-loop results due to their corrections up to $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$. The renormalization scale dependence is investigated by choosing $\mu = \xi\mu_t$, with ξ variable.

ξ	$\Delta\bar\rho/\Delta\rho^{(1)} - 1$ [%]		$\bar\delta_u/\delta_u^{(1)} - 1$ [%]		$\bar\delta_{WWH}/\delta_{WWH}^{(1)} - 1$ [%]		$\bar\delta_{ZZH}/\delta_{ZZH}^{(1)} - 1$ [%]	
	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$
1/4	-15.63	-11.15	-10.70	-8.25	-12.96	-8.67	-24.09	-15.15
1/2	-10.89	-11.55	-6.87	-8.22	-8.71	-9.09	-17.77	-16.56
1	-8.96	-11.19	-5.63	-7.79	-7.15	-8.86	-14.69	-16.51
2	-8.64	-10.88	-5.82	-7.56	-7.11	-8.71	-13.49	-16.16
4	-9.24	-10.85	-6.81	-7.70	-7.92	-8.84	-13.41	-15.93

In Tables 1 and 2, we investigate the ξ dependence of δ_u, δ_{WWH}, and δ_{ZZH} and their $\overline{\rm MS}$ counterparts, respectively. For comparison, we also include the results for $\Delta\rho$ and $\Delta\bar\rho$. To be specific, we consistently evaluate these quantities to leading and next-to-leading order in QCD and study their relative deviations from their respective one-loop values, which we denote by the superscript (1), *e.g.*, $\Delta\rho^{(1)} = N_c X_t$, *etc.* Notice that the on-shell and $\overline{\mathrm{MS}}$ results coincide at one loop. In our $\mathcal{O}(\alpha_s)$ analysis, we use the one-loop formula for $\alpha_s(\mu)$ with $\Lambda^{(6)}_{\overline{MS}} = 41$ MeV and omit the $\mathcal{O}(\alpha_s^2)$ terms in Eqs. (3) and (40). For the time being, let us concentrate on the entries for $\xi = 1$ and assess the effect of the QCD corrections as well as their scheme dependence. We observe that, in both schemes, the QCD corrections are throughout negative, even for $\bar{\delta}_u$ and $\bar{\delta}_{WWH}$, where the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ terms are in part positive. This is due to the fact that we consistently compute all QCD parameters, *i.e.*, $\alpha_s(\mu)$, $m_t(\mu)$, and μ_t, to the orders under consideration. The reduction in x_t, which occurs as an overall factor in the $\overline{\mathrm{MS}}$ formulae, happens to overcompensate the positive effect of these particular coefficients. Inclusion of the $\mathcal{O}(\alpha_s^2)$ terms in $(\Delta\rho, \delta_u, \delta_{WWH}, \delta_{ZZH})$ increases the size of the QCD corrections by $(17, 31, 16, 5)\%$, respectively. In the $\overline{\mathrm{MS}}$ case, the increments amount to $(25, 38, 24, 12)\%$ of the respective $\mathcal{O}(\alpha_s)$ corrections. As might be expected, the scheme dependence of the QCD corrections to this quadruplet of quantities is dramatically reduced, by $(68, 55, 71, 80)\%$, as we pass from $\mathcal{O}(\alpha_s)$ to $\mathcal{O}(\alpha_s^2)$. Let us now also include the other ξ values in our consideration. Within each scheme, we determine the scale dependence of the QCD correction to a given quantity by comparing its largest and smallest values in the interval $1/4 \leq \xi \leq 4$. As expected, the scale dependence is drastically decreased when we take the $\mathcal{O}(\alpha_s^2)$ terms into account, namely by $(66, 37, 68, 87)\%$ and $(89, 87, 93, 86)\%$ in the on-shell and $\overline{\mathrm{MS}}$ schemes, respectively. The exceptionally small reduction of the scale dependence in the case of δ_u is due to the fact that δ_u has the smallest $\mathcal{O}(\alpha_s)$ term and the largest $\mathcal{O}(\alpha_s^2)$ term of all four on-shell quantities. Figure 2 nicely visualizes how the scheme and scale dependences of the four quantities under consideration are diminished if the next-to-leading-order QCD corrections are incorporated.

In the remainder of this section, we shall stick to the on-shell scheme. In Eqs. (14), (24), (30), (33), (34), and (38), we have presented correction factors for various Higgs-boson production cross sections and decay rates in terms of $\Delta\rho$, δ_u, δ_{WWH}, and δ_{ZZH}. It is instructive to cast these correction factors into the generic form

$$K = 1 + C_1 \Delta\rho^{(1)} \left[1 + C_2 a \left(1 + \frac{7}{4} aL\right) + C_3 a^2\right], \tag{41}$$

where C_i $(i = 1, 2, 3)$ are numerical coefficients. Notice that we have kept the full μ dependence in Eq. (41). We could have written Eqs. (6), (13), (23), and (29) in the same way. The fact that the coefficient of aL is universal may be understood by observing that K represents a physical observable, which must be RG invariant through the order of our calculation, and that, to leading order of QCD, K only implicitly depends on μ, via a. In fact, the coefficient of aL is nothing but β_0 of Eq. (2). The outcome of this decomposition is displayed in Table 3. In the case of $K_3^{(\ell\ell)}$, we have treated $x = B/A$, where A and B are defined in Eq. (36), as an additional expansion parameter and discarded terms of $\mathcal{O}(x^2)$. This is justified because, in practice, $|x| \ll 1$, *e.g.*, for $\sqrt{s} = 300$ GeV and $M_H = 100$ GeV, we find $x \approx -5.233 \cdot 10^{-2}$. While in the case of the three basic corrections, $K_{\ell\ell H}$, K_{WWH},

Table 3: Coefficients of the correction factors in the form of Eq. (41) for the various Higgs-boson decay rates and production cross sections discussed in the text. In the last line, $x = B/A$, where A and B are given by Eq. (36), and terms of $\mathcal{O}(x^2)$ have been neglected.

K	C_1	C_2	C_3
$K_{\ell\ell H}$	$7/3$	-1.797	-16.201
K_{WWH}	$-5/3$	-2.284	-10.816
K_{ZZH}	$-5/3$	-4.684	-6.847
$K_1^{(\nu)}$	$-2/3$	-7.420	4.774
$K_1^{(\ell)}$	-1.272	-5.249	-4.445
$K_2^{(\nu\nu)}$	$1/3$	6.261	-53.330
$K_2^{(\nu\ell)}$	-0.272	-14.025	32.824
$K_2^{(\ell\ell)}$	-0.878	-6.323	0.113
$K_3^{(\ell\ell)}$	$-0.878 - 2.353\,x$	$-6.323 + 9.281\,x$	$0.113 - 39.416\,x$

and K_{ZZH}, C_2 and C_3 are both negative, this is not in general so. In fact, in all composite corrections, except for $K_1^{(\ell)}$, the $\mathcal{O}(\alpha_s^2)$ terms partially compensate the $\mathcal{O}(\alpha_s)$ ones. In $K_2^{(\nu\nu)}$, we even find a counterexample to the heuristic rule [10] that, in the G_F formulation of the on-shell scheme, the $\mathcal{O}(G_F M_t^2)$ terms get screened by their QCD corrections. In the latter case, we also encounter a gigantic value of C_3. Both features may be ascribed to the fact, that, in $\mathcal{O}(G_F M_t^2)$, the δ_{ZZH} and $\Delta\rho$ terms of $K_2^{(\nu\nu)}$ largely cancel. The extraordinarily large value of C_3 in $K_2^{(\nu\ell)}$ is also accompanied by a suppression of C_1. The C_1 values of $K_2^{(\ell\ell)}$ and $K_3^{(\ell\ell)}$ are relatively small, too. We are thus in the fortunate position that the leading high-M_t corrections to the $2 \to 3$ and $1 \to 4$ processes of Higgs-boson production and decay with a ZZH coupling, for which full one-loop calculations have not yet been performed, are throughout quite small. Thus, there is hope that the subleading fermionic corrections to these processes will not drastically impair the situation. However, the IBA does not provide us with any information on the bosonic corrections.

Table 4: Full one-loop weak corrections (in %) to various Higgs-boson decay rates and production cross sections and their $\mathcal{O}(G_F M_t^2)$ terms. In the last line, we have used $\sqrt{s} = 175$ GeV.

Observable	M_H [GeV]	$\mathcal{O}(\alpha)$ weak [%]	$\mathcal{O}(G_F M_t^2)$ [%]
$\Gamma(H \to \tau^+\tau^-)$	75	1.792	2.369
$\Gamma(H \to \nu\bar{\nu}Z)$	105	1.275	-0.677
$\Gamma(H \to \ell^+\ell^- Z)$	105	-1.220	-1.292
$\Gamma(Z \to \nu\bar{\nu}H)$	65	0.024	-0.677
$\Gamma(Z \to \ell^+\ell^- H)$	65	0.296	-1.292
$\sigma(e^+e^- \to ZH)$	75	-2.293	-1.292

In this context, it is interesting to revisit processes for which the full one-loop weak corrections are known and to investigate in how far the $\mathcal{O}(G_F M_t^2)$ terms play a dominant rôle there. Here, we are only interested in reactions which already proceed at tree level. Specifically, we shall consider $Z \to f\bar{f}H$ [33] for $M_H = 65$ GeV, $H \to \tau^+\tau^-$ [22] and $e^+e^- \to ZH$ [31] for $M_H = 75$ GeV, and $H \to f\bar{f}Z$ [6] for $M_H = 105$ GeV, where $f = \nu, \ell$. Our analysis of $\sigma(e^+e^- \to ZH)$ will refer to LEP2 energy, $\sqrt{s} = 175$ GeV. In all these cases, the quantumelectrodynamical (QED) and weak corrections are separately finite and gauge independent at one loop. In Table 4, we compare the full one-loop weak corrections to these processes with their $\mathcal{O}(G_F M_t^2)$ terms. In the case of $H \to \tau^+\tau^-$, $H \to \ell^+\ell^- Z$, and $e^+e^- \to ZH$, the $\mathcal{O}(G_F M_t^2)$ terms give a reasonably good account of the full corrections, while they come out with the wrong sign in the other cases. However, the full calculations for $M_H = 65$ GeV give very small results anyway. On the other hand, the $\mathcal{O}(G_F M_t^2)$ term for $H \to \nu\bar{\nu}Z$ is suppressed due to a partial cancellation between δ_{ZZH} and $\Delta\rho$ in $K_1^{(\nu)}$ and cannot be expected to dominate the full correction. Whenever the full correction is known, it should be included on the right-hand side of Eq. (41) with the $\mathcal{O}(G_F M_t^2)$ term subtracted. In conclusion, the radiative corrections considered in Table 4 all appear to be well under control.

5 Conclusions

In this paper, we have presented the three-loop $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections to the effective Lagrangians for the interactions of light Higgs bosons with pairs of charged leptons, W bosons, and Z bosons in the SM. While the demand for corrections in this order is certainly more urgent in the gauge sector [13], where precision test are presently being carried out, our analysis is also interesting from a theoretical point of view, since it allows us to recognize a universal pattern. In addition to $\Delta\rho$, we have now three more independent observables with quadratic M_t dependence at our disposal for which the QCD expansion is known up to next-to-leading order, namely δ_u, δ_{WWH}, and δ_{ZZH}. In the on-shell scheme of electroweak and QCD renormalization, these four electroweak parameters exhibit striking common properties. In fact, the leading- and next-to-leading-order QCD corrections act in the same direction and screen the $\mathcal{O}(G_F M_t^2)$ terms. Even the sets of α_s/π and $(\alpha_s/\pi)^2$ coefficients each lie in the same ball park. For the choice $\mu = M_t$, the coefficients of α_s/π range between -1.797 and -4.684, and those of $(\alpha_s/\pi)^2$ between -6.847 and -16.201. If we compare this with the corresponding coefficients of the ratio μ_t^2/M_t^2, which are -2.667 and -11.140, then it becomes apparent that the use of the top-quark pole mass is the origin of these similarities. Here, $\mu_t = m_t(\mu_t)$, for which we have presented a closed two-loop formula. If we express the QCD expansions in terms of μ_t rather than M_t and choose $\mu = \mu_t$, then the coefficients of α_s/π and $(\alpha_s/\pi)^2$ nicely group themselves around zero; they range from -2.017 to 0.870 and from -3.970 to 1.344, respectively. This indicates that the perturbation expansions converge more rapidly—before they start to diverge as they are generally believed to do beyond a certain order—if we renormalize the top-quark mass according to the $\overline{\mathrm{MS}}$ scheme. Of course, we have to bear in mind that the QCD perturbation series are believed to diverge beyond a certain order. Without going into details, we would like to mention that the study of renormalons [39] offers a possible theoretical explanation of this observation. Since the on-shell and $\overline{\mathrm{MS}}$ results coincide in lowest order, this does, of course, not imply that the QCD corrections are any smaller in the $\overline{\mathrm{MS}}$ scheme. It just means that, as a rule, the $\mathcal{O}(G_F M_t^2)$ terms with M_t replaced by the two-loop expression for μ_t are likely to provide fair approximations for the full three-loop results. Furthermore, we have demonstrated that, similarly to $\Delta\rho$, the scheme and scale dependences of δ_u, δ_{WWH}, and δ_{ZZH} are considerably reduced when the next-to-leading-order QCD corrections are taken into account. Armed with this information, we have made rather precise predictions for a variety of production and decay processes of low-mass Higgs bosons at present and future e^+e^- colliders. In all the cases considered here, the radiative corrections appear to be well under control now.

ACKNOWLEDGEMENTS

We would like to thank Bill Bardeen, Kostja Chetyrkin, and Michael Spira for very useful discussions. One of us (BAK) is indebted to the FNAL Theory Group for inviting him as a Guest Scientist. He is also grateful to the Phenomenology Department of the University of Wisconsin at Madison for the great hospitality extended to him during a visit when a major part of his work on this project was carried out.

References

[1] F. Abe *et al.* (CDF Collaboration), Phys. Rev. Lett. **74** (1995) 2626;
S. Abachi *et al.* (D0 Collaboration), Phys. Rev. Lett. **74** (1995) 2632.

[2] J.D. Bjorken, in *Weak Interactions at High Energy and the Production of New Particles: Proceedings of Summer Institute on Particle Physics*, August 2–13, 1976, edited by M.C. Zipf, SLAC Report No. 198 (1976) p. 1.

[3] P. Janot, lecture delivered at *First General Meeting of the LEP2 Workshop*, CERN, Geneva, Switzerland, 2–3 February 1995.

[4] J. Ellis, M.K. Gaillard, and D.V. Nanopoulos, Nucl. Phys. **B 106** (1976) 292.

[5] B.L. Ioffe and V.A. Khoze, Fiz. Elem. Chastits At. Yadra **9** (1978) 118 [Sov. J. Part. Nucl. **9** (1978) 50].

[6] B.A. Kniehl, Phys. Rep. **240** (1994) 211.

[7] B.A. Kniehl and A. Sirlin, Phys. Lett. **B 318** (1993) 367;
B.A. Kniehl, Phys. Rev. **D 50** (1994) 3314;
A. Djouadi and P. Gambino, *ibid.* **D 51** (1995) 218.

[8] B.A. Kniehl and M. Spira, Nucl. Phys. **B 432** (1994) 39.

[9] A. Kwiatkowski and M. Steinhauser, Phys. Lett. **B 338** (1994) 66; (E) **B 342** (1995) 455.

[10] B.A. Kniehl and M. Spira, Nucl. Phys. **B 443** (1995) 37.

[11] B.A. Kniehl and M. Spira, Report Nos. DESY 95–041, FERMILAB–PUB–95/081–T, MPI/PhT/95–21, hep–ph/9505225, Z. Phys. C (in press).

[12] A. Djouadi and P. Gambino, Phys. Rev. Lett. **73** (1994) 2528.

[13] L. Avdeev, J. Fleischer, S. Mikhailov, and O. Tarasov, Phys. Lett. **B 336** (1994) 560; (E) **B 349** (1995) 597;
K.G. Chetyrkin, J.H. Kühn, and M. Steinhauser, *ibid.* **B 351** (1995) 331; in *Proceedings of the Ringberg Workshop on Perspectives of Electroweak Interactions in e^+e^- Collisions*, Tegernsee, Germany, February 5–8, 1995, edited by B.A. Kniehl (World Scientific, Singapore, 1995).

[14] M. Consoli, W. Hollik, and F. Jegerlehner, in *Z Physics at LEP 1*, edited by G. Altarelli, R. Kleiss, and C. Verzegnassi, CERN Yellow Report No. 89–08 (September 1989) Vol. 1, p. 7;
W.F.L. Hollik, Fortschr. Phys. **38** (1990) 165;
F. Halzen, B.A. Kniehl, and M.L. Stong, in *Particle Physics: VI Jorge André Swieca Summer School*, Campos de Jordão, Brasil, 14–26 January, 1991, edited by O.J.P. Éboli, M. Gomes, and A. Santoro (World Scientific, Singapore, 1992) p. 219; Z. Phys. **C 58** (1993) 119.

[15] K-I. Aoki, Z. Hioki, R. Kawabe, M. Konuma, and T. Muta, Prog. Theor. Phys. Suppl. **73** (1982) 1;
M. Böhm, H. Spiesberger, and W. Hollik, Fortschr. Phys. **34** (1986) 687.

[16] A. Sirlin, Phys. Rev. **D 22** (1980) 971;
F. Halzen and B.A. Kniehl, Nucl. Phys. **B 353** (1991) 567.

[17] W.A. Bardeen, A.J. Buras, D.W. Duke, and T. Muta, Phys. Rev. **D 18** (1978) 3998.

[18] J.A.M. Vermaseren, *Symbolic Manipulation with FORM*, (Computer Algebra Netherlands, Amsterdam, 1991).

[19] F.V. Tkachov, Phys. Lett. **100 B** (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B 192** (1981) 159;
D.J. Broadhurst, Z. Phys. **C 54** (1992) 54.

[20] D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343; Phys. Rev. **D 8** (1973) 3633;
H.D. Politzer, Phys. Rev. Lett. **30** (1973) 1346;
D.R.T. Jones, Nucl. Phys. **B 75** (1974) 531;
W.E. Caswell, Phys. Rev. Lett. **33** (1974) 244.

[21] N. Gray, D.J. Broadhurst, W. Grafe, and K. Schilcher, Z. Phys. **C 48** (1990) 673; D.J. Broadhurst, N. Gray, and K. Schilcher, Z. Phys. **C 52** (1991) 111.

[22] J. Fleischer and F. Jegerlehner, Phys. Rev. **D 23** (1981) 2001; D.Yu. Bardin, B.M. Vilenskiĭ, and P.Kh. Khristov, Yad. Fiz. **53** (1991) 240 [Sov. J. Nucl. Phys. **53** (1991) 152]; B.A. Kniehl, Nucl. Phys. **B 376** (1992) 3; A. Dabelstein and W. Hollik, Z. Phys. **C 53** (1992) 507.

[23] Z. Hioki, Phys. Lett. **B 224** (1989) 417; (E) **B 228** (1989) 560; B.A. Kniehl, Nucl. Phys. **B 352** (1991) 1.

[24] A. Djouadi and P. Gambino, Phys. Rev. **D 49** (1994) 3499; *ibid.* **D 49** (1994) 4705.

[25] A.I. Vaĭnshteĭn, M.B. Voloshin, V.I. Zakharov, and M.A. Shifman, Yad. Fiz. **30** (1979) 1368 [Sov. J. Nucl. Phys. **30** (1979) 711]; A.I. Vaĭnshteĭn, V.I. Zakharov, and M.A. Shifman, Usp. Fiz. Nauk **131** (1980) 537 [Sov. Phys. Usp. **23** (1980) 429]; M.B. Voloshin, Yad. Fiz. **44** (1986) 738 [Sov. J. Nucl. Phys. **44** (1986) 478]; M.A. Shifman, Usp. Fiz. Nauk **157** (1989) 561 [Sov. Phys. Usp. **32** (1989) 289].

[26] L.B. Okun, *Leptons and Quarks*, (North-Holland, Amsterdam, 1982) p. 229; J.F. Gunion, H.E. Haber, G. Kane, and S. Dawson, *The Higgs Hunter's Guide*, (Addison-Wesley, Redwood City, 1990) p. 40.

[27] W. Kilian, Report Nos. DESY 95–075 and hep–ph/9505309 (May 1995).

[28] B.A. Kniehl, Nucl. Phys. **B 357** (1991) 439.

[29] T.G. Rizzo, Phys. Rev. **D 22** (1980) 722; W.-Y. Keung and W.J. Marciano, *ibid.* **D 30** (1984) 248.

[30] B.A. Kniehl, Phys. Lett. **B 244** (1990) 537; A. Grau, G. Pancheri, and R.J.N. Phillips, *ibid.* **B 251** (1990) 293.

[31] B.A. Kniehl, Z. Phys. **C 55** (1992) 605.

[32] B.A. Kniehl, Phys. Rev. **D 42** (1990) 3100.

[33] B.A. Kniehl, Phys. Lett. **B 282** (1992) 249.

[34] E. Gross, B.A. Kniehl, and G. Wolf, Z. Phys. **C 63** (1994) 417; (E) **C 66** (1995) 321.

[35] T. Asaka and K. Hikasa, Phys. Lett. **B 345** (1995) 36.

[36] M.G. Alviggi *et al.* (LEP Electroweak Working Group), Report Nos. LEPEWWG /95–01, ALEPH 95–038 PHYSIC 95–036, DELPHI 95–37 PHYS 482, L3 Note 1736, OPAL Technical Note TN284 (March 1995).

[37] S. Bethke, in *Proceedings of the Tennessee International Symposium on Radiative Corrections: Status and Outlook*, Gatlinburg, Tennessee, June 27–July 1, 1994, edited by B.F.L. Ward (World Scientific, Singapore, 1994).

[38] B.A. Kniehl, in *Reports of the Working Group on Precision Calculations for the Z Resonance*, edited by D.Yu. Bardin, W. Hollik, and G. Passarino, CERN Yellow Report No. 95–03 (March 1995) p. 299.

[39] M. Beneke and V.M. Braun, Phys. Lett. **B 348** (1995) 513;
M. Neubert, Phys. Rev. **D 51** (1995) 5924; Report Nos. CERN–TH.7524/94 and hep–ph/9502264 (revised April 1995);
P. Ball, M. Beneke, and V.M. Braun, Report Nos. CERN–TH/95–26, UM–TH–95–3, and hep–ph/9502300 (February 1995);
K. Philippides and A. Sirlin, Report Nos. NYU–TH–95/03/01 and hep–ph/9503434 (March 1995);
P. Gambino and A. Sirlin, Report Nos. NYU–TH–95/05/02 and hep–ph/9505426 (May 1995).

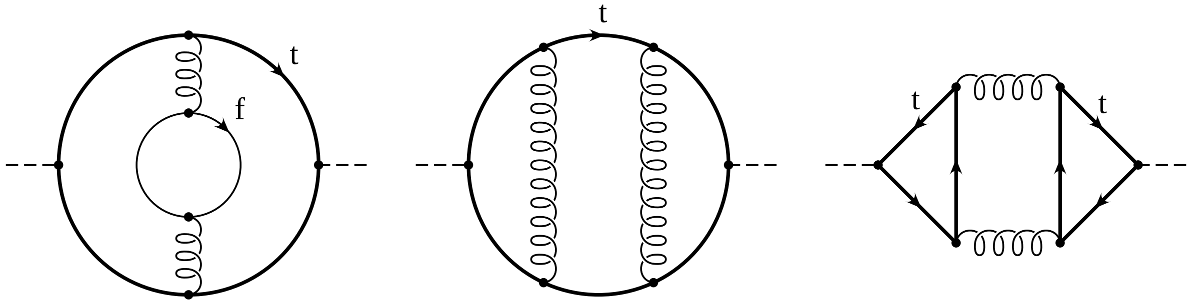

Figure 1: Typical Feynman diagrams pertinent to $\Pi_{HH}(q^2)$ in $\mathcal{O}(\alpha_s^2 G_F M_t^2)$. f stands for any quark.

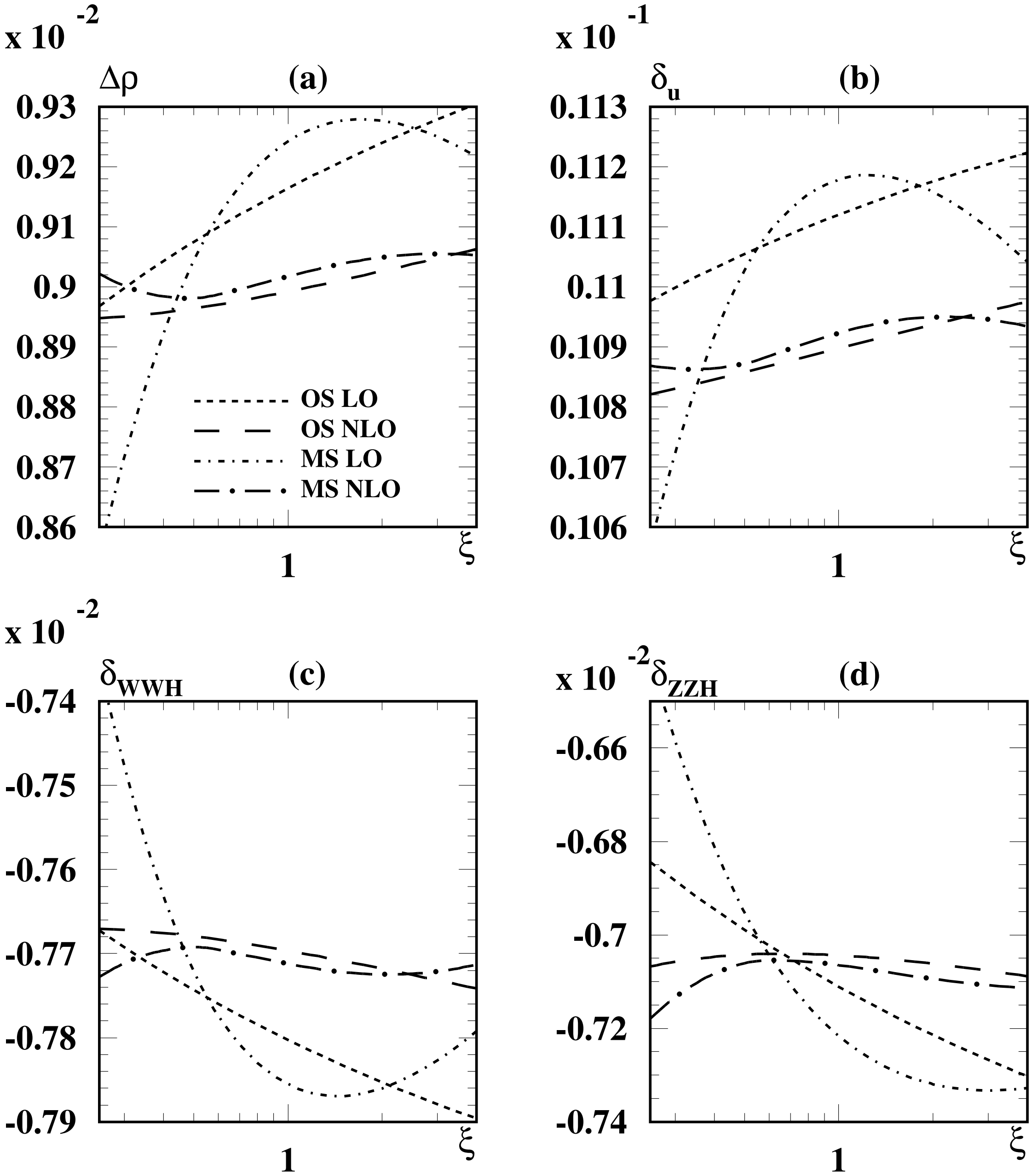

Figure 2: ξ dependences of (a) $\Delta\rho$, (b) δ_u, (c) δ_{WWH}, and (d) δ_{ZZH} to $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ in the on-shell and $\overline{\rm MS}$ schemes.