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# THREE-LOOP QCD CORRECTIONS TO $\delta\rho$, $\Delta r$ and $\Delta\kappa$

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ABSTRACT

QCD corrections to electroweak observables are reviewed. Recent results on contributions from the top-bottom doublet of $\mathcal{O}(\alpha_s^2)$ to $\delta\rho$, $\Delta r$ and $\Delta\kappa$ are presented. It is demonstrated that the first three terms in the expansion in $M_Z^2/M_t^2$ provide an excellent approximation to the exact result. Calculational techniques are briefly discussed.

## 1. Introduction

The precision of measurements of electroweak observables at LEP, SLC and the TEVATRON has increased steadily during the past years [1]. Theoretical predictions for these quantities on the basis of the Standard Model (SM) exhibit a strong dependence on the mass of the top quark. The remarkable agreement between the recent value of $M_t$ published by the CDF and D0 collaborations [2] of $M_t = 176 \pm 8 \pm 10$ GeV and $M_t = 199^{+19}_{-21} \pm 22$ GeV on one side and the value deduced indirectly from precision measurements of $M_t = 173^{+12+18}_{-13-20}$ GeV on the other side can be considered as striking demonstration of the importance of quantum corrections and the predictive power of precision measurements combined with theoretical calculations. Once the top mass is measured at hadron or electron positron colliders with sufficient precision, one may subsequently aim towards a determination of the Higgs mass $M_H$ with similar methods. As a consequence of the relatively weak $M_H$ dependence fairly precise predictions of the $M_t - M_H$ interdependence are required: A shift of $M_t$ of 2 GeV corresponds to a shift of about 15 MeV in $M_W$. Conversely, for $M_W$ and $M_Z$ fixed, the same shift of 2 GeV in $M_t$ leads to a shift of about 25% in the prediction for $M_H$. From these considerations it becomes clear that a relative accuracy in $\Delta r$ or $\delta\rho$ of about 2 % will be required from theory in order to match the anticipated experimental precision. This justifies the effort to perform a calculation of QCD corrections to order $\alpha_s^2$, requiring the evaluation of massive three-loop diagrams.

The high value of the top quark mass allows to identify observables which are enhanced by a factor $G_F M_t^2$ or even to go one step further and perform an expansion in $1/M_t^2$. Apart of the $Zb\bar{b}$ vertex, the $W$ and $Z$ boson self energies induce correction terms $\propto M_t^2$. These are collected in the $\rho$ parameter

$$\delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}. \tag{1}$$

Recently, the calculation of $\delta\rho$ up to $\mathcal{O}(\alpha_s^2)$ was performed independently by two groups [3].

The $\rho$ parameter governs the leading contribution of two important observables, namely $\Delta r$ and the effective mixing angle $\sin^2\bar{\Theta}$. It has become customary to parametrize the radiative corrections between the electroweak parameters $M_W$, $M_Z$, $\alpha$ and $G_F$ through the quantity $\Delta r$ [5]:

$$M_W^2 = \frac{M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}M_Z^2 G_F(1-\Delta r)}}\right). \tag{2}$$

$\Delta r$ can be expressed in terms of the selfenergies $\Pi_{WW}(q^2), \Pi_{ZZ}(q^2), \Pi_{\gamma\gamma}(q^2)$ and $\Pi_{\gamma Z}(q^2)$. If we restrict ourselves to the contribution from the (t,b)-doublet, $\Delta r$ is given by:

$$\Delta\bar{r}_{tb} = \frac{c^2}{s^2}\mathrm{Re}\left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}\right) + \tilde{\Pi}_{\gamma\gamma}(M_Z^2) + \frac{1}{M_W^2}\left(\Pi_{WW}(0) - \mathrm{Re}\Pi_{WW}(M_W^2)\right) \tag{3}$$

with $\tilde{\Pi}_{\gamma\gamma}(q^2) = \Pi_{\gamma\gamma}(q^2)/q^2$. The quantity $\Delta\bar{r}_{tb}$ is introduced in order to indicate the treatment of the photon polarization: To circumvent the mass singularities from the approximation of vanishing bottom mass $\tilde{\Pi}_{\gamma\gamma}(q^2)$ is evaluated at $q^2 = M_Z^2$. The difference is accounted for by dispersion relations with input from the actual measurement of $\sigma(e^+e^- \to hadrons)$ in the low energy region [4].

The effective weak mixing angle governs in particular the asymmetries [6] in $Z$ boson production and decay. It is related to $\sin^2\Theta \equiv 1 - M_W^2/M_Z^2$ through a correction factor

$$\sin^2\bar{\Theta} = (1 + \Delta\kappa)\sin^2\Theta. \tag{4}$$

$\Delta\kappa$ can be expressed in terms of two point functions as follows

$$\Delta\kappa_{tb} = -\frac{c}{s}\frac{\Pi_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{c^2}{s^2}\mathrm{Re}\left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}\right). \tag{5}$$

It is possible to expand $\Delta r$ and $\sin^2\bar{\Theta}$ in the small parameter $M_Z^2/M_t^2$ by evaluating the Taylor series of the self-energies using the techniques described in [3]. In principle this expansion can be performed up to arbitrary high powers in the inverse top mass and up to order $\alpha_s^2$. However, it will be demonstrated that already the first three terms provide an excellent approximation to the full answer [7].

In section 2 some technical details to the calculation are given. In section 3 we present the result and discuss briefly the numerical consequences.

## 2. Calculational Techniques

Three different types of integrals arise: Diagrams involving only light quarks lead to massless two-point functions. These are calculated with the help of the FORM [8] package MINCER [9]. They give rise to terms depending logarithmically on $M_Z^2/M_t^2$.

For the graphs where the massive top quark couples to the photon or the $W$ or $Z$ boson massive three-loop tadpole diagrams have to be evaluated. After the decomposition of the scalar products of the numerator into appropriate combinations of the denominator recurrence relations provided by the the integration-by-parts (IP) method [10] are used in order to reduce the calculation to a few simple integrals plus a small set of so-called master integrals. The IP method was first applied in [11] to three-loop tadpole diagrams. There the subclass relevant for the neutral bosons was considered. The recurrence relations for the $W$ graphs were derived in [3]. The three master diagrams are pictured in Fig. 1. The corresponding integrals are given by:

$$
\begin{aligned}
(a) &= \left(\frac{\mu^2}{M^2}\right)^{3\epsilon}\left[\frac{2}{\epsilon}\zeta(3)+6\zeta(3)-9\zeta(4)+2B_4\right]\\
(b) &= \left(\frac{\mu^2}{M^2}\right)^{3\epsilon}\left[\frac{2}{\epsilon}\zeta(3)+D_3\right]\\
(c) &= \left(\frac{\mu^2}{M^2}\right)^{3\epsilon}M^4\left[\frac{1}{\epsilon^3}+\frac{15}{4\epsilon^2}+\frac{3}{2\epsilon}\zeta(2)+\frac{65}{8\epsilon}+\frac{81}{4}S_2-\zeta(3)+\frac{45}{8}\zeta(2)+\frac{135}{16}\right]
\end{aligned}
$$

On the r.h.s. we followed the standard $\overline{\mathrm{MS}}$ practice and discarded terms proportional to $\gamma_E$ and $\log 4\pi$. The first master integral has been calculated analytically in [11]; the results for the last two integrals are given in [3] ($D_3$ is presently only known numerically). The values for the constants $B_4$, $S_2$ and $D_3$ are as follows:

$$
\begin{aligned}
B_4 &= 16\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{2}{3}\log^4 2-\frac{2}{3}\pi^2\log^2 2-\frac{13}{180}\pi^4=-1.76280\ldots\\
S_2 &= \frac{4}{9\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)=0.260434\ldots\\
D_3 &= -3.02700\ldots.
\end{aligned}
$$

For the calculation of $\delta\rho$ the external momentum has been set to zero. The evaluation of $\Delta\bar{r}_{tb}$ and $\Delta\kappa_{tb}$ require also polarization functions for external momentum $q^2=M_{W/Z}^2$. We approximate these functions through a Taylor expansion in $M_{W/Z}^2/M_t^2$. Using the d'Alembertian operator the expansion of the transverse part of the polarisation tensor is given by

$$
\Pi(0)+q^2\Pi'(0)=
$$

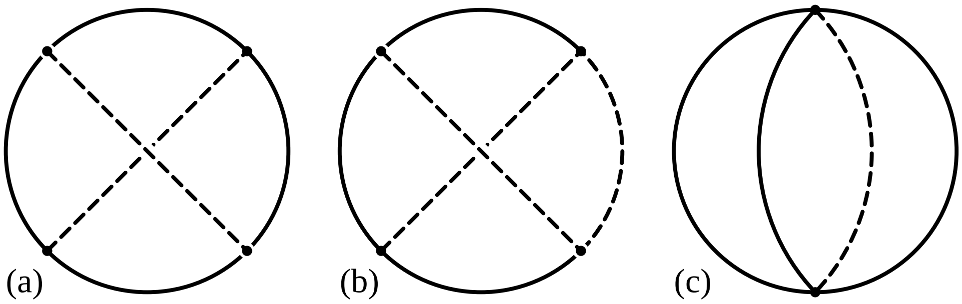

Fig. 1. The master diagrams. Solid lines represent massive scalar propagators, dashed lines massless ones.

$$\frac{1}{D-1}\Bigg\{g_{\mu\nu}\Pi^{\mu\nu}(0)+\frac{q^2}{2D}\left(\Box g_{\mu\nu}\Pi^{\mu\nu}(q)\right)|_{q=0}$$
$$-\frac{1}{q^2}\left[\frac{q^2}{2D}\left(\Box q_\mu q_\nu\Pi^{\mu\nu}(q)\right)|_{q=0}+\frac{(q^2)^2}{8D(D+2)}\left(\Box^2 q_\mu q_\nu \Pi^{\mu\nu}(q)\right)|_{q=0}\right]\Bigg\}. \qquad (6)$$

A similar expression holds for $\Pi''(0)$.

The third class of integrals consists of the double triangle diagram and the double bubble diagram where the external quark is massless and the second quark is the top. In order to get a correct expansion in $1/M_t$ these diagrams have to be treated with the heavy mass expansion technique [12]. Besides the expansion of the whole diagram in the external momentum also all hard subgraphs have to be considered. A hard subgraph is by definition one-particle irreducible w.r.t. the light lines and contains all top propagators. These subdiagrams have to be expanded in their external momentum (and eventually small masses) before the momentum integration is performed. So the three-loop graphs factorize into a one-loop times a two-loop integral.

## 3. Results and Discussion

As already mentioned above the $\rho$ parameter is the leading part of both $\Delta\bar{r}_{tb}$ and $\Delta\kappa_{tb}$. So we first list the result for $\delta\rho$. After performing charge and mass renormalization and setting $C_A=3$, $C_F=4/3$ and $\mu^2=\bar{m}_t^2$ $\delta\rho^{\overline{\mathrm{MS}}}$ reads:

$$\delta\rho_{\overline{\mathrm{MS}}}=3x_t$$
$$\Bigg\{1+\frac{\alpha_s}{4\pi}\left(8-\frac{16}{3}\zeta(2)\right)$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[n_f\left(-\frac{50}{3}-\frac{64}{9}\zeta(3)+\frac{112}{9}\zeta(2)\right)$$
$$+\frac{26459}{81}+882S_2-\frac{8}{9}D_3-\frac{5072}{27}\zeta(3)+\frac{1144}{9}\zeta(4)-\frac{25064}{81}\zeta(2)-\frac{16}{9}B_4\Bigg]\Bigg\} \qquad (7)$$

with $n_f = 6$ denotes the total number of quarks. The variable $x_t$ is defined as $x_t = G_F \bar{m}_t^2 / 8\sqrt{2}\pi^2$. In the following we list the results for $\Delta\bar{r}_{tb}$ and $\Delta\kappa_{tb}$ in order $\alpha_s^0$, $\alpha_s^1$ and $\alpha_s^2$ including also the $M_Z^2/\bar{m}_t^2$ terms:

$$
\begin{aligned}
\Delta\bar{r}_{tb}^{\overline{\rm MS}} &= -\frac{c^2}{s^2}\delta\rho_{\overline{\rm MS}} - 3\frac{c^2}{s^2}x_t\frac{M_Z^2}{\bar{m}_t^2} \\
&\quad \Bigg\{\frac{1}{3} + \frac{8}{9}s^2\bar{l}_Z - \frac{16}{27}s^2 - \frac{2}{3}\bar{l}_Z + \frac{M_Z^2}{\bar{m}_t^2}\Bigg( - \frac{2}{5} + \frac{103}{90}s^2 - s^4\Bigg) \\
&\quad + \frac{\alpha_s}{4\pi}\Bigg(\frac{88}{9} + \frac{128}{9}s^2\zeta(3) - \frac{128}{27}s^2\zeta(2) + \frac{32}{9}s^2\bar{l}_Z - \frac{496}{27}s^2 - \frac{32}{3}\zeta(3) + \frac{64}{27}\zeta(2) - \frac{8}{3}\bar{l}_Z \\
&\qquad + \frac{M_Z^2}{\bar{m}_t^2}\Bigg(\frac{4138}{1215} + 8s^2\zeta(2) - \frac{40361}{3645}s^2 - \frac{16}{3}s^4\zeta(2) + \frac{46}{9}s^4 - \frac{8}{3}\zeta(2)\Bigg)\Bigg) \\
&\quad + \Bigg(\frac{\alpha_s}{4\pi}\Bigg)^2\Bigg[n_f\Bigg( - \frac{4480}{243} + \frac{256}{27}s^2\zeta(3)\bar{l}_Z - \frac{2944}{81}s^2\zeta(3) + \frac{448}{27}s^2\zeta(2) - \frac{352}{27}s^2\bar{l}_Z \\
&\qquad + \frac{32}{27}s^2\bar{l}_Z^2 + \frac{2696}{243}s^2 - \frac{64}{9}\zeta(3)\bar{l}_Z + \frac{2080}{81}\zeta(3) - \frac{176}{27}\zeta(2) + \frac{88}{9}\bar{l}_Z - \frac{8}{9}\bar{l}_Z^2 \\
&\qquad + \frac{M_Z^2}{\bar{m}_t^2}\Bigg( - \frac{2095}{1458} + \frac{32}{3}s^2\zeta(3) - \frac{68}{27}s^2\zeta(2) + \frac{87587}{21870}s^2 - \frac{64}{9}s^4\zeta(3) \\
&\qquad + \frac{136}{81}s^4\zeta(2) - \frac{407}{81}s^4 - \frac{32}{9}\zeta(3) + \frac{68}{81}\zeta(2)\Bigg)\Bigg) \\
&\qquad + \frac{94957}{243} + 856S_2s^2 - 428S_2 + \frac{256}{81}D_3s^2 - \frac{128}{81}D_3 - \frac{4480}{27}s^2\zeta(3)\bar{l}_Z \\
&\qquad + \frac{12016}{27}s^2\zeta(3) + \frac{12032}{81}s^2\zeta(4) - \frac{3200}{27}s^2\zeta(5) - \frac{52816}{243}s^2\zeta(2) - \frac{256}{27}s^2B_4 \\
&\qquad + \frac{688}{3}s^2\bar{l}_Z - \frac{560}{27}s^2\bar{l}_Z^2 - \frac{24164}{81}s^2 + \frac{1120}{9}\zeta(3)\bar{l}_Z - \frac{8117}{18}\zeta(3) - \frac{6016}{81}\zeta(4) \\
&\qquad + \frac{800}{9}\zeta(5) + \frac{22736}{243}\zeta(2) + \frac{128}{27}B_4 - \frac{1400}{9}\bar{l}_Z + \frac{116}{9}\bar{l}_Z^2 \\
&\qquad + \frac{M_Z^2}{\bar{m}_t^2}\Bigg(\frac{1440889423}{17496000} + \frac{1913}{2}S_2s^2 - \frac{1913}{3}S_2s^4 - \frac{1913}{6}S_2 + \frac{8}{3}D_3s^2 + \frac{5632}{6075}s^2\bar{l}_Z \\
&\qquad - \frac{16}{9}D_3s^4 - \frac{8}{9}D_3 - \frac{4346699}{7290}s^2\zeta(3) - \frac{536}{3}s^2\zeta(4) + \frac{593473}{2430}s^2\zeta(2) - \frac{128}{1215}s^2\bar{l}_Z^2 \\
&\qquad + \frac{2345403277}{13122000}s^2 + \frac{20812}{81}s^4\zeta(3) + \frac{1072}{9}s^4\zeta(4) - \frac{591937}{3645}s^4\zeta(2) + \frac{284077}{5832}s^4 \\
&\qquad + \frac{4169129}{51840}\zeta(3) + \frac{536}{9}\zeta(4) - \frac{595393}{7290}\zeta(2) - \frac{7024}{6075}\bar{l}_Z + \frac{32}{405}\bar{l}_Z^2\Bigg)\Bigg]\Bigg\}, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\kappa_{tb}^{\overline{\rm MS}} &= \frac{c^2}{s^2}\delta\rho_{\overline{\rm MS}} - 3\frac{c^2}{s^2}x_t\frac{M_Z^2}{\bar{m}_t^2} \\
&\quad \Bigg\{ - \frac{1}{3} - \frac{4}{9}s^2\bar{l}_Z + \frac{8}{27}s^2 + \frac{2}{3}\bar{l}_Z + \frac{M_Z^2}{\bar{m}_t^2}\Bigg(\frac{2}{5} - \frac{37}{45}s^2 + \frac{1}{2}s^4\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_s}{4\pi}\Bigg(-\frac{88}{9}-\frac{64}{9}s^2\zeta(3)+\frac{64}{27}s^2\zeta(2)-\frac{16}{9}s^2\bar{l}_Z+\frac{248}{27}s^2+\frac{32}{3}\zeta(3)-\frac{64}{27}\zeta(2)+\frac{8}{3}\bar{l}_Z\\
&\quad+\frac{M_Z^2}{\bar{m}_t^2}\Big(-\frac{4138}{1215}-\frac{16}{3}s^2\zeta(2)+\frac{24838}{3645}s^2+\frac{8}{3}s^4\zeta(2)-\frac{23}{9}s^4+\frac{8}{3}\zeta(2)\Big)\Bigg)\\
&+\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[n_f\Bigg(\frac{4480}{243}-\frac{128}{27}s^2\zeta(3)\bar{l}_Z+\frac{1472}{81}s^2\zeta(3)-\frac{224}{27}s^2\zeta(2)+\frac{176}{27}s^2\bar{l}_Z\\
&\quad-\frac{16}{27}s^2\bar{l}_Z^2-\frac{1348}{243}s^2+\frac{64}{9}\zeta(3)\bar{l}_Z-\frac{2080}{81}\zeta(3)+\frac{176}{27}\zeta(2)-\frac{88}{9}\bar{l}_Z+\frac{8}{9}\bar{l}_Z^2\\
&\quad+\frac{M_Z^2}{\bar{m}_t^2}\Big(\frac{2095}{1458}-\frac{64}{9}s^2\zeta(3)+\frac{136}{81}s^2\zeta(2)-\frac{35633}{10935}s^2+\frac{32}{9}s^4\zeta(3)-\frac{68}{81}s^4\zeta(2)\\
&\quad+\frac{407}{162}s^4+\frac{32}{9}\zeta(3)-\frac{68}{81}\zeta(2)\Big)\Bigg)\\
&\quad-\frac{94957}{243}-428S_2s^2+428S_2-\frac{128}{81}D_3s^2+\frac{128}{81}D_3+\frac{2240}{27}s^2\zeta(3)\bar{l}_Z\\
&\quad-\frac{6008}{27}s^2\zeta(3)-\frac{6016}{81}s^2\zeta(4)+\frac{1600}{27}s^2\zeta(5)+\frac{26408}{243}s^2\zeta(2)+\frac{128}{27}s^2B_4\\
&\quad-\frac{344}{3}s^2\bar{l}_Z+\frac{280}{27}s^2\bar{l}_Z^2+\frac{12082}{81}s^2-\frac{1120}{9}\zeta(3)\bar{l}_Z+\frac{8117}{18}\zeta(3)+\frac{6016}{81}\zeta(4)\\
&\quad-\frac{800}{9}\zeta(5)-\frac{22736}{243}\zeta(2)-\frac{128}{27}B_4+\frac{1400}{9}\bar{l}_Z-\frac{116}{9}\bar{l}_Z^2\\
&\quad+\frac{M_Z^2}{\bar{m}_t^2}\Big(-\frac{1440889423}{17496000}-\frac{1913}{3}S_2s^2+\frac{1913}{6}S_2s^4+\frac{1913}{6}S_2-\frac{16}{9}D_3s^2\\
&\quad+\frac{8}{9}D_3s^4+\frac{8}{9}D_3+\frac{5283239}{14580}s^2\zeta(3)+\frac{1072}{9}s^2\zeta(4)-\frac{593089}{3645}s^2\zeta(2)-\frac{2816}{6075}s^2\bar{l}_Z\\
&\quad+\frac{64}{1215}s^2\bar{l}_Z^2-\frac{506454163}{6561000}s^2-\frac{10406}{81}s^4\zeta(3)-\frac{536}{9}s^4\zeta(4)+\frac{591937}{7290}s^4\zeta(2)\\
&\quad-\frac{284077}{11664}s^4-\frac{4169129}{51840}\zeta(3)-\frac{536}{9}\zeta(4)+\frac{595393}{7290}\zeta(2)+\frac{7024}{6075}\bar{l}_Z-\frac{32}{405}\bar{l}_Z^2\Big)\Bigg]\\
&+i\pi\Bigg[-\frac{4}{9}s^2+\frac{16}{27}s^4+\frac{\alpha_s}{4\pi}\Big(-\frac{16}{9}s^2+\frac{64}{27}s^4\Big)\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\Bigg(n_f\Big(\frac{176}{27}s^2-\frac{32}{27}s^2\bar{l}_Z-\frac{128}{27}s^2\zeta(3)-\frac{704}{81}s^4+\frac{128}{81}s^4\bar{l}_Z+\frac{512}{81}s^4\zeta(3)\Big)\\
&\qquad-\frac{344}{3}s^2+\frac{560}{27}s^2\bar{l}_Z+\frac{2240}{27}s^2\zeta(3)+\frac{1376}{9}s^4-\frac{2240}{81}s^4\bar{l}_Z-\frac{8960}{81}s^4\zeta(3)\\
&\qquad+\frac{M_Z^2}{\bar{m}_t^2}\Big(\frac{128}{1215}s^2\bar{l}_Z-\frac{2816}{6075}s^2-\frac{512}{3645}s^4\bar{l}_Z+\frac{11264}{18225}s^4\Big)\Bigg)\Bigg]\Bigg\} \qquad (9)
\end{aligned}
$$

with $s^2 \equiv 1 - M_W^2/M_Z^2$ and $\bar{l}_Z \equiv \ln M_Z^2/\bar{m}_t^2$. The OS result can be received with the help of the relation between the running and the OS top mass [13]. The final result, after setting $n_f = 6$, reads ($\mu^2 = M_t^2$):

$$
\begin{aligned}
\Delta\bar{r}_{tb}^{OS} = & -\frac{c^2}{s^2} 3X_t \; \Bigg\{ 1 + \frac{M_Z^2}{M_t^2}\Big(0.3333 - 0.6667\, l_Z + (-0.5926 + 0.8889\, l_Z)s^2\Big) \\
& \quad + \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big( - 0.4 + 1.144 s^2 - s^4\Big) \\
& + \frac{\alpha_s}{\pi}\Big[ - 2.8599 + \frac{M_Z^2}{M_t^2}\Big( - 1.5640 - 0.6667\, l_Z + (0.1022 + 0.8889\, l_Z)s^2\Big) \\
& \quad + \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big( - 1.312 + 3.573 s^2 - 3.582 s^4\Big)\Big] \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \Big[ - 14.594 + \frac{M_Z^2}{M_t^2}\Big( - 17.224 + 0.08829\, l_Z + 0.4722\, l_Z^2 \\
& \quad + (22.6367 + 1.2527\, l_Z - 0.8519\, l_Z^2)s^2\Big) + \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big( - 7.7781 - 0.07226\, l_Z \\
& \quad + 0.004938\, l_Z^2 + (21.497 + 0.05794\, l_Z - 0.006584\, l_Z^2)s^2 - 21.0799 s^4\Big)\Big]\Bigg\} \\
\Delta\kappa_{tb}^{OS} = & -\frac{c^2}{s^2} 3X_t \; \Bigg\{ - 1 + \frac{M_Z^2}{M_t^2}\Big( - 0.3333 + 0.6667\, l_Z + (0.2963 - 0.4444\, l_Z)s^2\Big) \\
& \quad + \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big(0.4 - 0.8222 s^2 + 0.5 s^4\Big) \\
& + \frac{\alpha_s}{\pi}\Big[2.8599 + \frac{M_Z^2}{M_t^2}\Big(1.564 + 0.6667\, l_Z + (-0.051103 - 0.4444\, l_Z)s^2\Big) \\
& \quad \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big(1.312 - 2.682 s^2 + 1.791 s^4\Big)\Big] \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \Big[14.594 + \frac{M_Z^2}{M_t^2}\Big(17.224 - 0.08829\, l_Z - 0.47222\, l_Z^2 \\
& \quad + (-11.3184 - 0.6263\, l_Z + 0.4259\, l_Z^2)s^2\Big) + \left(\frac{M_Z^2}{M_t^2}\right)^2 \Big(7.7781 + 0.072263\, l_Z \\
& \quad - 0.004938\, l_Z^2 + (-16.0186 - 0.02897\, l_Z + 0.003292\, l_Z^2)s^2 + 10.54 s^4\Big)\Big] \\
& + i\frac{M_Z^2}{M_t^2}\Bigg[ - 1.396 s^2 + 1.862 s^4 + \frac{\alpha_s}{\pi}(-1.396 s^2 + 1.862 s^4) \\
& \quad + \left(\frac{\alpha_s}{\pi}\right)^2 \Big[(-1.968 + 2.676\, l_Z)s^2 + (2.6235 - 3.5682\, l_Z)s^4 \\
& \qquad + \frac{M_Z^2}{M_t^2}\Big((-0.09102 + 0.02069\, l_Z)s^2 + (0.1214 - 0.02758\, l_Z)s^4\Big)\Big]\Bigg]\Bigg\},
\end{aligned}
$$

where $l_Z = \ln M_Z^2/M_t^2$.

Finally we want to discuss the numerical effects. The exact one- and two-loop results for $\Delta\bar{r}_{tb}$ and $\Delta\kappa_{tb}$ are known [14] and pictured in Figs. 2 and 3 (solid line), respectively, together with the first three terms of the expansion. Already the lowest order terms of $\mathcal{O}(G_F M_t^2)$ (dotted line) and $\mathcal{O}(G_F M_Z^2)$ (dashed) provide an excellent approximation, at least for $M_t > 150$ GeV. In this region the addition of the $\mathcal{O}(G_F M_Z^4/M_t^2)$ terms leads to a result which is almost indistinguishable from the exact curve. This consideration justifies the restriction of the $\mathcal{O}(\alpha_s^2)$ corrections to the leading and the next-to-leading terms. Also the next-to-next-to-leading terms $\propto M_Z^2/M_t^2$ are available. The complete result for $\Delta\bar{r}_{tb}$ and $\sin^2\bar{\Theta}$ to $\mathcal{O}(\alpha_s^2)$ is therefore at hand.

Table 1. Numerical results for the $M_t^2$, the constant plus logarithmic and the $1/M_t^2$ contributions. For the numerical evaluation of $\delta\sin^2\bar{\Theta}/\sin^2\bar{\Theta}$ only the real part of $\Delta\kappa_{tb}$ is taken.

| | $M_t^2$ | $M_t^2$ + const. | $M_t^2$ + const. $+1/M_t^2$ |
|---|---|---|---|
| $\delta M_W/M_W$ (OS) | 0.00677 | 0.00838 | 0.00825 |
| $\delta\sin^2\bar{\Theta}/\sin^2\bar{\Theta}$ (OS) | -0.01618 | -0.01832 | -0.01814 |
| $\delta M_W/M_W$ ($\overline{\mathrm{MS}}$) | 0.00674 | 0.00833 | 0.00821 |
| $\delta\sin^2\bar{\Theta}/\sin^2\bar{\Theta}$ ($\overline{\mathrm{MS}}$) | -0.01610 | -0.01823 | -0.01804 |

Table 2. The change in $M_W$ separated according to powers in $\alpha_s$ and $M_t$ in the on-shell scheme.

| $\delta M_W$ in MeV | $\alpha_s^0$ | $\alpha_s^1$ | $\alpha_s^2$ |
|---|---|---|---|
| $M_t^2$ | 611.9 | -61.3 | -10.9 |
| const. | 136.6 | -6.0 | -2.6 |
| $1/M_t^2$ | -9.0 | -1.0 | -0.2 |

The numerical effects on $M_W$ and $\sin^2\bar{\Theta}$ are given in Table 1. The numbers are obtained with the following input data: $\alpha_s(M_t^2) = 0.1092$ (corresponding to $\alpha_s^{(5)} = 0.120$), $\bar{m}_t = 165$ GeV ($\bar{m}_t$ is the solution of the equation $\mu^2 = \bar{m}_t(\mu^2)$), $M_t = 175$ GeV, $M_Z = 91.188$ GeV, $M_H = 300$ GeV, $\alpha = 1/137.04$ and $G_F = 1.16639\,10^{-5}$ GeV$^{-2}$. For $\Delta r$ in Eq. (2) we used (besides our corrections) the complete $\mathcal{O}(\alpha)$ result [15] including the contributions from the bosonic diagrams, the vertex and box diagrams and also the from the light fermions with $\Delta\alpha = 0.05940$ [4].

In Table 2 the contributions are listed separately according to powers of $\alpha_s$ and $M_t$. In the OS scheme the three-loop result induces a shift in $M_W$ of about $-13.7$ MeV. The terms of order $M_t^2/M_Z^2$ alone accounts for for only $-10.9$ MeV. Inclusion of the next two terms increase the shift by $-2.8$ MeV.

To summarize: The three-loop corrections of $\mathcal{O}(\alpha_s^2)$ to $\Delta r$ and $\Delta\kappa$ from the top-bottom doublet have been calculated, using an expansion in $1/M_t^2$. Already the first three terms lead to an accurrate prediction.

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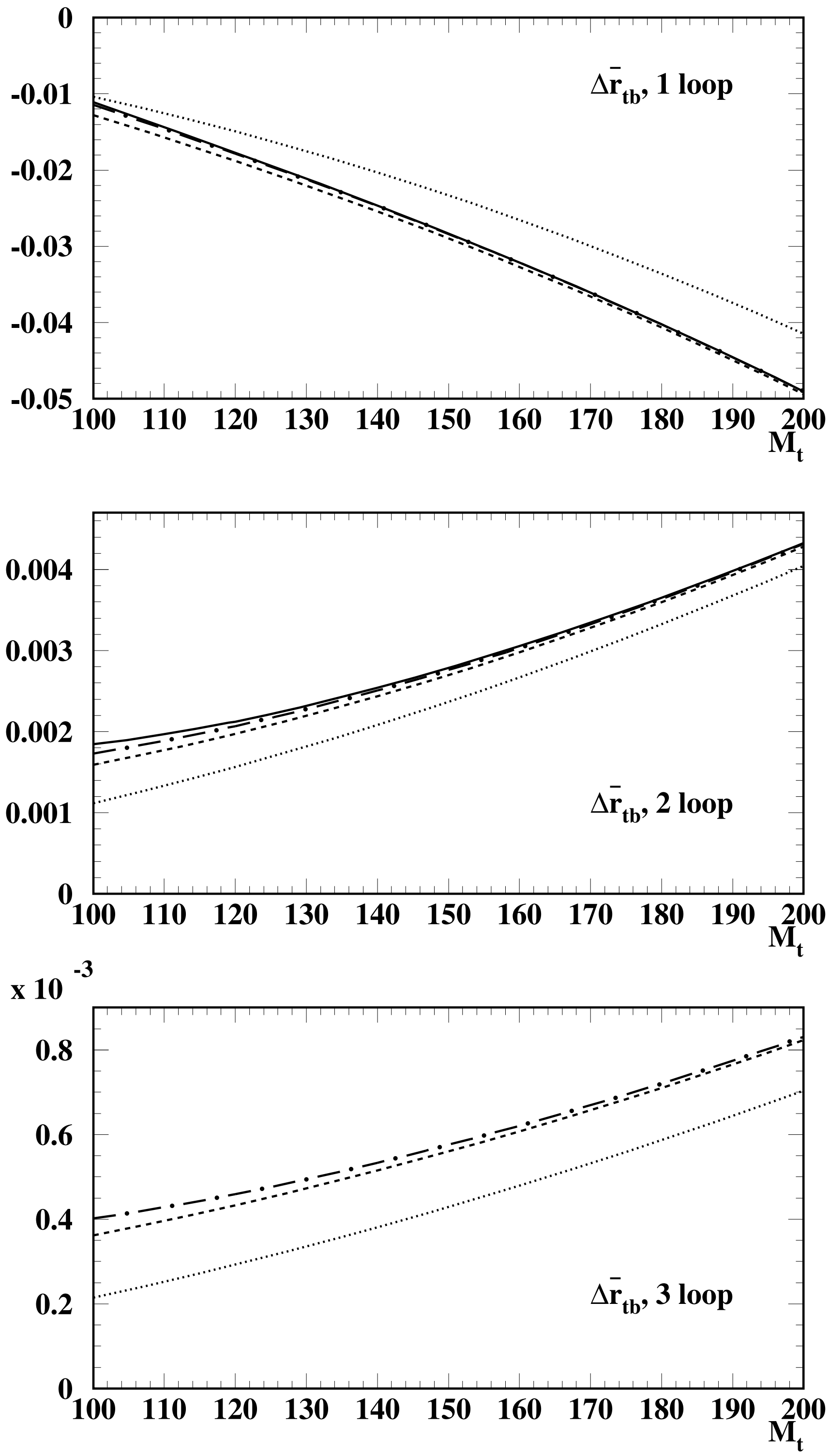


Fig. 2. $\Delta\bar{r}_{tb}$ as function of $M_t$. The dotted, dashed and dash-dotted curves correspond to an increasing number of terms in the approximation. In the figures a value $s^2 = 0.2321$ is chosen.

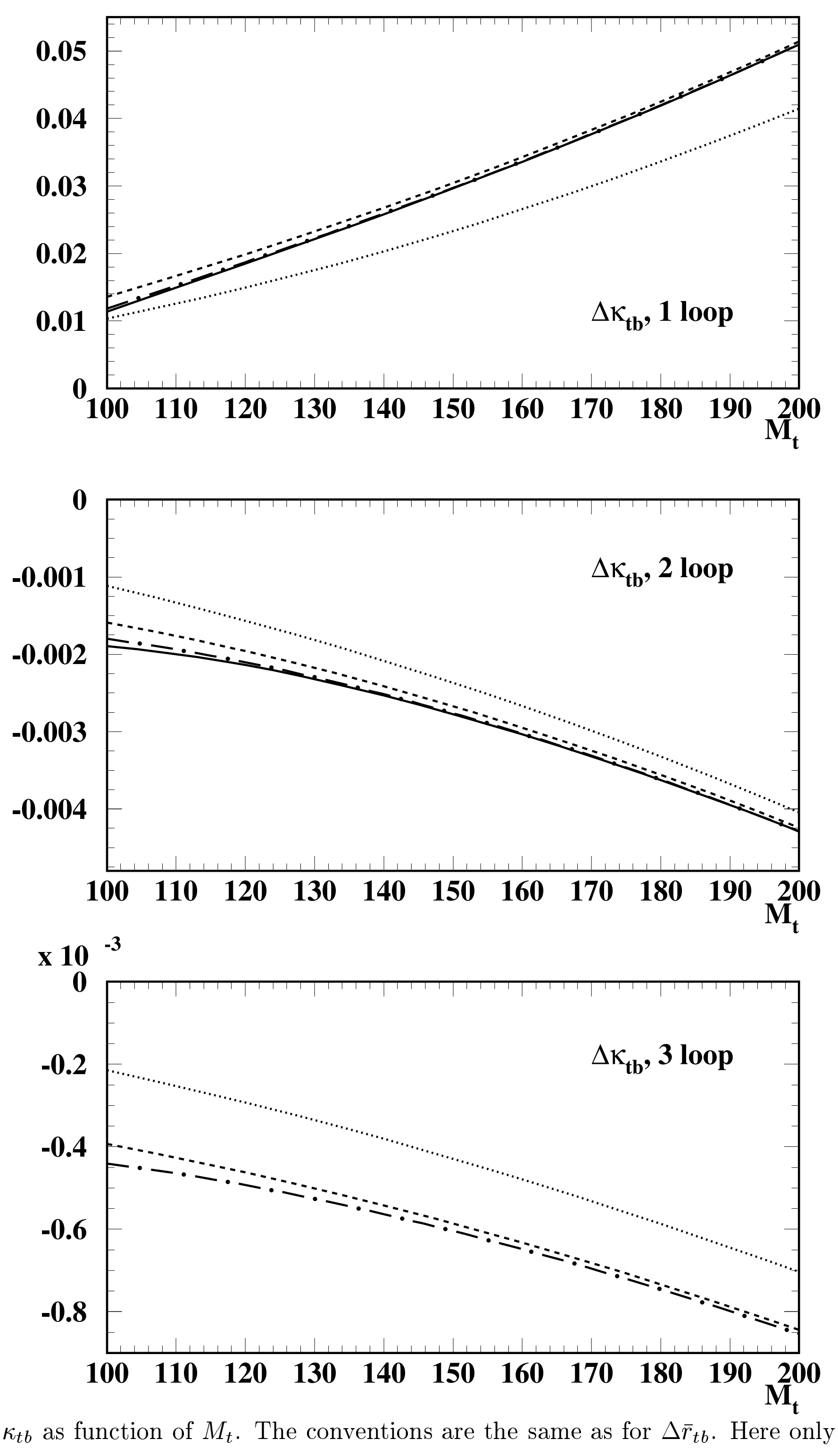


Fig. 3. $\Delta\kappa_{tb}$ as function of $M_t$. The conventions are the same as for $\Delta\bar{r}_{tb}$. Here only the real part is plotted.