hep-ph/9410332
TTP 94-21

# ON THE NEW METHOD OF COMPUTING TWO-LOOP MASSIVE DIAGRAMS [1]

Andrzej Czarnecki

Institut für Theoretische Teilchenphysik
Universität Karlsruhe
D-76128 Karlsruhe, Germany
e-mail: ac@ttpux2.physik.uni-karlsruhe.de

## INTRODUCTION

The improving precision of experiments in the high energy physics motivates theoretical studies of quantum corrections to various processes. In the two-loop approximation this is connected with great computational difficulties, especially if there are several mass scales involved in the process, which is a typical situation in the case of electroweak or mixed chromodynamic and electroweak corrections. Recently a new method has been proposed for the evaluation of scalar two-loop vertex and propagator functions [1, 2]. It has also been shown that a similar approach works even for the four-point functions [3]. In this talk I present a few examples which illustrate the principle of this method.

The aim will be to obtain a double integral representation which is suitable for numerical evaluation. In the following section I will derive it for a special case of the vertex function with zero momentum transfer. The next section refers to the general case of a planar vertex function with space-like values of external momenta, and the last one shows an example of dealing with ultraviolet divergent diagrams. The examples of two-loop functions to be considered in this paper are depicted in Fig. 1(a,b,c).

## VERTEX FUNCTION AT ZERO MOMENTUM TRANSFER

While ref. [2] describes the general method of computing the two-loop vertex function, here the method is illustrated with the special case of zero-momentum transfer

[1]Submitted to the proceedings of the NATO Advanced Study Institute "Frontiers in Particle Physics", Cargèse, August 1994.

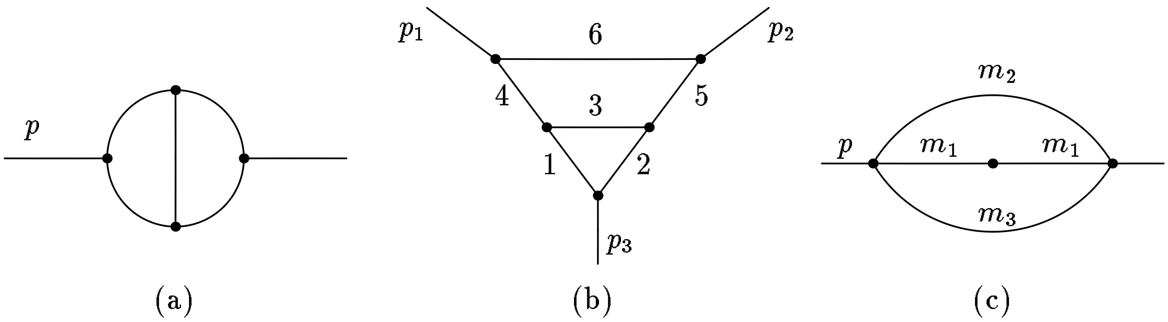


Figure 1: Examples of two-loop diagrams: a) Master two-point diagram, b) Planar vertex, c) Sunrise diagram with a zero-momentum insertion

which is in fact a two-point function. The principle remains the same, but the computation becomes much simpler and it is easy to write down explicit formulas. The diagram and numbering of lines is depicted in Fig. 1(b). The four momenta in the rest frame of the external particle are ($p_i$ are outgoing)

$$\begin{aligned} p_2^\mu &= -p_1^\mu = (q,0,0,0), \\ l^\mu &= (l_+ + l_-, l_+ - l_-, \vec{l}_\perp), \\ k^\mu &= (k_+ + k_-, k_+ - k_-, \vec{k}_\perp), \end{aligned} \tag{1}$$

and the momentum $k$ runs along the lines (4,3,5,6) and $l$ along (1,3,2). We shall see later that one obtains different, but equivalent, formulas if one uses a different routing of the momenta. The two-loop function of Fig. 1(b) is

$$V_0(q^2) = \iint d^4k d^4l \frac{1}{P_1P_2P_3P_4P_5P_6}. \tag{2}$$

With the present choice of momenta, and with $s \equiv l_\perp^2$ and $t \equiv k_\perp^2$, the explicit form of the propagators is

$$\begin{aligned} P_{1,2} &= (2l_+ - q)(2l_- - q) - s - m_{1,2}^2 + i\eta, \\ P_3 &= 4(l_+ + k_+)(l_- + k_-) - s - t - 2\sqrt{st}z - m_3^2 + i\eta, \\ P_{4,5} &= (2k_+ + q)(2k_- + q) - t - m_{4,5}^2 + i\eta, \\ P_6 &= 4k_+k_- - t - m_6^2 + i\eta. \end{aligned} \tag{3}$$

There is only one propagator, $P_3$, through which both internal momenta flow, and $z$ denotes the cosine of the angle between the two perpendicular momentum vectors $\vec{k}_\perp$ and $\vec{l}_\perp$. Integrations over the two angular variables describing the absolute and relative configuration of $\vec{k}_\perp$ and $\vec{l}_\perp$ can now be performed and we obtain from (2)

$$V_0(q^2) = -4\pi^2 \int dk_+ dl_+ ds dt dk_- dl_- \frac{1}{\sqrt{A^2 - B^2}} \frac{1}{P_1P_2P_4P_5P_6}, \tag{4}$$

with

$$\begin{aligned} A &= 4(l_+ + k_+)(l_- + k_-) - s - t - m_3^2 + i\eta, \\ B^2 &= 4st. \end{aligned} \tag{5}$$

The integrations over the $k_-$ and $l_-$ are done with help of contour integrals. It turns out that the singularities in $P_4$ and $P_5$ do not contribute, while $P_1$, $P_2$ and $P_6$ contribute only for $k_+$ and $l_+$ lying in a triangular region $T$ in the $k_+l_+$ plane delimited by the lines

$$\begin{aligned} k_+ + l_+ &= 0, \\ k_+ &= 0, \\ l_+ &= \frac{q}{2}. \end{aligned} \tag{6}$$

The function $V_0$ becomes now

$$\begin{aligned} V_0(q^2) = 2\pi^4 \iint_T \frac{dk_+ dl_+}{k_+(2l_+ - q)} \int_0^\infty dt \int_0^\infty ds \\ \times \sum_{\{i,j\}=\{1,2\},\{2,1\}} \frac{1}{P_i P_4 P_5} \frac{1}{\sqrt{A^2 - B^2}} \Bigg|_{k_- \to k_6, l_- \to l_j}, \end{aligned} \tag{7}$$

where

$$k_6 = \frac{t + m_6^2}{4k_+}, \qquad l_{1,2} = \frac{s + m_{1,2}^2}{2(2l_+ - q)} + \frac{q}{2}. \tag{8}$$

Substituting the explicit formulas for the propagators one obtains

$$\begin{aligned} V_0(q^2) = \frac{8\pi^4}{q^2(m_2^2 - m_1^2)} \iint_T dk_+ dl_+ \frac{k_+}{2l_+ - q} \int_0^\infty \frac{dt}{(t + t_4)(t + t_5)} \\ \times \int_0^\infty ds \left( \frac{1}{\sqrt{(at + b_2 + cs)^2 - 4st}} - \frac{1}{\sqrt{(at + b_1 + cs)^2 - 4st}} \right), \end{aligned} \tag{9}$$

with

$$\begin{aligned} a &= \frac{l_+}{k_+}, \\ b_{1,2} &= 2(k_+ + l_+) \left( \frac{m_6^2}{2k_+} + \frac{m_{1,2}^2}{2l_+ - q} + q \right) - m_3^2, \\ c &= \frac{2k_+ + q}{2l_+ - q}, \\ t_{4,5} &= \frac{2k_+}{q} \left\{ (2k_+ + q) \left( \frac{m_6^2}{2k_+} + q \right) - m_{4,5}^2 \right\}. \end{aligned} \tag{10}$$

For the sake of simplicity let us assume that $q^2$ lies below all thresholds so that the function $V_0$ is real. In such case the integration over $s$ is elementary

$$\begin{aligned} \int_0^\infty ds \left( \frac{1}{\sqrt{(at + b_2 + cs)^2 - 4st}} - \frac{1}{\sqrt{(at + b_1 + cs)^2 - 4st}} \right) \\ = \frac{1}{c} \ln \frac{t(1 - ca) - cb_2}{t(1 - ca) - cb_1}, \end{aligned} \tag{11}$$

and in the integration over $t$ one encounters dilogarithms,

$$\mathrm{Li}_2(x) = - \int_0^x \mathrm{d}y \, \frac{\ln|1 - y|}{y}. \tag{12}$$

The final result is

$$V_0(q^2) = \frac{4\pi^4}{q(m_2^2-m_1^2)(m_5^2-m_4^2)} \int_{-q/2}^{0} \frac{dk_+}{2k_+ + q} \int_{-k_+}^{q/2} dl_+$$
$$\times \left\{ \ln\frac{t_4}{t_5}\ln\frac{t_2}{t_1} + \mathrm{Li}_2\left(1-\frac{t_5}{t_2}\right) - \mathrm{Li}_2\left(1-\frac{t_4}{t_2}\right) - \mathrm{Li}_2\left(1-\frac{t_5}{t_1}\right) + \mathrm{Li}_2\left(1-\frac{t_4}{t_1}\right)\right\}, \tag{13}$$

with $t_{1,2} = -cb_{1,2}/(1-ca)$.

The complication which arises in the case of the non-zero momentum transfer consists in the fact that the propagators $P_{1,2}$ after substitution of the appropriate value for $l_-$ do not have the simple form $\pm(m_2^2 - m_1^2)$ but retain dependence on the variables $l_+$ and $s$. This leads to a more complicated form for the integrations over $s$ and $t$, and the final formula contains dilogarithms as well as Clausen functions. There are also two additional residues which contribute and each contribution in general comes from a different triangle in the $k_+l_+$ plane.

Since the final two integrations over $k_+$ and $l_+$ are to be done numerically it is useful to have an alternative formula which provides a cross check and a test of accuracy. Such formula can be derived by choosing the internal momenta in such way that $k$ runs through the lines $(1,2,5,6,4)$ and $l$ through $(1,2,3)$.

With this choice the propagators are

$$\begin{aligned}
P_{1,2} &= (2k_+ + 2l_+ + q)(2k_- + 2l_- + q) - s - t - 2\sqrt{st}z - m_{1,2}^2 + i\eta, \\
P_3 &= 4l_+l_- - s - m_3^2 + i\eta, \\
P_{4,5} &= (2k_+ + q)(2k_- + q) - t - m_{4,5}^2 + i\eta, \\
P_6 &= 4k_+k_- - t - m_6^2 + i\eta.
\end{aligned} \tag{14}$$

There are now two propagators which depend on $z$: $P_1$ and $P_2$, but only one combination of propagators ($P_3$ and $P_6$) whose singularities contribute to the contour integrations over $k_-$ and $l_-$. The triangular region of the integration over $k_+$ and $l_+$ is now limited by the lines

$$\begin{aligned}
k_+ + l_+ &= -\frac{q}{2}, \\
l_+ &= 0, \\
k_+ &= 0.
\end{aligned} \tag{15}$$

After the integration over the angular variables and over $k_-$ and $l_-$ we get

$$V_i(q^2) = \frac{\pi^4}{m_2^2 - m_1^2} \int \frac{dk_+ dl_+}{k_+ l_+} \int_0^\infty \frac{dt}{P_4 P_5}$$
$$\times \int_0^\infty ds \left( \frac{1}{\sqrt{A_1'^2 - B^2}} - \frac{1}{\sqrt{A_2'^2 - B^2}} \right)\Bigg|_{k_- \to k_6, l_- \to l_3} \tag{16}$$

with

$$\begin{aligned}
k_6 &= \frac{t + m_6^2}{4k_+}, \qquad l_3 = \frac{s + m_3^2}{4l_+}, \\
A_{1,2}' &= (2k_+ + 2l_+ + q)\left(\frac{t+m_6^2}{2k_+} + \frac{s+m_3^2}{2l_+} + q\right) - s - t - m_{1,2}^2.
\end{aligned} \tag{17}$$

The integrations over $s$ and $t$ proceed in exactly the same way as in the previous calculation. Finally, we can make the shift $l_+ \to l_+ - q/2$. It turns out that this change of variables not only makes the region of integration in the $k_+ l_+$ plane equal to the triangle $T$ defined in (6), but the whole formula for $V_i$ becomes almost the same as formula (13), the only difference being the coefficients $b_i$, which in the present case are

$$b'_{1,2} = 2(k_+ + l_+)\left(\frac{m_6^2}{2k_+} + \frac{m_3^2}{2l_+ - q} + q\right) - m_{1,2}^2. \tag{18}$$

The equivalence of the two formulas can be checked after integrating over $k_+$ and $l_+$. It provides an excellent cross check for the numerical calculation.

In practical calculations one can encounter a mass configuration in which $m_1 = m_2$. In this case the formula simplifies:

$$V_0(q^2, m_1 = m_2) = -\frac{4\pi^4}{q^2(m_5^2 - m_4^2)} \times \int_{-q/2}^{0} dk_+ \int_{-k_+}^{q/2} dl_+ \frac{k_+}{k_+ + l_+}\left(\frac{1}{t_5 - t_0}\ln\frac{t_5}{t_0} - \frac{1}{t_4 - t_0}\ln\frac{t_4}{t_0}\right), \tag{19}$$

where $t_0 = -cb'_1/(1 - ca)$.

## SPACE-LIKE EXTERNAL MOMENTA

If the external momenta have space-like values the computation of the propagator and vertex diagrams is greatly simplified since the internal particles do not become on-shell. In particular we can easily check the analytical results obtained for the two-point and the planar three-point functions when all internal particles are massless. While the result for the two-point function (see Fig. 1(a)) has been know for long time [4, 5], the much more complex formulas for the vertex functions (of both planar and crossed topologies) have been obtained only very recently [6]. We present here numerical evaluation of the vertex function with all internal masses equal $m$ and space-like external momenta. In the limit $m \to 0$ we reproduce the result of [6].

For the numerical calculation it is convenient to choose such reference frame that the external outgoing momenta become (according to the notation of Fig. 1(b))

$$\begin{aligned} p_1^\mu &= (e, q_1, 0, 0), \\ p_2^\mu &= (-e, q_2, 0, 0), \\ p_3^\mu &= (0, -q_1 - q_2, 0, 0). \end{aligned} \tag{20}$$

Repeating the calculations described in the previous section we arrive at a double integral representation which is easy to evalulate numerically. Fig. 2 shows the ratio of the vertex function

$$V(p_1^2, p_2^2, p_3^2, m^2) = \iint \frac{d^4k\, d^4l}{P_1 P_2 P_3 P_4 P_5 P_6} \tag{21}$$

to the value of the vertex at zero internal masses

$$U(p_1^2, p_2^2, p_3^2) = \left(\frac{i\pi^2}{p_3^2}\right)^2 \Phi^{(2)}\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right), \tag{22}$$

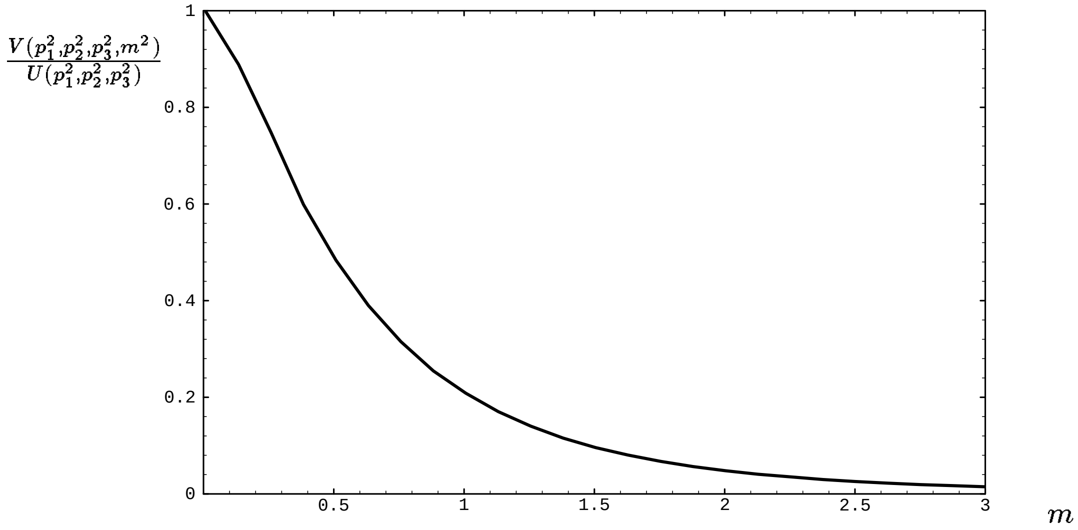


Figure 2: Vertex function for space-like external momenta $p_1^2 = -1$, $p_2^2 = -4$, $p_3^2 = -25$. All internal masses are equal $m$ and the function is divided by the result of Ussyukina and Davydychev which corresponds to the vertex with $m = 0$.

where the function $\Phi^{(2)}$ has been derived in [6]. In formula (21) $P_i$ denote propagators defined analogously to the formula (3)). For the purpose of numerical calculation we choose one arbitrary configuration of external momenta $p_1^2 = -1$, $p_2^2 = -4$, $p_3^2 = -25$. We see that for very small masses the ratio of the two formulas becomes unity which confirms the analytical result of Ussyukina and Davydychev.

## DIVERGENT INTEGRALS

The method of calculation of two-loop diagrams described here is limited to the four-dimensional space. In dealing with divergent integrals we first have to find another diagram with the same divergent part but simple enough to be computed analytically. The difference of the two diagrams can then be calculated in four dimensions and the final result is obtained by adding the analytical formula for the simpler diagram. Such procedure, based on the representation of the two-point functions proposed in [7], has been described in [8]. In the present section I illustrate an analogous procedure in the framework of the representation which works for both two- and three-point functions, with the example of the sunrise diagram with a zero momentum insertion in one of the propagators (Fig. 1(c)).

Considerable effort has been recently devoted to the investigation of this diagram. Asymptotic expansions have been derived in the papers [9, 10], and explicit expressions in terms of generalized hypergeometric, or Lauricella, functions were obtained in ref. [11]. The same diagram has also been analyzed in [12]. It has been noted in [13] that a general two-loop diagram can formally be expressed as a sunrise diagram with masses and the external momentum being functions of Feynman parameters over which one can integrate numerically. The latter reference gave a very convenient one-dimensional integral representation for this diagram.

The value of the sunrise diagram is

$$S(p^2, m_1, m_2, m_3) = (\pi e^{\gamma_E})^{2\omega} \iint d^D k d^D l \frac{1}{P_1^2 P_2 P_3} \tag{23}$$

with $\gamma_E$ being Euler's constant and

$$\begin{aligned} P_1 &= (l+k+p)^2 - m_1^2 + i\eta, \\ P_2 &= l^2 - m_2^2 + i\eta, \\ P_3 &= k^2 - m_3^2 + i\eta, \end{aligned} \tag{24}$$

and since the sunrise diagram is ultraviolet divergent we have to compute it in $D \equiv 4 - 2\omega$ dimensions.

It has been shown in the previous sections that the triangular regions over which one has to perform the final two integrations numerically are determined only by the values of external momenta, and are independent of masses of particles inside the diagram. Therefore it is convenient to choose for the subtraction a diagram which differs from the diagram we are interested in only by the values of internal masses. In the present case we choose a diagram with vanishing $m_2$ and $m_3$ which can be computed analytically

$$\begin{aligned} S(p^2, m_1, 0, 0) &= -\pi^4 \left[ \frac{1}{2\omega^2} + \frac{1}{2\omega}(1 - 2\ln m_1^2) - \frac{1}{2} + \frac{\pi^2}{4} \right. \\ &\quad \left. + \ln m_1^2 \left( \ln m_1^2 - 1 \right) + \mathrm{Li}_2 \left( \frac{p^2}{m_1^2} \right) + \frac{p^2 - m_1^2}{p^2} \ln \left( \frac{m_1^2 - p^2}{m_1^2} \right) \right] + O(\omega) \end{aligned} \tag{25}$$

and the value of a diagram with arbitrary masses can be expressed by

$$S(p^2, m_1, m_2, m_3) = S(p^2, m_1, 0, 0) + \Delta(p^2, m_1, m_2, m_3) \tag{26}$$

where $\Delta(p^2, m_1, m_2, m_3) \equiv \Delta$ is free from divergences and can be computed using our method. For simplicity we only consider the case of $p^2 < m_1^2$ where both diagrams are real.

After the integration over angular variables as in (4) and over $k_-$ and $l_-$ with help of contour integrals we obtain

$$\Delta = \pi^4 \iint_{\mathcal{T}} \frac{dk_+ dl_+}{k_+ l_+} \iint_0^\infty ds dt \left( \frac{A}{(A^2 - B^2)^{3/2}} - \frac{A_0}{(A_0^2 - B^2)^{3/2}} \right), \tag{27}$$

with

$$\begin{aligned} A &= (2l_+ + 2k_+ + p) \left( \frac{m_2^2 + s}{2l_+} + \frac{m_3^2 + t}{2k_+} + p \right) - m_1^2 - s - t + i\eta \\ &\equiv at + b + cs \\ B^2 &= 4st \end{aligned} \tag{28}$$

and the subscript 0 means that we take $m_2 = m_3 = 0$. The region of $k_-$ and $l_-$ integration is a triangle $\mathcal{T}$ delimited by the lines $k_+ = 0$, $l_+ = 0$ and $l_+ + k_+ = -p/2$. The integrations over $s$ and $t$ are easy

$$\int_0^\infty \mathrm{d}s \frac{at + b + cs}{[(at + b + cs)^2 - 4st]^{3/2}} = \frac{1}{(1 - ac)t - bc} \qquad \text{for } a, b, c < 0 \tag{29}$$

and finally we arrive at

$$\Delta \;=\; -4\pi^4 \int_{-p/2}^{0} dk_+ \int_{-k_+}^{0} \frac{dl_+}{p(2l_+ + 2k_+ + p)} \ln \frac{p(2l_+ + 2k_+ + p) - m_1^2}{\left(p + \frac{m_2^2}{2l_+} + \frac{m_3^2}{2k_+}\right)(2l_+ + 2k_+ + p) - m_1^2}. \tag{30}$$

Thus we have found a double integral representation of the sunrise diagram. One of the $k_+$, $l_+$ integrations can still be carried out, and since the argument of the logarithm is a polynomial of the second degree the result will in general involve dilogarithms of complex arguments even below the threshold. For the purpose of numerical evaluation it may be convenient to work with a double-integral, but explicitly real representation.

**ACKNOWLEDGMENTS**

I thank D. Broadhurst, K.G. Chetyrkin, and A.I. Davydychev for discussion and advice, and B. Krause and M. Steinhauser for checking some of the formulas. I am very grateful to the organizers of the Cargèse Summer Institute for the opportunity to take part in this great event. I thank Graduiertenkolleg Elementarteilchenphysik at the University of Karlsruhe for support.

# References

[1] D. Kreimer, Phys. Lett. **B292** (1992) 341.

[2] A. Czarnecki, U. Kilian, and D. Kreimer, *New representation of two-loop propagator and vertex functions*, hep-ph/9405423, in press in Nucl. Phys. B.

[3] D. Kreimer, *A short note on two-loop box functions*, hep-ph/9407234.

[4] J.L. Rosner, Ann. Phys. **44** (1967) 11.

[5] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159.

[6] N.I. Ussyukina and A.I. Davydychev, Phys. Lett. **B298** (1993) 363; Yad. Fiz. **56** (1993) 172; Phys. Lett. **B332** (1994) 159.

[7] D. Kreimer, Phys. Lett. **B273** (1992) 277.

[8] F.A. Berends and J.B. Tausk, Nulc. Phys. **B421** (1994) 456.

[9] A.I. Davydychev and J.B. Tausk, Nucl. Phys. **B397** (1993) 123.

[10] A.I. Davydychev, V.A. Smirnov, and J.B. Tausk, Nucl. Phys. **B410** (1993) 325.

[11] F.A. Berends, M. Buza, M. Böhm, and R. Scharf, Zeit. Phys. **C63** (1994) 227. S. Bauberger, F.A. Berends, M. Böhm, and M. Buza, hep-ph/9409388.

[12] F.A. Lunev, *On evaluation of two-loop self-energy diagram with three propagators*, hep-th/9408161.

[13] A. Ghinculov and J.J. van der Bij, *Massive two-loop diagrams: the Higgs propagator*, hep-ph/9405418.