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RADIATION OF HEAVY QUARKS

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Abstract

The rate for the production of massive fermions radiated off a pair of massless fermions is calculated analytically. Combined with the analytic calculation of the corresponding virtual contribution one obtains the order α^2 correction to the cross section which is induced by the real and virtual radiation of a pair of massive fermions by massless fermions. Approximations valid for large and for small masses are derived and shown to agree with earlier calculations.

1. Introduction

A large arsenal of approximations and expansions has been developed for QCD calculations in those cases where masses are large or small compared to the characteristic energy of the problem. There is, however, only a fairly limited number of three loop problems with arbitrary nonvanishing masses and energies where answers can be obtained in closed analytical form. The results are first of all useful in their own right; in addition they provide valuable tests of the approximation methods, strengthening our confidence in these methods in those cases where no exact result is available.

In this paper the imaginary part of the three loop amplitude depicted in Fig.1 will be calculated in closed form. It corresponds to the rate for the production of a pair of massless fermions through an external current, with additional radiation of a (real or virtual) pair of fermions of mass m. The calculation will be performed in the framework of QED; the transition to QCD will be accomplished at the end of the paper. The calculational technique is based on the fact that all relevant integrals can be performed in two steps: First one evaluates effectively a two loop diagram; the result is subsequently integrated with a fairly simple weight function. This technique has been used previously [1, 2] to calculate individually the contributions of the two fermion and the four fermion cuts to the imaginary part. There the limit $s \gg m^2$ was considered and only logarithmically enhanced or constant terms were kept. In [3] the same approach was used to evaluate the vertex corrections (including of course the appropriate counterterms) for arbitrary m^2/s. In this paper a similar formula is derived for the four fermion cut, completing thus the $\mathcal{O}(\alpha^2)$ evaluation of this admittedly rather simple diagram.

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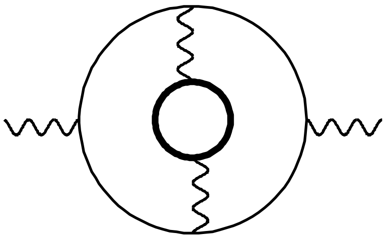

Figure 1: *Characteristic Feynman diagram describing the production of a pair of massless and a (virtual or real) pair of massive fermions.*

The bulk of the paper is formulated for QED as a reference theory, with α defined as usual at $q^2 = 0$ and with fermions of unit charge. The definition of m is irrelevant in the order under consideration. At the end of the paper the transition to QCD will be performed and the $\overline{\rm MS}$ definition of the coupling constant employed.

2. Four fermion contribution

The four fermion cut through the diagram Fig.1 corresponds essentially to the part of the four fermion production cross section where the light fermion q with mass zero is coupled to the external virtual photon or Z boson and the heavy fermion Q with mass m is radiated off the light fermion through a virtual photon. For the present problem it is convenient to decompose the four fermion phase space $(q\bar{q}Q\bar{Q})$ integration into the integration over the three particle phase space $q\bar{q}(Q\bar{Q})$, where $(Q\bar{Q})$ denotes a system of fixed invariant mass s', and a (trivial) two particle phase space integration for the $(Q\bar{Q})$ system. This leaves the final integration over s'.

The first step corresponds to the calculation of the rate for the decay of a vector boson of mass $\sqrt{s}$ into a vector boson of mass $\sqrt{s'}$ plus a pair of massless fermions $q\bar{q}$. The two particle system gives rise to the familiar $Q\bar{Q}$ threshold factor $\beta(3-\beta^2)/2$. One thus arrives at the following integral [2]

$$\begin{aligned}
\frac{\sigma_{q\bar{q}Q\bar{Q}}}{\sigma_{q\bar{q}}(\mathrm{Born})} \equiv R_{q\bar{q}Q\bar{Q}} &= \left(\frac{\alpha}{\pi}\right)^2 \varrho^R , \\
\varrho^R &= \frac{1}{3}\int_{4x}^{1} \frac{\mathrm{d}u}{u}\,(1+\frac{2x}{u})\sqrt{1-\frac{4x}{u}}\;\times \\
&\quad \Big\{\frac{1}{2}(1+u)^2\ln^2 u + \frac{1}{2}(3+4u+3u^2)\ln u + \frac{5}{2}(1-u^2) \\
&\quad -2(1+u)^2\left[\mathrm{Li}_2(-u)+\ln(1+u)\ln u+\zeta(2)/2\right]\Big\}, \qquad (1)
\end{aligned}$$

where $u = s'/s$ and $x = m^2/s$. This integral can be solved in a straightforward way numerically, and after some effort, also analytically in terms of polylogarithms:

$$
\begin{aligned}
\varrho^R = \quad & \frac{4}{3}\left(1-6x^2\right)\left\{\frac{1}{2}\,\mathrm{Li}_3(\frac{1-w}{2})-\frac{1}{2}\,\mathrm{Li}_3(\frac{1+w}{2})\right.\\
& +\mathrm{Li}_3(\frac{1+w}{1+a})-\mathrm{Li}_3(\frac{1-w}{1-a})+\mathrm{Li}_3(\frac{1+w}{1-a})-\mathrm{Li}_3(\frac{1-w}{1+a})\\
& \left.+\frac{1}{2}\ln(\frac{1+w}{1-w})\left[\zeta(2)-\frac{1}{12}\ln^2(\frac{1+w}{1-w})+\frac{1}{2}\ln^2(\frac{a-1}{a+1})-\frac{1}{2}\ln(\frac{1+w}{2})\ln(\frac{1-w}{2})\right]\right\}\\
& +\frac{1}{9}\,a\,(19+46x)\,\Big(\mathrm{Li}_2(\frac{1+w}{1+a})+\mathrm{Li}_2(\frac{1-w}{1-a})-\mathrm{Li}_2(\frac{1+w}{1-a})-\mathrm{Li}_2(\frac{1-w}{1+a})\\
& \qquad +\ln(\frac{a-1}{a+1})\,\ln(\frac{1+w}{1-w})\Big) \qquad (2)\\
& +4\left(\frac{19}{72}+x+x^2\right)\left(\mathrm{Li}_2(-\frac{1+w}{1-w})-\mathrm{Li}_2(-\frac{1-w}{1+w})-\ln x\,\ln(\frac{1+w}{1-w})\right)\\
& +7\left(\frac{73}{189}+\frac{74}{63}x+x^2\right)\ln(\frac{1+w}{1-w})-\frac{1}{3}\left(\frac{2123}{108}+\frac{2489}{54}x\right)\,w\,,
\end{aligned}
$$

where

$$
a=\sqrt{1+4x}\,,\quad w=\sqrt{1-4x}\,. \qquad (3)
$$

This formula is the main result of this paper. The function ϱ^R is evidently zero for $m>\sqrt{s}/2$, corresponding to $x>1/4$. It can be expanded in the limit of small masses, $x=m^2/s\ll 1$. Including terms of order x^3 one obtains

$$
\begin{aligned}
\varrho^R \;=\; & \frac{1}{9}\left[-\frac{1}{2}\ln^3 x-\frac{19}{4}\ln^2 x+\left(6\zeta(2)-\frac{73}{3}\right)\ln x+15\zeta(3)+19\zeta(2)-\frac{2123}{36}\right]\\
& +\frac{4}{3}\,x\left[-\frac{3}{2}\ln^2 x-3\ln x+6\zeta(2)-12\right]\\
& +\frac{1}{3}x^2\left[\ln^3 x-6\ln^2 x+12\left(-\zeta(2)+1\right)\ln x-30\zeta(3)+24\zeta(2)\right] \qquad (4)\\
& +\frac{2}{9}x^3\left[-4\ln^2 x+\frac{28}{3}\ln x+12\zeta(2)+\frac{37}{9}\right]+\mathcal{O}(x^4)\,.
\end{aligned}
$$

The exact result for ϱ^R and approximations including successively higher orders in $x=m^2/s$ are displayed in Fig.2a. The curve including terms up to m^4/s^2 already provides an excellent approximation.

3. Vertex correction

The vertex correction has been calculated in [3]. Through the two particle cut it leads to the following contribution to R:

$$
\delta R_{q\bar q}=\left(\frac{\alpha}{\pi}\right)^2\varrho^V\,, \qquad (5)
$$

with

$$
\begin{aligned}
\varrho^V \;=\; & \frac{2}{3}\left(1-6x^2\right)\left(\mathrm{Li}_3(A^2)-\zeta(3)-2\zeta(2)\ln A+\frac{2}{3}\ln^3 A\right)\\
& +\frac{1}{9}\,(19+46x)\,\sqrt{1+4x}\,\left(\mathrm{Li}_2(A^2)-\zeta(2)+\ln^2 A\right) \qquad (6)\\
& +\frac{5}{36}\left(\frac{53}{3}+44x\right)\ln x+\frac{3355}{648}+\frac{119}{9}x\,,
\end{aligned}
$$

where $A = (\sqrt{1+4x}-1)/\sqrt{4x}$. The function ϱ^V is simply twice the real part of the form factor $F_2^{(f)}$ defined in [3] in the region $x = 1/4r > 0$. The vertex correction is present for large and small m^2/s as well. The leading term in the heavy mass expansion has also been calculated in [4] employing a completely different technique

$$\varrho^V \approx \frac{s}{m^2}\frac{1}{45}\left(\ln\frac{m^2}{s}+\frac{22}{5}\right) . \tag{7}$$

As previously discussed in [3] (see Fig. 2 of [3]), the heavy mass expansion provides an excellent approximation to the full answer from $m^2 \gg s$ even down to the threshold $4m^2 = s$. This justifies, for example, the use of eq.(7) for the contribution of virtual top quarks to the Z decay rate for the full top mass range. The quality of the expansion is demonstrated in Fig.3 where the full analytic result for ϱ^V is compared to the approximation in the range $s/m^2 \leq 4$. For small $x = m^2/s$, on the other hand, one obtains (terms up to $\mathcal{O}(x)$ can be found also in [3])

$$\begin{aligned}
\varrho^V &= \frac{1}{9}\left[\frac{1}{2}\ln^3 x+\frac{19}{4}\ln^2 x+\left(-6\,\zeta(2)+\frac{265}{12}\right)\ln x-6\zeta(3)-19\zeta(2)+\frac{3355}{72}\right] \\
&\quad +\frac{4}{3}x\left[\frac{3}{2}\ln^2 x+3\ln x-6\zeta(2)+12\right] \\
&\quad +\frac{1}{3}x^2\left[-\ln^3 x+6\ln^2 x+\left(12\zeta(2)-\frac{33}{2}\right)\ln x+12\zeta(3)-24\zeta(2)+\frac{39}{2}\right] \\
&\quad +\frac{2}{9}x^3\left[2\ln^2 x-\frac{14}{3}\ln x-8\zeta(2)-\frac{1}{3}\right]+\mathcal{O}(x^4) .
\end{aligned} \tag{8}$$

In Fig.2b the function ϱ^V is displayed in the range from $x = 1/4$ down to $1/4 \times 10^{-2}$, together with approximations including successively higher orders. Again one concludes that the expansion up to m^4/s^2 provides an excellent approximation in the full mass range for nearly massless quarks as well as close to threshold. Upon adding ϱ^R and ϱ^V and performing the same approximations as above, one obtains

$$\begin{aligned}
\varrho^V+\varrho^R &= \left[-\frac{1}{4}\ln x+\zeta(3)-\frac{11}{8}\right]+x^2\left[-\frac{3}{2}\ln x-6\zeta(3)+\frac{13}{2}\right] \\
&\quad +x^3\left[-\frac{4}{9}\ln^2 x+\frac{28}{27}\ln x+\frac{8}{9}\zeta(2)+\frac{68}{81}\right]+\mathcal{O}(x^4) .
\end{aligned} \tag{9}$$

The quadratic and cubic logarithms compensate as is evident from eqs.(4) and (8). A linear logarithm whose origin will be discussed in a moment remains. The sum $\varrho^V + \varrho^R$ is shown in Fig.4a, together with the approximations discussed before. Terms proportional m^2/s are absent in the approximations, whence dotted and dashed-dotted curves coincide. The absence of these terms has been observed and dicussed in [5] in the context of QCD.

4. QCD

It has become customary to formulate the QCD result in the $\overline{\mathrm{MS}}$ renormalization scheme which is better adapted to the limit $m \to 0$. To perform this transition (still in the abelian theory) corrections of order α and α^2 must be redistributed.

With α defined as fine structure constant in the usual way one finds

$$R = 1+\frac{3}{4}\frac{\alpha}{\pi}+\left(\frac{\alpha}{\pi}\right)^2\left[\left(\varrho^R+\varrho^V\right)+...\right] . \tag{10}$$

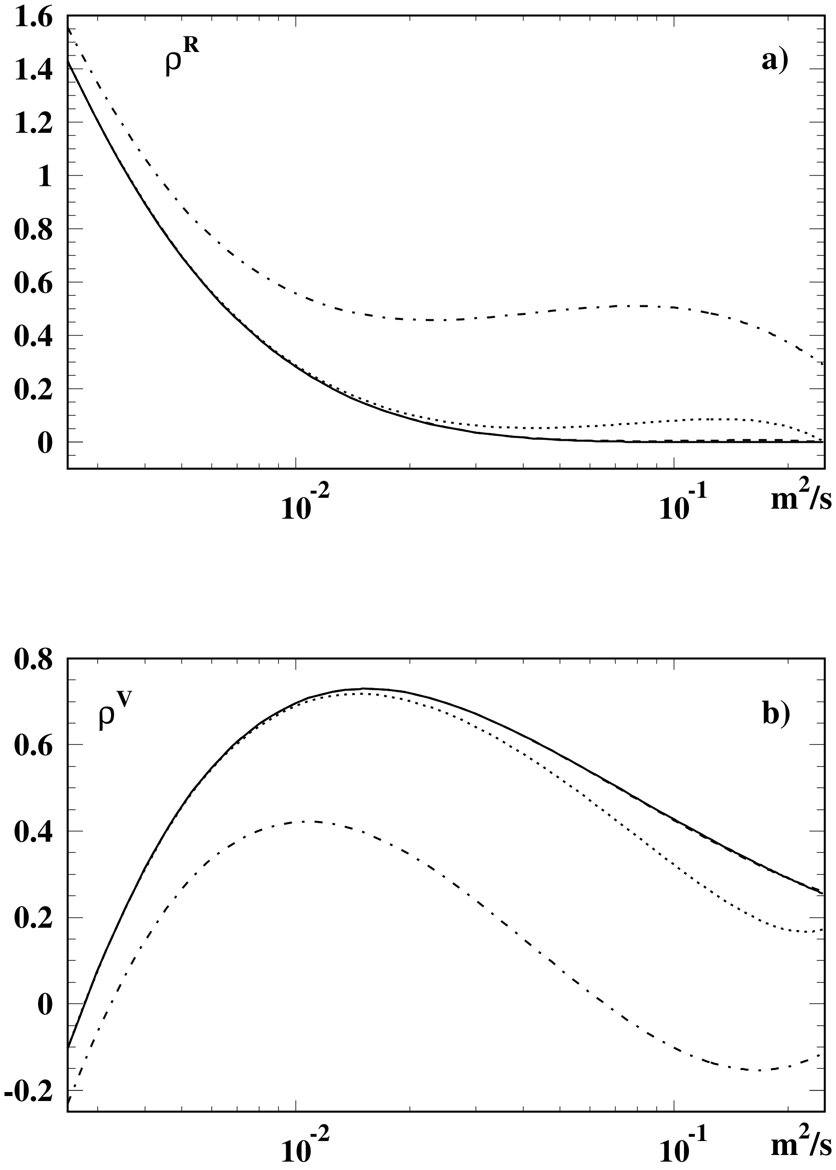

Figure 2: *a) The function* ϱ^R *describing the production of four fermions in the region* $0 < x = m^2/s < 1/4$*. Solid line: exact result; dashed-dotted line: logarithmic and constant terms only; dotted line: including* m^2/s *corrections; dashed line: including* m^2/s *and* m^4/s^2 *corrections. b) Corresponding curves for* ϱ^V *describing virtual corrections.*

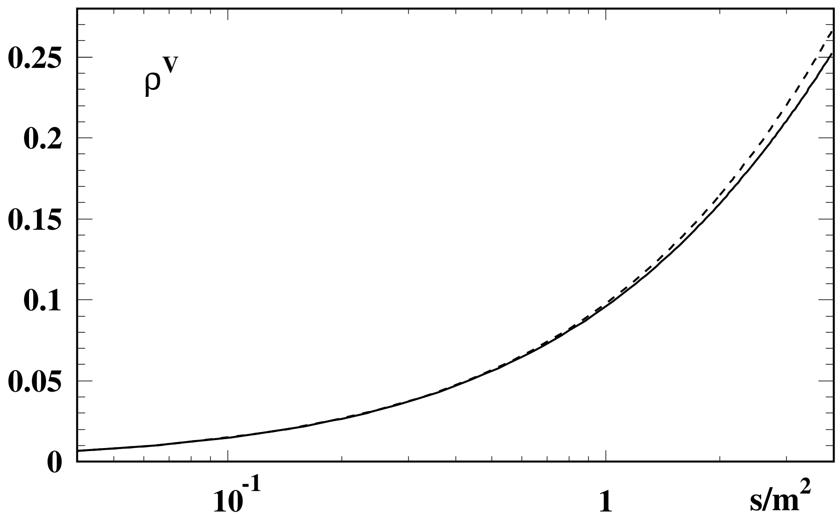

Figure 3: *The function ϱ^V describing virtual corrections in the range $s/m^2 < 4$ (solid curve) and the approximation eq.(7) (dashed curve).*

The dots indicate other contributions of $\mathcal{O}(\alpha^2)$ (for example from two photon exchange or other fermion loops) which are irrelevant for the present discussion. The fine structure constant is related to the coupling constant in the $\overline{\mathrm{MS}}$ scheme at scale μ^2 through

$$\alpha = \alpha_{\overline{\mathrm{MS}}}(\mu^2)\left(1 + \frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi}\frac{1}{3}\ln\frac{m^2}{\mu^2}\right) + \mathcal{O}(\alpha^3) \tag{11}$$

which implies [2]

$$R = 1 + \frac{3}{4}\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi} + \left(\frac{\alpha_{\overline{\mathrm{MS}}}(\mu^2)}{\pi}\right)^2\left[\left(\varrho^R + \varrho^V + \frac{1}{4}\ln\frac{m^2}{\mu^2}\right) + ...\right]. \tag{12}$$

The combination $\varrho^R + \varrho^V + \frac{1}{4}\ln m^2/s$ thus gives the contribution from the (real plus virtual) radiation of a massive fermion pair to the total rate or cross section in the $\overline{\mathrm{MS}}$ scheme at scale $\mu^2 = s$. It is shown in Fig.4b together with the three approximations introduced above.

The group theoretical transition from QED (U(1)) to QCD (SU(3)) is easily performed by multiplying the first order correction by

$$\frac{1}{3}\mathrm{Tr}\left(\frac{\lambda^a\lambda^a}{4}\right) = \frac{4}{3}, \tag{13}$$

and the second order term by

$$\frac{1}{3}\mathrm{Tr}\left(\frac{\lambda^a\lambda^b}{4}\right)\left(\frac{\lambda^a\lambda^b}{4}\right) = \frac{2}{3}. \tag{14}$$

Adopting as renormalization scale $\mu^2 = s$ one arrives at

$$R_{\mathrm{QCD}} = 1 + \frac{\alpha_{\mathrm{S}}(s)}{\pi} + \left(\frac{\alpha_{\mathrm{S}}(s)}{\pi}\right)^2\left[\frac{2}{3}\left(\varrho^R + \varrho^V + \frac{1}{4}\ln\frac{m^2}{s}\right) + ...\right]. \tag{15}$$

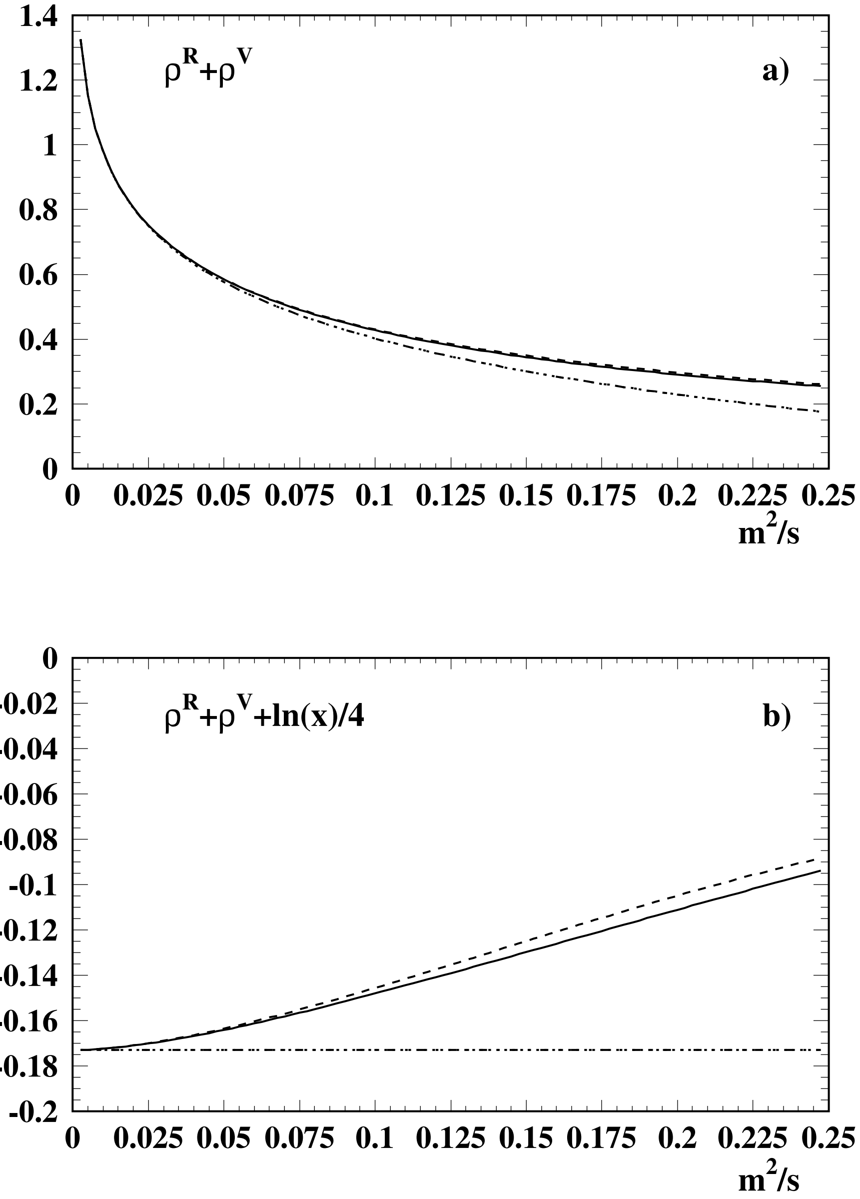

Figure 4: *a) The function $\varrho^R + \varrho^V$ as described in the text. Solid line: exact result; dashed-dotted line: logarithmic and constant terms only; dotted line: including m^2/s corrections; dashed line: including m^2/s and m^4/s^2 corrections. b) The function $\varrho^R + \varrho^V + \ln(x)/4$ and its approximations describing the same result in the $\overline{\mathrm{MS}}$ scheme.*

The second order correction factor in the small m limit

$$\frac{2}{3}\left(\varrho^R + \varrho^V + \frac{1}{4}\ln\frac{m^2}{s}\right) = \frac{2}{3}\zeta(3) - \frac{11}{12} + x^2\left(-\ln x - 4\zeta(3) + \frac{13}{3}\right) + x^3\left(-\frac{8}{27}\ln^2 x + \frac{56}{81}\ln x + \frac{16}{27}\zeta(2) + \frac{136}{243}\right) + \mathcal{O}(x^4), \tag{16}$$

is well behaved. The constant term coincides with the corresponding result given in [6] (see also [2]), the quartic mass term with the results of [7].

To summarize: The rate for the production of massive fermions radiated off a pair of massless fermions has been calculated analytically. Combined with the analytic calculation of the corresponding virtual contribution one obtains the correction to the total cross section of order α^2 which is induced by real and virtual radiation of a pair of massive fermions. The transition to QCD and to the $\overline{\mathrm{MS}}$ scheme allows the comparison with approximate formulae valid for large or small m^2/s. Earlier results for $\alpha_s^2 m^4/s^2$ corrections have been confirmed. These leading terms provide an excellent approximation to the full anwer.

Acknowledgement: We would like to thank K. Chetyrkin for helpful discussions.

Note added: After completion of this calculation we received a paper by D.E. Soper and L.R. Surguladze [8], where the combination $2/3(\varrho^R + \varrho^V + \frac{1}{4}\ln\frac{m^2}{s})$ has been calculated numerically.

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