TTP94–08*
June 1994

Quartic Mass Corrections to R_{had}

K.G.Chetyrkin[1], J.H.Kühn[2]

Institut für Theoretische Teilchenphysik
Universität Karlsruhe
Kaiserstr. 12, Postfach 6980
D-76128 Karlsruhe, Germany

Abstract

The influence of nonvanishing quark masses on the total cross section in electron positron collisions and on the Z decay rate is calculated. The corrections are expanded in m^2/s and α_s. Methods similar to those applied for the quadratic mass terms allow to derive the corrections of order $\alpha_s m^4/s^2$ and $\alpha_s^2 m^4/s^2$. Coefficients which depend logarithmically on m^2/s and which cannot be absorbed in a running quark mass arise in order α_s^2. The implications of these results on electron positron annihilation cross sections at LEP and at lower energies in particular between the charm and the bottom threshold and for energies several GeV above the $b\bar{b}$ threshold are discussed.

*The complete postscript file of this preprint, including figures, is available via anonymous ftp at ttpux2.physik.uni-karlsruhe.de (129.13.102.139) as /ttp94-08/ttp94-08.ps

[1]On leave from Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, 117312, Russia

[2]Work supported by BMFT contract 056KA93P

1 Introduction

The precise measurement of the cross section for hadron production in electron positron annihilation leads to an accurate determination of the strong coupling constant α_s. The method is free from a variety of theoretical uncertainties and ambiguities which beset many other observables of (still) superior statistical accuracy. The extraction of α_s from the ratio R between the rates for hadron and lepton pair production is based on the calculation of perturbative QCD corrections. These are available up to third order in α_s in the massless limit [1, 2]. Mass corrections were calculated for small m^2/s up to order α_s^3 for vector and up to order α_s^2 for axial vector current induced reactions [3, 4]. In these papers it was furthermore demonstrated that all large logarithms which arise in the expansion coefficients can be absorbed in the running quark mass. The same comment applies to the contribution of "singlet" terms which are relevant for the axial part of the Z decay rate [5, 6, 7, 8].

In principle one would of course prefer to control the mass dependence of the cross section for arbitrary m^2/s. To date this has been achieved in first order in α_s only. Higher orders in α_s were calculated for the first term in the m^2/s expansion, allowing for calculational techniques based on massless propagator type integrals. In this work this program will be pursued further and m^4/s^2 terms will be calculated up to order α_s^2. These terms are fairly unimportant for Z decays, however, they will become relevant in the low energy region. Examples are measurements at a B meson factory just below or several GeV above the $b\bar{b}$ threshold, or alternatively around 5 GeV, an energy region that could be explored at the BEPC storage ring.

The α_s^2 calculation presented below is based on [9, 12]. There the operator product expansion of two point correlators was studied up to two loops, including therefore terms of first order in α_s. The expansion included power law suppressed terms up to operators of dimension four, which are induced by nonvanishing quark masses. Renormalisation group arguments, similar to those employed already in [3, 4], will allow to deduce the $\alpha_s^2 m^4$ terms. The calculation is performed for vector and axial vector current correlators. The first one is of course relevant for electron positron annihilation into heavy quarks at arbitrary energies, the second one for Z decays into b quarks and for top production at a future linear collider. In order α_s the m^4 terms can also be obtained from the Taylor expansion of the complete answer [13, 14]. This provides not only a (fairly trivial) cross check of the α_s term. The comparison with the complete answer in first order α_s may also indicate to which extent m^2 plus m^4 terms lead to a reliable estimate of the complete answer in order α_s^2.

2 Mass terms of first order in α_s

QCD corrections to vector and axial current correlators in order α_s and for arbitrary m^2/s were derived in [13]. Compact formulae are given in [14]. These are conventionally expressed in terms of the pole mass denoted by m in the following. It is straightforward to expand these results in m^2/s and one obtains – in obvious notation –

$$
\begin{aligned}
R_V &= 1 - 6\frac{m^4}{s^2} - 8\frac{m^6}{s^3} \\
&\quad + \frac{\alpha_s}{\pi}\left[1 + 12\frac{m^2}{s} + \left(10 - 24\log(\frac{m^2}{s})\right)\frac{m^4}{s^2}\right. \\
&\quad \left. - \frac{16}{27}\left(47 + 87\log(\frac{m^2}{s})\right)\frac{m^6}{s^3}\right],
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
R_A &= 1 - 6\frac{m^2}{s} + 6\frac{m^4}{s^2} + 4\frac{m^6}{s^3} \\
&\quad + \frac{\alpha_s}{\pi}\left[1 - \left(6 + 12\log(\frac{m^2}{s})\right)\frac{m^2}{s} + \left(-22 + 24\log(\frac{m^2}{s})\right)\frac{m^4}{s^2}\right. \\
&\quad \left. + \frac{8}{27}\left(41 + 42\log(\frac{m^2}{s})\right)\frac{m^6}{s^3}\right].
\end{aligned} \tag{2}
$$

The approximations to the correction function for the vector current correlator (including successively higher orders and without the factor α_s/π) are compared to the full result in Fig.1. For high energies, say for $2m_b/\sqrt{s}$ below 0.3, an excellent approximation is provided by the constant plus the m^2 term. In the region of $2m/\sqrt{s}$ above 0.3 the m^4 term becomes increasingly important. The inclusion of this term improves the agreement significantly and leads to an excellent approximation even up to $2m/\sqrt{s} \approx 0.7$ or 0.8. For the narrow region between 0.6 and 0.8 the agreement is further improved through the m^6 term. Higher powers in m which can be calculated in a straightforward way, lead only to a modest further improvement in the region close to threshold, say above 0.8. This region with its behaviour governed by the Coulomb singularity requires therefore a different treatment. The same considerations apply for the axial current correlator (Fig.1).

Inclusion of m^2 and m^4 terms hence allows to extend the QCD prediction from the high energy limit down to fairly low center of mass energy values, perhaps 4 to 5 GeV above the $b\bar{b}$ threshold, or from about 2 GeV above the charm threshold up to just below $\Upsilon(4S)$. Similarly to the discussion in [3, 4] for the m^2 terms, the logarithms accompanying the m^4 terms can also be absorbed through a redefinition of m in terms of the $\overline{MS}$ mass [15, 16] at scale s

$$
m^2 = \overline{m}^2(s)\left(1 + \frac{\alpha_s}{\pi}(-2\log m^2/s + 8/3)\right) \tag{3}
$$

and one obtains[3] ($\overline{m} \equiv \overline{m}(s)$)

$$
\begin{aligned}
R_V &= 1 - 6\frac{\overline{m}^4}{s^2} - 8\frac{\overline{m}^6}{s^3} \\
&\quad + \frac{\alpha_s}{\pi}\left[1 + 12\frac{\overline{m}^2}{s} - 22\frac{\overline{m}^4}{s^2} - \frac{16}{27}\left(6\log(\frac{\overline{m}^2}{s}) + 155\right)\frac{\overline{m}^6}{s^3}\right],
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
R_A &= 1 - 6\frac{\overline{m}^2}{s} + 6\frac{\overline{m}^4}{s^2} + 4\frac{\overline{m}^6}{s^3} \\
&\quad + \frac{\alpha_s}{\pi}\left[1 - 22\frac{\overline{m}^2}{s} + 10\frac{\overline{m}^4}{s^2} + \frac{8}{27}\left(-39\log(\frac{\overline{m}^2}{s}) + 149\right)\frac{\overline{m}^6}{s^3}\right].
\end{aligned} \tag{5}
$$

This resummation is possible for the second and fourth powers of m in first order α_s and in fact for m^2 corrections to R_V and R_A in all orders of α_s. However, logarithmic terms persist in the m^4 corrections, starting from $\mathcal{O}(\alpha_s^2)$, as discussed in the next section. Higher order terms in the mass expansion, starting from m^6, evidently exhibit superficially "incurable" logs already in order α_s. In fact, these logs may be also summed up (see [18, 12] and also [19]). Unlike the quadratic mass term, the resulting expression will contain some explicit dependence on the strong coupling constant $\alpha_s(m_q)$ taken at the scale equal to the quark mass. The numerical effect of such a summation proves to be rather small unless the ratio s/m_q^2 is chosen too large.

3 Corrections of order $\alpha_s^2 m^4$

Motivated by the fact that the first few terms in the m^2/s expansion provide already an excellent approximation to the complete answer in order α_s, we now proceed to the evaluation of the three loop corrections. The calculation makes use of the properties of the operator product expansion as discussed in [9, 12]. To allow for a comprehensive presentation, the relevant formulae are repeated whenever necessary.

1. Operator Product Expansion (OPE)

The calculation is based on the operator product expansion of the T-product of two vector or axial vector currents

$$
T_{\mu\nu}(q, J) = i\int T(J_\mu(x)J_\nu^+(0))\mathrm{e}^{iqx}\mathrm{d}x, \tag{6}
$$

with $J_\mu = \overline{u}\gamma_\mu d$ (or $J_\mu = \overline{u}\gamma_\mu\gamma_5 d$). Here u and d are just two generically different quarks with masses m_u and m_d. Quarks which are not coupled to the external current will influence the result in order α_s^2 through their coupling to the gluon field.

[3]The m^4 terms given below are in disagreement with those quoted in [17].

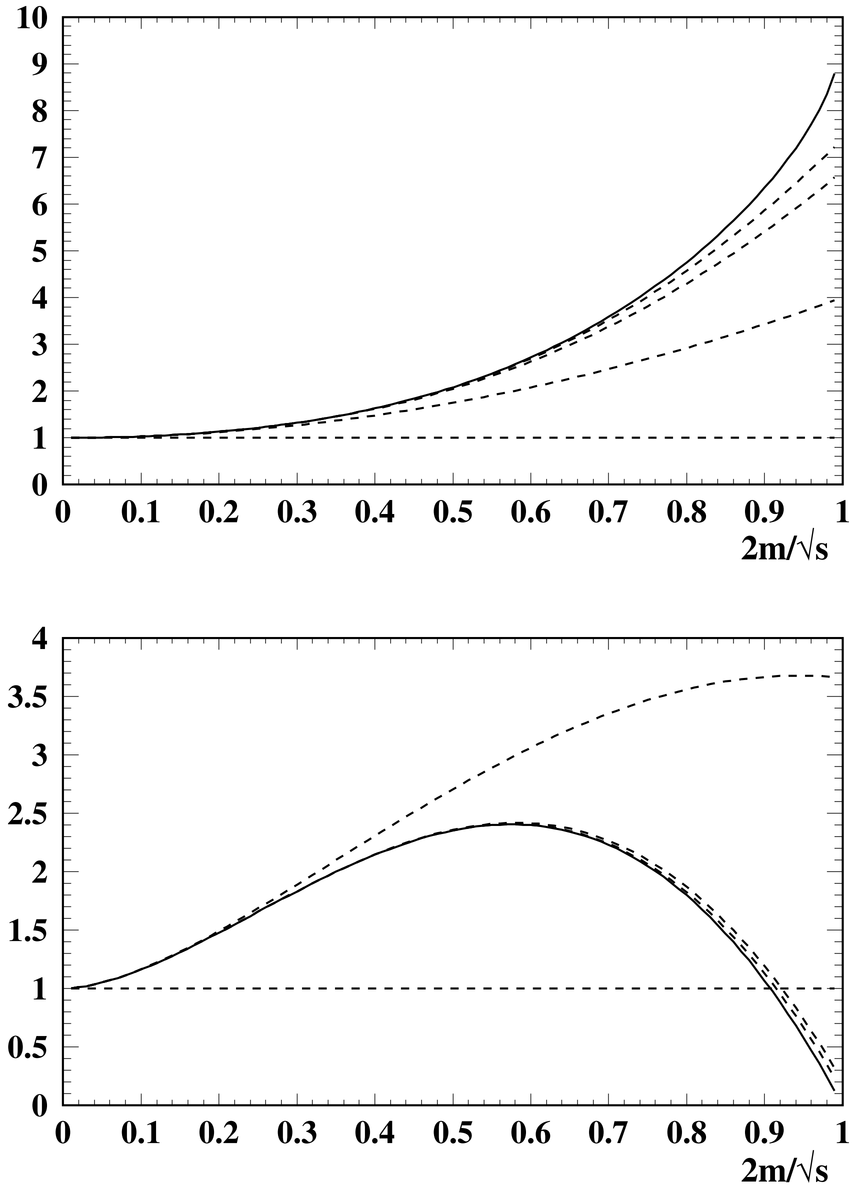

Figure 1: Comparison between the complete $\mathcal{O}(\alpha_s)$ correction function (solid line) and approximations of increasing order (dashed lines) in m^2 for vector (upper Figure) and axial vector current (lower Figure) induced rates.

The result may be immediately transformed to the case of the electromagnetic current of a heavy, say, t (or b) quark.

The asymptotic behaviour of this (operator valued) function for $Q^2 = -q^2 \to \infty$ is given by an OPE of the form (Different powers of Q^2 may be studied separately. Only terms proportional to Lorentz scalar operators of dimension four are kept to derive the m_q^4 corrections as eventually we are interested in the function (6) sandwiched between the vacuum states.)

$$T_{\mu\nu} \underset{q^2\to\infty}{=} \frac{1}{Q^4}\sum_n \left\{ (q_\mu q_\nu - g_{\mu\nu}q^2) \overset{T}{C}_n (Q^2,\mu^2,\alpha_s) + q_\mu q_\nu \overset{L}{C}_n (Q^2,\mu^2,\alpha_s) \right\} O_n + \dots \tag{7}$$

As demonstrated in [20, 9] the following operators of dimension four may in general appear at the rhs. of (7),

$$\begin{aligned}
&O_1 = G^2_{\mu\nu},\ O_2^{ij} = m_i \overline{q}_j q_j,\ O_3^i = \overline{q}_i (i \hat{\overleftrightarrow{D}} /2 - m_i) q_i, \\
&O_4 = A^a_\nu \left(\nabla^{ab}_\mu G^b_{\mu\nu} + g \sum_i \overline{q}_i \frac{\lambda^a}{2} \gamma_\nu q_i \right) - \partial_\mu \overline{c}^a \partial_\mu c^a, \\
&O_5 = (\nabla^{ab}_\mu \partial_\mu \overline{c}^b) c^a, \\
&O_6^{ij} = m_i^2 m_j^2, \\
&O_6^i = m_u m_d m_i^2.
\end{aligned} \tag{8}$$

The (gauge non-invariant) operators O_4 and O_5 are non-physical; they appear since the gauge invariance of the Lagrangian is broken through gauge-fixing. Matrix elements of these operators vanish for physical and thus by definition gauge invariant states. Hence they do not contribute in the calculation below. They must, however, be taken into account in the calculation of the coefficient functions (CF's) multiplying the physical operators. Once these CF's are known, one may freely ignore the non-physical operators.

In [9] the following results were obtained for the (transversal) coefficient functions:

$$\overset{T}{C}_1 = \frac{\alpha_s}{12\pi}(1 + \frac{7}{6}\frac{\alpha_s}{\pi}), \tag{9}$$

$$\begin{aligned}
&\sum \overset{T}{C}{}_2^k\, O_2^k = \\
&-\frac{\alpha_s}{\pi}\left(1 + \frac{\alpha_s}{\pi}\left[-\frac{1}{6}fL - \frac{1}{6}f + \frac{11}{4}L + \frac{29}{6}\right]\right)(m_u \overline{u}u + m_d \overline{d}d) \\
&\pm \left(1 + \frac{4\alpha_s}{3\pi}\left[1 + \frac{3\alpha_s}{4\pi}\left(-\frac{2}{9}fL - \frac{7}{27}f + \frac{11}{3}L + \frac{191}{18}\right)\right]\right)(m_u \overline{d}d + m_d \overline{u}u) \\
&+(\frac{\alpha_s}{\pi})^2\left(\frac{4}{3}\zeta(3) + \frac{L}{3} - 1\right)\sum_i m_i \overline{q}_i q_i,
\end{aligned} \tag{10}$$

$$\sum \overset{T}{C}_6\, O_6 =$$
$$\frac{3}{16\pi^2}\left\{\frac{\alpha_s}{\pi}\left(\frac{152}{9}-\frac{32}{3}\zeta(3)\right)m_u^2 m_d^2-\left(2+\frac{\alpha_s}{\pi}(4+4L)\right)(m_u^4+m_d^4)\right. \tag{11}$$
$$\left.\mp\left[4L+\frac{\alpha_s}{\pi}\left(\frac{56}{3}-16\zeta(3)+\frac{32}{3}L+8L^2\right)\right](m_u^3 m_d+m_d^3 m_u)\right\},$$

where $L = \ln \mu^2/Q^2$, $\zeta(3) = 1.202\ldots$, f stands for the number of flavours; the upper (lower) sign refers to $\overline{u}\gamma_\mu d$ ($\overline{u}\gamma_\mu\gamma_5 d$).

From these results one may find for example the CF's for $T_{\mu\nu}(q, J)$ with $J_\mu = \overline{b}\gamma_\mu b$ or $\overline{b}\gamma_\mu\gamma_5 b$ through the substitutions $d \to b,\ \ u \to b$ and $m_d \to m_b,\ \ m_u \to m_b$.

The starting point of the calculation was the nondiagonal $\bar{u}d$ current. Singlet contributions are therefore absent by construction. These would contribute in order α_s^2 to the axial part of $\Gamma(Z \to \bar{b}b)$ and will not be considered in this work. Their relative importance will be discussed in section 4.

It should be stressed that the generic OPE (7) determines the asymptotic behaviour of any Green function with an insertion of the T-product $T_{\mu\nu}(q, J)$, in particular of the correlator

$$\Pi_{\mu\nu}(q) = \langle T_{\mu\nu}(q, J)\rangle. \tag{12}$$

In particular, as demonstrated in [12] the expansion (7) bears all the information about the large Q behaviour of the correlator (12) in perturbation theory, provided the Vacuum Expectation Values (VEV's) of local operators are evaluated and renormalized according to the standard minimal subtraction prescription [21] or its modification — the $\overline{MS}$ scheme — suggested in [22]. (No normal ordering of the operator products!) The above results allow to find the absorptive part of the polarization operator $\Pi(Q^2)$ defined in the standard manner

$$\Pi_{\mu\nu}(q) = -g_{\mu\nu}q^2\Pi(Q^2) + \ldots.$$

once the $\overline{MS}$ renormalized values for the vacuum expectations values $\langle G^2\rangle$ and $\langle\overline{q}q\rangle$ are known.

However, this nice feature of the minimal subtraction scheme has its price: In schemes without normal ordering the renormalization properties of composite operators are more involved. For instance, the operator $m_i\overline{\psi}_j\psi_j$ mixes under renormalization group transformations with "operators" proportional to the unit "operator" times polynomials of fourth order in the quark masses. For our aims it will be necessary to know the anomalous dimension matrix of all physical operators in the list (8). It has been shown in [20, 12] (for earlier related works see [23, 24, 25, 26]) that this matrix can be expressed, to all orders in perturbation theory, in terms of the QCD β-function, the quark mass anomalous dimension $\gamma_m(\alpha_s)$ and the so-called vacuum anomalous dimension γ_0. The explicit expression for the anomalous dimension matrix will be presented below.

In the calculation of $R(s)$ only the logarithmic parts of the CF's need to be taken into account (the constant terms have no absorptive parts at $s > 0$!). Three remarkable observations can be made:

- In first order of α_s only trivial operators (proportional to the unit operator times some combination of quark masses) contribute to $R(s)$ since only these depend on L. Hence $R(s)$ may be found directly from the coefficient function (11). This fact directly leads to the absence of the mass logarithms in $R(s)$ in order α_s as the latter may appear only from VEV's of non-trivial operators. This is in agreement with the expansion of the exact result discussed in section 2. Furthermore, the argument does not apply to terms of order $\alpha_s m^6$, again in agreement with the expansion calculated in section 2.
- To find the correction of order $\alpha_s^2 m_q^4/s^2$ to $R(s)$ there is no need to compute three-loop contributions of order α_s^2 to the VEV's $\langle G^2 \rangle$ and $\langle \overline{q}q \rangle$. Only already known two-loop terms of order α_s (see below) are required.
- The calculations of [9] were performed at the two-loop level and thus their results as expressed by (9)-(11) do not include terms of order α_s^2 for the CF's $\mathbf{C}_6^n$. Fortunately, the terms proportional to L may be inferred from the renormalization group invariance in analogy to [3].

2. Renormalization Group (RG) Functions

The following notation will be employed:

- The effective *couplant* is defined as $a(\mu) \equiv \frac{\alpha_s(\mu)}{\pi}$ and the number of flavours is denoted by f.
- The β-function and the anomalous dimension of the mass γ_m are defined through

$$\mu^2 \frac{d}{d\mu^2} \left(a(\mu)\right) = a\beta(a) \equiv -a \sum_{i \geq 0} \beta_i \left(a\right)^{i+1}, \tag{13}$$

$$\mu^2 \frac{d}{d\mu^2} m(\mu) = m(\mu)\gamma_m(\alpha_s) \equiv -m \sum_{i \geq 0} \gamma_m^i \left(a\right)^{i+1}. \tag{14}$$

with the expansion coefficients up to the three loop order [10, 11]

$$\begin{gathered} \beta_0 = \left(11 - \frac{2}{3} f\right)/4, \quad \beta_1 = \left(102 - \frac{38}{3} f\right)/16, \\ \beta_2 = \left(\frac{2857}{2} - \frac{5033}{18} f + \frac{325}{54} f^2\right)/64, \end{gathered} \tag{15}$$

$$\gamma_m^0 = 1, \quad \gamma_m^1 = \left(\frac{202}{3} - \frac{20}{9}f\right)/16,$$
$$\gamma_m^2 = \left(1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] f - \frac{140}{81}f^2\right)/64. \tag{16}$$

- The anomalous dimension of the vacuum energy is[4]

$$\gamma_0(a,m) = \gamma_0^d(a)\sum_i m_i^4 + \gamma_0^{nd}(a)\sum_{i\neq j} m_i^2 m_j^2, \tag{17}$$

with

$$\gamma_0^d(a) = -\frac{3}{16\pi^2}\left[1 + \frac{4}{3}a + \left(\frac{313}{72} - \frac{5}{12}f - \frac{2}{3}\zeta(3)\right)a^2\right],$$
$$\gamma_0^{nd}(a) = \frac{3}{8\pi^2}a^2. \tag{18}$$

3. Operator mixing

It has been found in [12] that the anomalous dimensions of the (physical) operators of dimension four contributing to (7) may be presented as follows[5]:

$$\mu^2\frac{d}{d\mu^2}O_1 = -\left(a\frac{\partial}{\partial a}\beta\right)O_1 + 4\left(a\frac{\partial}{\partial a}\gamma_m\right)\sum_i O_2^{ii} + 4a\frac{\partial}{\partial a}\gamma_0,$$
$$\mu^2\frac{d}{d\mu^2}O_2^{ij} = -m_i\frac{\partial}{\partial m_j}\gamma_0, \tag{19}$$
$$\mu^2\frac{d}{d\mu^2}O_6 = 4\gamma_m O_6.$$

where the last equation is of course a direct consequence of the definition of the quark mass anomalous dimension.

4. RG constraints on the coefficient functions

In our particular case of the polarization operator $\Pi(Q^2)$ the OPE (7) assumes the form

$$\Pi(Q^2) \underset{q^2\to\infty}{=\!=\!=} O(1) + O(m^2/Q^2)$$
$$+\frac{\mathbf{C}_1 O_1}{Q^4} + \sum_i \frac{\mathbf{C}_2^{i,i}O_2^{ii}}{Q^4} + \frac{\mathbf{C}_2^{ud}O_2^{ud}}{Q^4} + \frac{\mathbf{C}_2^{du}O_2^{du}}{Q^4} + \frac{\mathbf{C}_6(\alpha_s, L, m_i)}{Q^4}, \tag{20}$$

---

[4]The leading term and the first correction in the expression for γ_0^d were calculated in [12]. Recently one of us [27] extended the calculation to include the correction of order α_s^2.

[5]The differences between (19) and eq.(3.4) of [12] originate from a different overall sign in the definition of γ_{ij} and the different normalization of the operator O_1.

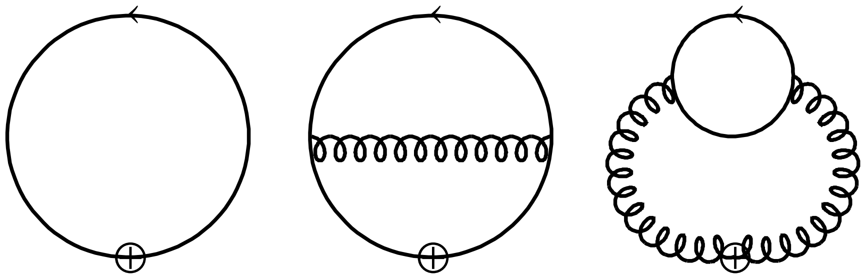

Figure 2: Diagrams giving rise to nonzero vacuum expectation values of the composite operators $\overline{\psi}\psi$ and $G^2_{\mu\nu}$ in the lowest orders of perturbation theory.

where

$$\mathbf{C}_6(\alpha_s, L, m_i) = \sum_{ij} \mathbf{C}_6^{ij} O_6^{ij} + \sum_i \mathbf{C}_6^i O_6^i. \tag{21}$$

The condition of the RG invariance implies

$$\mu^2 \frac{d}{d\mu^2} \sum_n \mathbf{C}_n O_n = 0, \tag{22}$$

or, explicitly,

$$\begin{gathered} \frac{\partial Cbf_6}{\partial L} = -4\gamma_m \mathbf{C}_6 - (a\beta)\frac{\partial \mathbf{C}_6}{\partial a} \\ -\mathbf{C}_1 4a \frac{\partial \gamma_0}{\partial a} + \sum_i \mathbf{C}_2^{ii} m_i \frac{\partial}{\partial m_i}\gamma_0 + \mathbf{C}_2^{ud} m_u \frac{\partial}{\partial m_d}\gamma_0 + \mathbf{C}_2^{du} m_d \frac{\partial}{\partial m_u}\gamma_0. \end{gathered} \tag{23}$$

The structure of the last equation allows obviously to find the logarithmic terms of order α_s^2 in $\mathbf{C}_6$ in terms of the function $\mathbf{C}_6$ and $\mathbf{C}_1$ which are to be known *completely* up to and including the order α_s contributions and the coefficient functions $\mathbf{C}_2^n$ which should be known in the order α_s^2. An inspection of (9)-(11) immediately reveals that we indeed know these coefficient functions with the accuracy required.

5. Calculation of VEV's of O_1 and O_2

Diagrams contributing to VEV's of O_1 and O_2 in order α_s are shown in Fig.2. The VEV of O_2 was computed in [28, 12] and reads

$$< O_2^{ij} >= \frac{3m_i m_j^3}{4\pi^2}\left[1 + \ln(\frac{\mu^2}{m_j^2}) + 2a(\mu)\left(\ln^2(\frac{\mu^2}{m_j^2}) + \frac{5}{3}\ln(\frac{\mu^2}{m_j^2}) + \frac{5}{3}\right)\right]. \tag{24}$$

The perturbative "gluon condensate" was computed in [29] (eq. (B.1a)):

$$< O_1 >= -\frac{a(\mu)}{2\pi^2}\sum_i \left[9 + 8\ln(\frac{\mu^2}{m_i^2}) + 3\ln^2(\frac{\mu^2}{m_i^2})\right] m_i^4. \tag{25}$$

We have confirmed the correctness of the result (25) by an independent calculation.

6. Calculation of the $\alpha_s^2 m^4$ corrections

Now we are in a position to find the corrections of order $\alpha_s^2 m^4/s^2$ to the both spectral functions R_V and R_A. Indeed, simply integrating the rhs of (23) one gets the missing logarithmic terms of order α_s^2 in $\mathbf{C}_6$:

$$
\begin{aligned}
&\sum \overset{T}{C_6}\, O_6 = \\
&\frac{3}{16\pi^2}\left\{\left[a\left(\frac{152}{9}-\frac{32}{3}\zeta(3)\right)+a^2L\left(114-72\zeta(3)+\frac{16}{9}f\zeta(3)-\frac{76}{27}f\right)\right]m_u^2m_d^2\right.\\
&-\left[2+a(4+4L)+a^2L\left(36+8L-\frac{10}{9}f\right)\right](m_u^4+m_d^4)\\
&\mp\left[4L+a\left(\frac{56}{3}-16\zeta(3)+\frac{32}{3}L+8L^2\right)\right.\\
&\left.-a^2L\left(\frac{4}{9}fL^2+\frac{22}{9}fL-\frac{8}{3}f\zeta(3)+\frac{157}{27}f-18L^2-77L+\frac{332}{3}\zeta(3)-\frac{3617}{18}\right)\right]\times\\
&(m_u^3m_d+m_d^3m_u)\\
&\left.-a^2L\left(\frac{2}{3}L+\frac{16}{3}\zeta(3)-\frac{40}{9}\right)\sum_i m_i^4 \pm a^2 16 L m_u m_d \sum_i m_i^2\right\}.
\end{aligned}
\tag{26}
$$

Note that the term proportional to $\sum_i m_i^4$ in the above result was independently found via a direct calculation in [30].

The next step is to use eqs. (24,25,26) to find the corrections of order $\alpha_s^2 m_q^4/s^2$ to the spectral densities of the vector and axial vector correlators. The results read (below we set for brevity the $\overline{\mathrm{MS}}$ normalization scale $\mu=\sqrt{s}$ and $\overline{m}_u(s)=\overline{m}_d(s)=\overline{m}$)

$$
\begin{aligned}
R_V &= 1+O(\overline{m}^2/s)-6\frac{\overline{m}^4}{s^2}\left(1+\frac{11}{3}a\right)\\
&+a^2\frac{\overline{m}^4}{s^2}\left[f\left(\frac{1}{3}\log\left(\frac{\overline{m}^2}{s}\right)-\frac{2}{3}\pi^2-\frac{8}{3}\zeta(3)+\frac{143}{18}\right)\right.\\
&-\frac{11}{2}\log\left(\frac{\overline{m}^2}{s}\right)+27\pi^2+112\zeta(3)-\frac{3173}{12}+12\sum_i\frac{\overline{m}_i^2}{\overline{m}^2}\\
&\left.+\left(\frac{13}{3}-4\zeta(3)\right)\sum_i\frac{\overline{m}_i^4}{\overline{m}^4}-\sum_i\frac{\overline{m}_i^4}{\overline{m}^4}\log\left(\frac{\overline{m}_i^2}{s}\right)\right],
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
R_A &= 1+O(\overline{m}^2/s)+6\frac{\overline{m}^4}{s^2}\left(1+\frac{5}{3}a\right)\\
&+a^2\frac{\overline{m}^4}{s^2}\left[f\left(-\frac{7}{3}\log\left(\frac{\overline{m}^2}{s}\right)+\frac{2}{3}\pi^2+\frac{16}{3}\zeta(3)-\frac{41}{6}\right)\right.
\end{aligned}
$$

$$+\frac{77}{2}\log\left(\frac{\overline{m}^2}{s}\right) - 27\pi^2 - 220\zeta(3) + \frac{3533}{12} - 12\sum_i \frac{\overline{m}_i^2}{\overline{m}^2}$$
$$+\left(\frac{13}{3} - 4\zeta(3)\right)\sum_i \frac{\overline{m}_i^4}{\overline{m}^4} - \sum_i \frac{\overline{m}_i^4}{\overline{m}^4}\log\left(\frac{\overline{m}_i{}^2}{s}\right)\Bigg]. \tag{28}$$

Note that the sum over i includes also the quark coupled to the external current and with mass denoted by m. Hence in the case with one heavy quark u of mass m $(d \equiv u)$ one should set $\sum_i \frac{\overline{m}_i^4}{\overline{m}^4} = 1$ and $\sum_i \frac{\overline{m}_i^2}{\overline{m}^2} = 1$. In the opposite case when one considers the correlator of light (massless) quarks the heavy quark appears only through its coupling to gluons. There one finds:

$$R_V = R_A = 1 + O(\overline{m}^2/s) + a^2\frac{\overline{m}^4}{s^2}\left[\frac{13}{3} - \log\left(\frac{\overline{m}^2}{s}\right) - 4\zeta(3)\right]. \tag{29}$$

Numerically, eqs. (27,28) look as follows

$$\begin{aligned} R_V &= 1 + O(\overline{m}^2/s) - 6\frac{\overline{m}^4}{s^2}\left(1 + \frac{11}{3}a\right) \\ &\quad + a^2\frac{\overline{m}^4}{s^2}\Bigg[f\left(\frac{1}{3}\log\left(\frac{\overline{m}^2}{s}\right) - 1.841\right) \\ &\quad - \frac{11}{2}\log\left(\frac{\overline{m}^2}{s}\right) + 136.693 + 12\sum_i \frac{\overline{m}_i^2}{\overline{m}^2} \\ &\quad - 0.475\sum_i \frac{\overline{m}_i^4}{\overline{m}^4} - \sum_i \frac{\overline{m}_i^4}{\overline{m}^4}\log\left(\frac{\overline{m}_i{}^2}{s}\right)\Bigg], \end{aligned} \tag{30}$$

$$\begin{aligned} R_A &= 1 + O(\overline{m}^2/s) + 6\frac{\overline{m}^4}{s^2}\left(1 + \frac{5}{3}a\right) \\ &\quad + a^2\frac{\overline{m}^4}{s^2}\Bigg[f\left(-\frac{7}{3}\log\left(\frac{\overline{m}^2}{s}\right) + 6.157\right) \\ &\quad + \frac{77}{2}\log\left(\frac{\overline{m}^2}{s}\right) - 236.515 - 12\sum_i \frac{\overline{m}_i^2}{\overline{m}^2} \\ &\quad - 0.475\sum_i \frac{\overline{m}_i^4}{\overline{m}^4} - \sum_i \frac{\overline{m}_i^4}{\overline{m}^4}\log\left(\frac{\overline{m}_i{}^2}{s}\right)\Bigg]. \end{aligned} \tag{31}$$

4 Discussion

The Z decay rate is hardly affected by the m^4 contributions. The lowest order term evaluated with $\overline{m} = 2.6$ GeV leads to a relative suppression (enhancement) of about 5×10^{-6} for the vector (axial vector) current induced $Z \to b\bar{b}$ rate. Terms of increasing order in α_s become successively smaller. It is worth noting, however, that the corresponding series, evaluated in the onshell scheme, leads to terms which are larger by about one order of magnitude and of oscillatory signs. The m_b^4 correction to $\Gamma(Z \to q\bar{q})$ which starts in order α_s^2 is evidently even smaller. From these considerations it is clear that m^4 corrections to the Z decay rate are well under control — despite the still missing singlet piece — and that they can be neglected for all practical purposes.

The situation is different in the low energy region, say several GeV above the charm or the bottom threshold. For definiteness the second case will be considered and for simplicity all other masses will be put to zero[6]. The contributions to R^V from m^4 terms are presented in Fig.3 as functions of $2m/\sqrt{s}$ in the range from 0.05 to 1. As input parameters $m_{\mathrm{pole}} = 4.7$GeV and $\Lambda_{\overline{MS}} = 235$MeV, corresponding to $\alpha_s(m_Z^2) = .12$ have been chosen. Corrections of higher orders are added successively. The prediction is fairly stable with increasing order in α_s as a consequence of the fact that most large logarithms were absorbed in the running mass. The relative magnitude of the sequence of terms from the m^2 expansion is displayed in Fig.4. The curves for m^0 and m^2 are based on corrections up to third order in α_s with the m^2 term starting at first order. The m^4 curve receives corrections from order zero to two.

Of course, very close to threshold, say above 0.75 (corresponding to $\sqrt{s}$ below 13 GeV) the approximation is expected to break down, as indicated already in Fig.1. Below the $b\bar{b}$ threshold, however, one may decouple the bottom quark and consider mass corrections from the charmed quark within the same formalism.

Also $R_{q\bar{q}}$ where q denotes a massless quark coupled to the external current is affected by virtual or real heavy quark radiation. The m^2 terms have been calculated in [3] and start from order α_s^3:

$$\delta R = -\left(\frac{\alpha_s}{\pi}\right)^3 \frac{4\overline{m}^2}{s}(15 - \frac{2}{3}f)(\frac{4}{3} - \zeta(3)) \tag{32}$$

The terms of order $\alpha_s^2 m^4$ were given above. Both lead to corrections of $\mathcal{O}(10^{-4})$, (evaluated at an energy $\sqrt{s}$ of 10 GeV) and can be neglected for all practical purposes.

Summary: QCD corrections to the vector and axial vector current induced rates and cross sections of order $\alpha_s m^4$ have been calculated. They have little effect on the Z decay rate. However, they are important for the analysis of hadron production in the low energy region.

Acknowledgement We would like to thank A. Kwiatkowski and T. Teubner for help in preparing the figures.

[6]A more detailed analysis of the phenomenological implications of these results for the low energy region will be presented in [19].

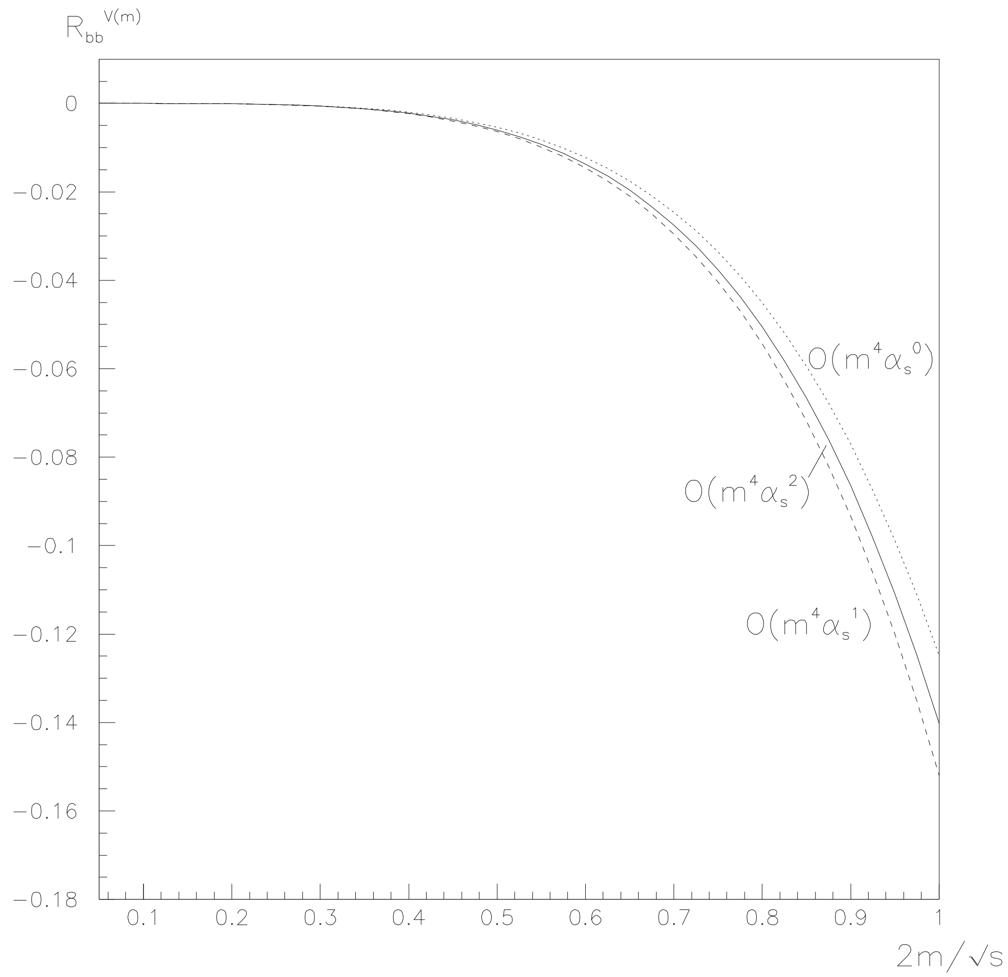

Figure 3: Contributions to R^V from m^4 terms including successively higher orders in α_s (order α_s^0/ α_s^1/ α_s^2 corresponding to dotted/ dashed/ solid lines) as functions of $2m_{\rm pole}/\sqrt{s}$.

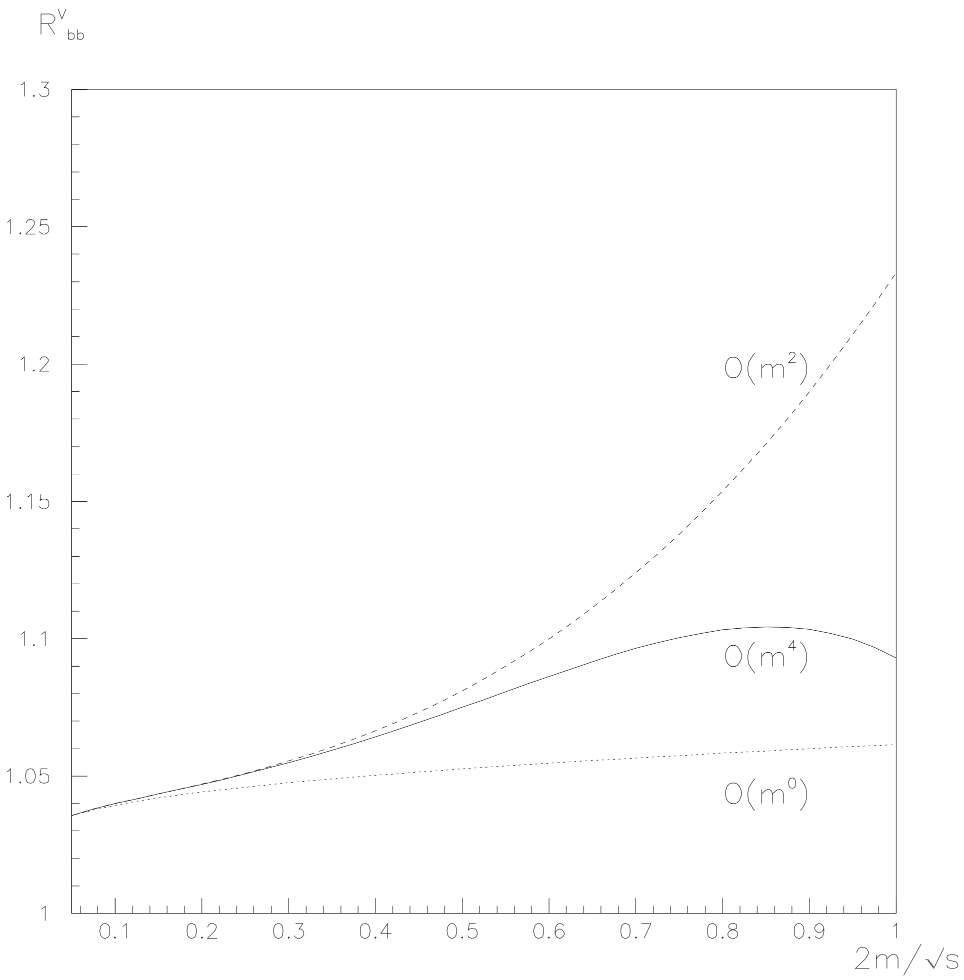

Figure 4: Predictions for R^V including successively higher orders in m^2.

References

[1] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Phys. Lett. B 85 (1979) 277;
M. Dine, J. Sapirstein, Phys. Rev. Lett. 43 (1979) 668;
W. Celmaster, R.J. Gonsalves, Phys. Rev. Lett. 44 (1980) 560.

[2] S.G. Gorishny, A.L. Kataev, S.A. Larin, Phys. Lett. 259 (1991) 144;
L.R. Surguladze, M.A. Samuel, Phys. Rev. Lett. 66 (1991) 560; erratum ibid, 2416.

[3] K.G. Chetyrkin, J.H. Kühn, Phys. Lett. B 248 (1990) 359.

[4] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Lett. B 282 (1992) 221.

[5] B.A. Kniehl, J.H. Kühn, Phys. Lett. B 224 (1990) 229;
Nucl. Phys. B 329 (1990) 547.

[6] K.G. Chetyrkin and J.H. Kühn, Phys. Lett. B308 (1993) 127.

[7] K.G. Chetyrkin and O.V. Tarasov, Phys. Lett. B 327 (1994) 114;
S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Phys. Lett. B 320 (1994) 159.

[8] K.G. Chetyrkin and A. Kwiatkowski, Phys. Lett. B 305 (1993) 285; Phys. Lett. B 319 (1993) 307.

[9] K.G. Chetyrkin, V.P. Spiridonov and S.G. Gorishny, Phys. Lett. 160B (1985) 149.

[10] O.V. Tarasov, A.A. Vladimirov, A.Yu. Zharkov, Phys. Lett. B 93 (1980) 429.

[11] O.V. Tarasov, preprint JINR P2-82-900 (1982).

[12] K.Chetyrkin, V.P. Spiridonov, Sov. J. Nucl. Phys. 47 (1988) 522.

[13] J. Schwinger, Particles, Sources and Fields, Vol.II, (Addison-Wesley, New York, 1973);
J. Jersak, E. Laermann and P.M. Zerwas, Phys. Lett. 98B (1981) 363;
T.H. Chang, K.J.F. Gaemers and W.L. van Neerven, Nucl. Phys. B202 (1980) 407.

[14] L.J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rep. 127C (1985) 1.

[15] R. Tarrach, Nucl. Phys. B183 (1981) 384.

[16] S. Narison, Phys. Lett. 197B (1987) 405.

[17] S. Narison, Phys. Lett. 260B (1991) 411.

[18] D.J. Broadhurst and S.C. Generalis, Open University preprint OUT-4102-12 (1984).

[19] K.G. Chetyrkin, J.H. Kühn, in preparation.

[20] V.P. Spiridonov, preprint INR P-0378 (1984).

[21] G. t' Hooft, Nucl. Phys. B 61 (1973) 455.

[22] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18 (1978) 3998.

[23] H. Kluber-Stern and J.B. Zuber, Phys. Rev. D18 (1975) 467.

[24] N.K. Nielsen, Nucl. Phys. B97 (1975) 527.

[25] N.K. Nielsen, Nucl. Phys. B120 (1977) 212.

[26] R. Tarrach, Nucl. Phys. B196 (1982) 45.

[27] K. Chetyrkin, in preparation.

[28] D.J. Broadhurst, Phys. Lett. 101B (1981) 423-426.

[29] E. Braaten, S. Narison and A. Pich, Nucl. Phys. B 373 (1992) 581.

[30] S.G. Gorishny and S.P. Spriridonov, unpublished notes (1985).