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SHORT AND LONG DISTANCE EFFECTS IN THE DECAY $\tau \to \pi\nu_\tau(\gamma)$

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Abstract

We calculate the radiative corrections to the decays $\tau \to M\nu_\tau$ and $\pi \to l\nu_l$, where the meson M is $M = \pi$ or K and the lepton l is $l = e$ or μ. We perform a complete calculation, which includes internal bremsstrahlung and structure dependent radiation in the radiative decays and point meson, hadronic structure dependent and short distance contributions in the virtual corrections. Our result for the radiative correction to the ratio $\Gamma(\tau \to \pi\nu_\tau(\gamma))/\Gamma(\pi \to \mu\nu_\mu(\gamma))$ is $\delta R_{\tau/\pi} = \left(0.16^{+0.09}_{-0.14}\right)\%$. For the ratio $\Gamma(\tau \to K\nu_\tau(\gamma))/\Gamma(K \to \mu\nu_\mu(\gamma))$, we obtain $\delta R_{\tau/K} = \left(0.90^{+0.17}_{-0.26}\right)\%$. For completeness we have also calculated the ratio of the electronic and muonic decay modes of the pion.

1 Introduction

In the calculation of radiative corrections to semileptonic (semihadronic) decays such as $\tau \to \pi\nu_\tau$ and $\pi \to \mu\nu_\mu$, one faces three different problems [1, 2]. As usual in radiative corrections, there are divergences, viz. first the infra-red (IR) divergences and second the ultra-violet (UV) divergences. The third problem, however, which is the central issue of this paper, is the treatment of the strong interaction.

The IR divergences are removed as usual by considering either radiative decays with hard photons (eg. , $\Gamma(\tau \to \pi\nu_\tau\gamma)$ with $E_\gamma > E_0$) or inclusive rates for decays into final states with and without photon (eg. $\Gamma(\tau \to \pi\nu_\tau) + \Gamma(\tau \to \pi\nu_\tau\gamma)$).

The UV divergences are removed by renormalization. The decay rate of $\pi \to \mu\nu_\mu$ is proportional to the pion decay constant f_π. In principle f_π is determined by the parameters of the standard model, but since we are not able to solve the nonperturbative regime of QCD, f_π has to be considered as an additional free parameter which has to be extracted from experimental data and therefore has to be renormalized order by order in perturbation theory. On the other hand, ratios as

$$R_{\tau/\pi} := \frac{\Gamma(\tau \to \pi\nu_\tau(\gamma))}{\Gamma(\pi \to \mu\nu_\mu(\gamma))} \tag{1}$$

and

$$R_{e/\mu} := \frac{\Gamma(\pi \to e\nu_e(\gamma))}{\Gamma(\pi \to \mu\nu_\mu(\gamma))} \tag{2}$$

are independent of f_π and therefore can be predicted by theory. Technically the UV divergences cancel in these ratios.

For a systematic treatment of the issue of strong interaction, a separation into different energy regimes should be made. If all momenta are very small compared to a typical hadronic resonance scale such as m_ρ, the matrix elements are fixed by low energy theorems of QCD. In this low energy regime the pion behaves like a pointlike particle, and its interactions with the photon are determined by scalar QED. The high energy regime, on the other hand, is dominated by the short distance corrections to the weak interaction, ie. by photonic corrections acting at the quark level. These two regimes are not adjacent to each other, but rather there is an intermediate region which is dominated by non-perturbative strong interaction, viz. by the physics of hadron resonances such as the ρ and the a_1 particles.

In a previous paper [3], we have calculated the corrections to $R_{\tau/\pi}$ within a model with an effective point pion field. Defining the radiative correction $\delta R_{\tau/\pi}$ by

$$R_{\tau/\pi} = R^{(0)}_{\tau/\pi}\Big(1 + \delta R_{\tau/\pi}\Big) \tag{3}$$

where

$$R^{(0)}_{\tau/\pi} = \frac{1}{2}\frac{m_\tau^3}{m_\pi m_\mu^2}\frac{\left(1 - m_\pi^2/m_\tau^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \tag{4}$$

denotes the prediction to the order $O(\alpha^0)$, we found a radiative correction

$$\delta R^{(P.M.)}_{\tau/\pi} = +1.05\%. \tag{5}$$

(P. M. denotes point meson). We added to this result another $+0.17\%$ arising from hadronic structure dependent radiation. So what is missing in [3] are the hadronic structure dependent effects in the virtual corrections and the short distance corrections.

As for the short distance correction, it was shown in Ref. [4] that the leading $O(\alpha)$ correction in the limit of a large Z boson mass, $m_Z^2 \to \infty$, is

$$\mathcal{M}_0 \to \left[1 + \frac{\alpha}{\pi}\ln\frac{m_Z}{\mu}\right]\mathcal{M}_0 \tag{6}$$

where $\mathcal{M}_0$ denotes the Born amplitude, m_Z acts as a cut-off and μ is an unspecified mass scale characteristic of the process. The difficulty is of course to find identify the scale μ.

If the relevant scale is μ_1 for the decay $\tau \to \pi\nu_\tau(\gamma)$ and μ_2 for $\pi \to \mu\nu_\mu(\gamma)$, the short distance contribution to the radiative correction of $R_{\tau/\pi}$ is

$$\frac{2\alpha}{\pi}\left(\ln\frac{m_Z}{\mu_1} - \ln\frac{m_Z}{\mu_2}\right) = \frac{2\alpha}{\pi}\ln\frac{\mu_2}{\mu_1} \tag{7}$$

and so only the difference in the scales μ_2 and μ_1 is relevant for the correction to $R_{\tau/\pi}$. In Ref. [5] Marciano and Sirlin give an estimate for $\delta R_{\tau/\pi}$ which is based on the short distance contribution only,

$$R^{(s.d.)}_{\tau/\pi} = R^{(0)}_{\tau/\pi}\frac{1 + 2\alpha/\pi\ln(m_Z/m_\tau)}{1 + \frac{3}{2}(\alpha/\pi)\ln(m_Z/m_\pi) + \frac{1}{2}(\alpha/\pi)\ln(m_Z/m_\rho)} \tag{8}$$

(s. d. denotes short distance) or

$$\delta R^{(s.d.)}_{\tau/\pi} = \frac{2\alpha}{\pi}\ln\left(\frac{m_\pi^{3/4}m_\rho^{1/4}}{m_\tau}\right) = -0.98\% \tag{9}$$

This result differs from the effective point meson result. But note that in the case of the pion decay this estimate extends the short distance physics to a very small scale of $\mu_2 = m_\pi^{3/4}m_\rho^{1/4} = 214$ MeV.

In a subsequent paper [6], Marciano included both the point meson and the short distance corrections to $\pi \to l\nu_l(\gamma)$, matched at the scale m_ρ. For the τ decay $\tau \to \pi\nu_\tau(\gamma)$, still only the short distance corrections were included. Reexpressing his prediction in terms of $\delta R_{\tau/\pi}$, we obtain from [6, 7]

$$\delta R_{\tau/\pi} = (0.55 \pm 1.)\% \tag{10}$$

where the $\pm 1\%$ is the author's estimate of the missing long distance corrections of $O(\alpha)$ to $\tau \to \pi\nu_\tau(\gamma)$. In a recent papper [8], the authors have further improved the calculation of the radiative pion decay by including the leading hadronic structure dependent effects (both in the radiative decay and in the virtual corrections) and by including the leading two-loop effects. Using the pion decay constant f_π extracted from this calculation to predict the tau decay, they obtain a new prediction for the rate, which can be rewritten in terms of $R_{\tau/\pi}$ as

$$\delta R_{\tau/\pi} = (0.67 \pm 1.)\% \tag{11}$$

where again the $\pm 1\%$ estimates the long distance corrections to $\tau \to \pi\nu_\tau(\gamma)$, which are still missing.

Comparing the numbers in (5,9,10,11), it becomes clear that a complete and systematic calculation of the full $O(\alpha)$ corrections would be important in order to obtain a reliable prediction. This is what we will present in this paper.

We have performed a systematic calculation of the radiative corrections to $R_{\tau/\pi}$ which includes all relevant contributions. In the calculation of the loops, we separate the integration over the momentum of the virtual photon k into the long distance region with $0 \le |k^2| \lesssim \mu^2_{cut}$ and into the short distance region with $\mu^2_{cut} \lesssim |k^2| \le m_Z^2$. The matching scale μ_{cut}, which separates long and short wavelengths, should be of the order $\mu_{cut} \sim O(1\text{ GeV})$. The long distance part includes the effective point meson and the hadronic structure dependent corrections. The latter is obtained by modifying the scalar QED coupling of the photon to the pion by vector meson dominance, and by adding loops which are proportional to the form factors F_V and F_A which determine the hadronic structure dependent radiation.

For the short distance corrections we first consider the leading logarithm, and second we give a complete calculation based on the parton model.

Our paper is organized as follows: In Sec. 2 we briefly review the amplitudes for the radiative decays and the parametrizations of the two form factors appearing in the hadronic structure dependent radiation. In Sec. 3 we make some general remarks on our treatment of the virtual corrections, concerning the W boson propagator and the separation into long and short distances. In Sec. 4 we calculate the point meson contribution. We have published the results for the radiative corrections in a model with a pointlike pion in [3]. Nevertheless we repeat them here for completeness and in order to give some more details and intermediate results, which are needed later in the calculation in order to combine the point meson correction with the other contributions. In Sec. 5 we consider the leading logarithm of the short distance contribution, and in Sec. 6 we calculate the non-leading corrections. In Sec. 7 we calculate the hadronic structure dependent loops . Then in Sec. 8 we explain how the different contributions combine to give the final result, which we evaluate numerically in Sec. 9. A summary and concluding remarks are given in Sec. 10.

2 The Radiative Decays

We consider the decays $\tau \to M\nu_\tau(\gamma)$ and $M \to l\nu_l(\gamma)$, where the meson M is the pion $M = \pi$ or the kaon $M = K$, and the lepton l is the muon or the electron, $l = \mu$ or e. The Born amplitudes are given by

$$\begin{aligned}
\mathcal{M}_0(\tau(s) \to M(p)\nu_\tau(q)) &= -G_F V_M f_M [\bar{u}_\nu(q)\not{p}\gamma_- u_\tau(s)] \\
\mathcal{M}_0(M(p) \to l(s)\nu_l(q)) &= G_F V_M f_M [\bar{u}_\nu(q)\not{p}\gamma_- v_l(s)] \qquad (12)
\end{aligned}$$

where

$$V_\pi = \cos\theta_C \qquad\qquad V_K = \sin\theta_C \qquad (13)$$

The matrix elements of the radiative decays $\tau \to M\nu_\tau\gamma$ [9, 10] and $M \to l\nu_l\gamma$ [11] can be written as the sums of two contributions, the internal bremsstrahlung (IB) and the structure dependent radiation (SD)

$$\mathcal{M}[\tau^-(s) \longrightarrow \nu_\tau(q)M^-(p)\gamma(k)] = \mathcal{M}_{IB} + \mathcal{M}_{SD} \qquad (14)$$

where

$$\begin{aligned}
\mathcal{M}_{IB} &= -G_F V_M e f_M m_\tau \left[\bar{u}_\nu(q)\gamma_+ \left(\frac{p\cdot\epsilon}{p\cdot k} + \frac{\not{k}\not{\epsilon}}{2s\cdot k} - \frac{s\cdot\epsilon}{s\cdot k}\right) u_\tau(s)\right] \\
\mathcal{M}_{SD} &= -\frac{G_F V_M e}{\sqrt{2}}\left\{ i\epsilon_{\mu\nu\rho\sigma}\left[\bar{u}_\nu(q)\gamma^\mu\gamma_- u_\tau(s)\right]\epsilon^\nu k^\rho p^\sigma \frac{F_V^{(M)}((k+p)^2)}{m_M}\right.
\end{aligned}$$

$$+\left[\bar{u}_\nu(q)\gamma_+\Big((p\cdot k)\not\epsilon-(\epsilon\cdot p)\not k\Big)u_\tau(s)\right]\frac{F_A^{(M)}((k+p)^2)}{m_M}\Bigg\} \tag{15}$$

Similarly

$$\mathcal{M}[M^+(p)\longrightarrow l^+(s)\nu_l(q)\gamma(k)]=\mathcal{M}'_{IB}+\mathcal{M}'_{SD} \tag{16}$$

where

$$\begin{aligned}\mathcal{M}'_{IB} &= -G_F V_M e f_M m_l\left[\bar{u}_\nu(q)\gamma_+\left(\frac{p\cdot\epsilon}{p\cdot k}-\frac{\not k\not\epsilon}{2s\cdot k}-\frac{s\cdot\epsilon}{s\cdot k}\right)v_l(s)\right]\\ \mathcal{M}'_{SD} &= \frac{G_F V_M e}{\sqrt{2}}\Bigg\{i\epsilon_{\mu\nu\rho\sigma}\left[\bar{u}_\nu(q)\gamma^\mu\gamma_- v_l(s)\right]\epsilon^\nu k^\rho p^\sigma\frac{F_V^{(M)}((k-p)^2)}{m_M}\\ &\quad+\left[\bar{u}_\nu(q)\gamma_+\Big((p\cdot k)\not\epsilon-(\epsilon\cdot p)\not k\Big)v_l(s)\right]\frac{F_A^{(M)}((k-p)^2)}{m_M}\Bigg\}\end{aligned} \tag{17}$$

The IB part is fixed by the QCD low energy theorems. It is exactly identical to the one which would be obtained for an elementary pointlike pion field, with electromagnetic interactions determined by scalar QED. The IB amplitude does not contain any unknown parameters beyond the meson decay constant f_M. The SD contribution, on the other hand, involves the two form factors $F_V^{(M)}(t)$ and $F_A^{(M)}(t)$, which describe the effects of non-perturbative strong interactions. Crossing symmetry implies that both the meson and the tau decays are described by the same analytical functions of the momentum transfer t, the difference being that in the τ decay, t varies from m_M^2 up to m_τ^2, and in radiative meson decay, $t=0\ldots m_M^2$. In Ref. [10] we have parametrized these form factors.

For the pionic case $M=\pi$, we gave

$$F_A^{(\pi)}(t)=F_A^{(\pi)}(0)\mathrm{BW}_{a_1}(t) \tag{18}$$

and

$$F_V^{(\pi)}(t)=F_V^{(\pi)}(0)\left[\mathrm{BW}_\rho(t)+\sigma\mathrm{BW}_{\rho'}(t)+\rho\mathrm{BW}_{\rho''}(t)\right]\frac{1}{1+\sigma+\rho} \tag{19}$$

$F_A^{(\pi)}(0)$ has been measured in radiative pion decay [12],

$$F_A^{(\pi)}(0)=+0.0116\pm 0.0016. \tag{20}$$

whereas $F_V^{(\pi)}(0)$ is related to the axial anomaly and predicted to be

$$F_V^{(\pi)}(0)=\frac{m_\pi}{4\sqrt{2}\pi^2 f_\pi}=+0.0270 \tag{21}$$

Note that here we use a relative sign $s:=\mathrm{sign}(f_\pi F_V(0))$ which is positive in our conventions, $s=+1$. In fact in [10], we used $s=-1$. However, as we have explained in [13], we now believe that $s=+1$ is the physical choice, which we will therefore use throughout this paper.

The Breit-Wigner resonance factors $\mathrm{BW}_X(t)$ are normalized according to $\mathrm{BW}_X(0) = 1$ and involve energy dependent widths

$$\mathrm{BW}_X(t) = \frac{m_X^2}{m_X^2 - t - i m_X \Gamma_X(t)} \tag{22}$$

or constants widths

$$\mathrm{BW}_X(t) = \frac{m_x^2 - i m_X \Gamma_X}{m_X^2 - t - i m_X \Gamma_X} \tag{23}$$

For the a_1 resonance, an energy dependent width based on the three pion and the pion-rho phase space has been calculated in [14],

$$\Gamma_{a_1}(t) = \frac{g(t)}{g(m_{a_1}^2)} \Gamma_{a_1} \tag{24}$$

with

$$g(t) = \begin{cases} 0 & \text{if } t < 9m_\pi^2 \\ 4.1(t - 9m_\pi^2)^3 [1 - 3.3(t - 9m_\pi^2) + 5.8(t - 9m_\pi^2)^2] & \text{if } 9m_\pi^2 \le t < (m_\rho + m_\pi)^2 \\ t(1.623 + 10.38/t^2 - 9.32/t^4 + 0.65/t^6) & \text{else} \end{cases} \tag{25}$$

(all numbers in appropriate powers of GeV)

For the ρ, ρ' and ρ'' resonances, an energy dependent width

$$\Gamma_X(t) = \begin{cases} 0 & \text{if } t \le 4m_\pi^2 \\ \frac{m_X}{\sqrt{t}} \left(\frac{\sqrt{t - 4m_\pi^2}}{\sqrt{m_X^2 - 4m_\pi^2}} \right)^3 \Gamma_X & \text{else} \end{cases} \tag{26}$$

can be derived from the P-wave two body phase space.

We fixed the two parameters σ and ρ in [10] by employing four constraints, viz. the QCD theorem on $\lim_{t \to \infty} F_{\pi\gamma}(t)$ [15] (where $F_V^{(\pi)}(t) = -m_\pi F_{\pi\gamma}(t)/\sqrt{2}$ [16]), the measurement of the slope $F'_{\pi\gamma}(t = 0)$ and the widths $\Gamma_{\rho \to \pi\gamma}$ and $\Gamma_{\rho' \to \pi\omega}$. The result was

$$\begin{aligned} \sigma &= +0.136 \\ \rho &= -0.051 \end{aligned} \tag{27}$$

We also compared to a dipole parametrization ($\sigma = 0.0584$, $\rho = 0$) and to a monopole form ($\sigma = \rho = 0$).

For the kaonic form factors ($M = K$), consider the parametrizations

$$\begin{aligned} F_V^{(K)}(t) &= F_V^{(K)}(0) \left[\mathrm{BW}_{K^\star}(t) + \sigma_K \mathrm{BW}_{K^{\star\prime}}(t) + \rho_K \mathrm{BW}_{K^{\star\prime\prime}}(t) \right] \frac{1}{1 + \sigma_K + \rho_K} \\ F_A^{(K)}(t) &= F_A^{(K)}(0) \mathrm{BW}_{K_{1(1270)}}(t) \end{aligned} \tag{28}$$

(where $K^\star = K^\star(892)$, $K^{\star\prime} = K^\star(1410)$ and $K^{\star\prime\prime} = K^\star(1680)$). In [10] we used the monopole parametrization $\sigma_K = \rho_K = 0$ of the vector form factor and constant widths for both the $K^\star$ and the K_1.

Flavour symmetry implies the following relations for the form factors at $t = 0$:

$$
\begin{aligned}
F_A^{(K)}(0) &= \frac{m_K}{m_\pi} F_A^{(\pi)}(0) = +0.0410 \\
F_V^{(K)}(0) &= \frac{m_K}{m_\pi} F_V^{(\pi)}(0) = +0.0955
\end{aligned} \tag{29}
$$

The measurement of $K \to l\nu_l\gamma$, on the other hand [12], gives

$$
\begin{aligned}
F_V^{(K)} + F_A^{(K)} &= +0.148 \pm 0.010 \\
F_V^{(K)} - F_A^{(K)} &\in [-0.3, 2.2]
\end{aligned} \tag{30}
$$

Taking the vector form factor from flavour symmetry and the anomaly and calculating the axial one from the measured sum and the value for $F_V^{(K)}(0)$, this results in

$$
\begin{aligned}
F_A^{(K)}(0) &= \frac{m_K}{m_\pi} F_A^{(K)}(0) = +0.0525 \pm 0.010 \\
F_V^{(K)}(0) &= \frac{m_K}{m_\pi} F_V^{(K)}(0) = +0.0955
\end{aligned} \tag{31}
$$

These are the values we will actually use.

3 General Considerations on the Treatment of the Virtual Corrections

The momentum dependence of the W propagator

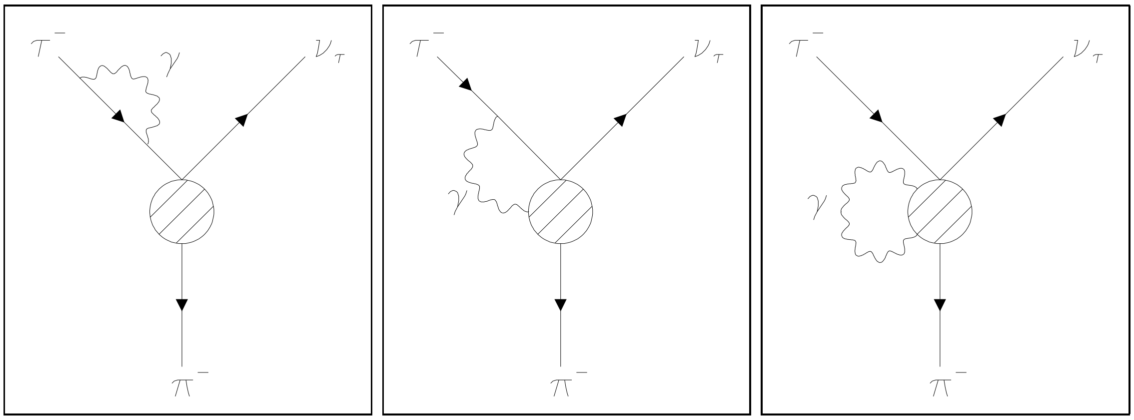
$$
\frac{1}{m_W^2} \frac{m_W^2}{m_W^2 - q^2} \tag{32}
$$

is determined by the familiar Feynman cut-off function $m_W^2/(m_W^2 - q^2)$. Thus we can as well use a local interaction with an UV cut-off equal to m_W in the calculation of the virtual corrections. In fact, it has been shown by Sirlin [18] that the radiative corrections of the order $G_F\alpha$ calculated within the full standard model with a single Higgs doublet are equal to the photonic corrections calculated with the local $V - A$ interaction and a cut-off equal to m_Z, except for very small contributions of the order $\alpha_s G_F \alpha$. While the photonic corrections are identical to those computed in the local $V - A$ theory with an effective cut-off equal to m_W, the non-photonic corrections lead to an additional contribution which depends on $\ln(m_Z/m_W)$ and θ_W. In the simplest electroweak model with a single Higgs doublet, where $\cos\theta_W = m_W/m_Z$, the photonic and the non-photonic corrections combine to give a result which is identical to the photonic correction obtained in the local theory with the cut-off equal to m_Z. As we will show, the actual value of the cut-off does not matter for our results on $R_{\tau/\pi}$, because it cancels in this ratio.

Thus we will use a local $V - A$ interaction and dimensional regularization, ie. the loop integrals are evaluated in $4 - \epsilon$ rather than in 4 space-time dimensions. We can translate our results into a momentum space cut-off by the replacement

$$
\Delta - \ln\frac{m^2}{\mu^2} \longrightarrow \ln\frac{m_Z^2}{m^2} \tag{33}
$$

Figure 1: The three different classes of radiative corrections



where

$$\Delta := \frac{2}{\epsilon} - \gamma_{Euler} + \ln 4\pi \tag{34}$$

The virtual corrections fall into three classes, see Fig. 1. The first class includes only a single diagram in which all couplings are known, and the third class is identical for $\tau \to \pi\nu_\tau$ and $\pi \to \mu\nu_\mu$, so these diagrams drop out in the ratio $R_{\tau/\pi}$. Therefore the most important class for the calculation of $R_{\tau/\pi}$ is the second one.

Now we want to separate the integration over the momentum of the virtual photon into two regions with small and large k_E^2, respectively [17]. We achieve this by splitting the photon propagator:

$$\frac{1}{k^2 - \lambda^2} = \underbrace{\frac{1}{k^2 - \lambda^2}\frac{\mu_{cut2}^2}{\mu_{cut}^2 - k^2}}_{\text{“long distance”}} + \underbrace{\frac{1}{k^2 - \mu_{cut}^2}}_{\text{“short distance”}} \tag{35}$$

Obviously the first part is important only for $|k^2| \lesssim \mu_{cut}^2$ and the second part only for $|k^2| \gtrsim \mu_{cut}^2$, so indeed they correspond to long and short distances, respectively. And so the photon propagator is divided into two parts, a regulated photon propagator with an effective cut-off μ_{cut} and a massive photon propagator with mass μ_{cut}. The scale μ_{cut} separates long and short wavelengths and should be of the order $\mu_{cut} \sim O(1\text{ GeV})$.

A photon with short wavelength ($|k^2| \gtrsim \mu_{cut}^2$) resolves the quarks in the pion. It interacts with the quarks which come into being in the initial process $\tau \to \nu_\tau \bar{u} d$, which subsesquently hadronize to form the pion (see Fig. 2). A photon with long wavelength ($|k^2| \lesssim \mu_{cut}^2$), on the other hand, has a small resolution and interacts with the pion as a whole or perhaps with some hadronic resonances (see Fig. 3). So according to Eqn. (35) we have to calculate the short distance diagrams such as in Fig. 2 with a massive photon propagator and the long distance diagrams such as in Fig. 3 with a regulated photon propagator.

In the next section we will start with the calculation of the corrections in a point pion model. This will give a first estimate, and it is part of the complete calculation.

Figure 2: Short distance corrections

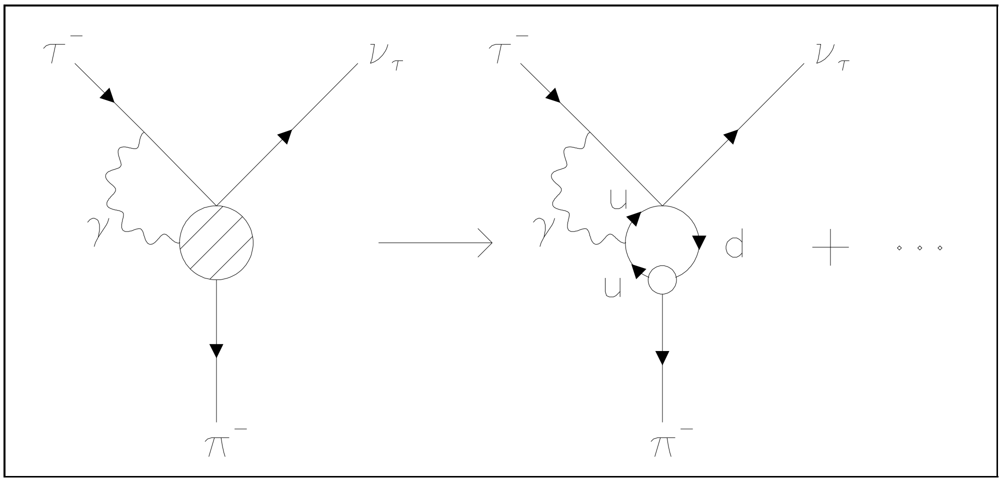
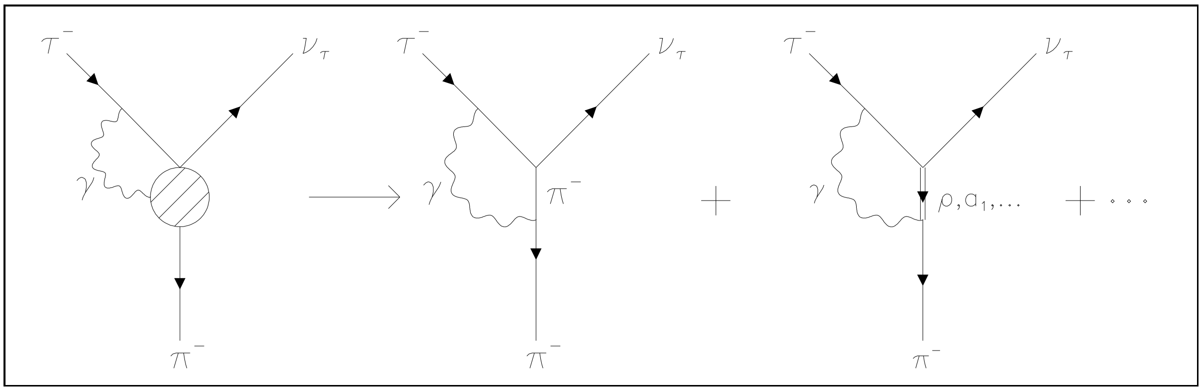


Figure 3: Long distance corrections



4 Point Meson Contribution

We will now calculate the corrections to $\tau \to \pi\nu_\tau$ and $\pi \to \mu\nu_\mu$ in a model with an effective point pion field, using a generalized photon propagator with a mass m^2

$$\frac{1}{k^2-\lambda^2} \to \frac{1}{k^2-m^2} \tag{36}$$

(λ denotes a small IR regulator mass, which in the end of the calculation is put equal to zero, whereas m denotes any finite mass.) It will become clear below why the consideration of such a generalized photon propagator is useful, see Eqns. (110)–(113).

For the case of the pion decay (and for $m^2 = \lambda^2$), these corrections have been calculated in [2]. Our calculation, however, differs in some technical details from that in [2]. First we use dimensional rather that cut-off regularization. Second Kinoshita replaces the vector-minus-axial vector current interaction

$$G_F \cos\theta_C f_\pi [\bar{\Psi}_l \gamma^\mu (1-\gamma_5)\Psi_\nu](i\partial_\mu - eA_\mu)\Phi_\pi \tag{37}$$

by the scalar-minus-pseudoscalar current interaction

$$G_F \cos\theta_C f_\pi m_l^{(0)} [\bar{\Psi}_l (1-\gamma_5)\Psi_\nu]\Phi_\pi \tag{38}$$

where $m_l^{(0)}$ denotes the bare lepton mass. We have performed the calculation both using Eqn. (37) and using Eqn. (38) and we have checked that they give identical results. Here we present the results using the $V-A$ form of Eqn. (37).

There are six Feynman diagrams for the virtual corrections to $\tau \to \pi\nu_\tau$, see Fig. 11, and of course there are six similar diagrams for $\pi \to \mu\nu_\mu$. The last diagram $\delta\mathcal{M}_T$ actually vanishes after pion mass renormalization.

The ratios of the virtual correction amplitudes over the Born amplitudes for both the tau and the meson decay can be expressed by the same functions of m_l^2 and m_M^2, if we use a general lepton mass m_l which denotes m_τ, m_μ or m_e, respectively. The result is

$$\begin{aligned}
\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &= \frac{\alpha}{4\pi}\left\{2B_0^M - B_1^M + 2m_l^2 C_0 - 2m_M^2 C_1\right\} \\
\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &= \frac{\alpha}{4\pi}\left\{1 - 4B_0^l - 2B_1^l\right\} \\
\frac{\delta\mathcal{M}_3}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &= \frac{\alpha}{4\pi}\left\{-B_0^M + B_1^M\right\} \\
\frac{\delta\mathcal{M}_4}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &= \frac{\alpha}{4\pi}\left\{\frac{1}{2}B_0^M - B_1^M + m_M^2[B_0'^M - B_1'^M]\right\} \\
\frac{\delta\mathcal{M}_5}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &= \frac{\alpha}{4\pi}\left\{\frac{1}{2} + B_1^l + m_l^2[4B_0'^l + 2B_1'^l]\right\}
\end{aligned} \tag{39}$$

where the B_i^l, B_i^π and C_i are given in terms of the standard n-point functions B and C:

$$\begin{aligned}
C_i &= C_i(m_M^2, m_l^2, 0, m_M, m, m_l) \\
B_i^l &= B_i(m_l^2, m_l, m) \\
B_i^M &= B_i(m_M^2, m_M, m)
\end{aligned} \tag{40}$$

(see App. 10 for our conventions). Thus

$$\begin{aligned}
\frac{\delta\mathcal{M}_{PML}}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) &:= \sum_{i=1}^{6}\frac{\delta\mathcal{M}_i}{\mathcal{M}_0} \\
&= \frac{\alpha}{4\pi}\left\{\frac{3}{2} + \frac{3}{2}B_0^M - B_1^M + m_M^2[B_0'^M - B_1'^M] - 4B_0^l - B_1^l\right. \\
&\qquad \left. + m_l^2[4B_0'^l + 2B_1'^l] + 2m_l^2 C_0 - 2m_M^2 C_1\right\}
\end{aligned} \tag{41}$$

("PML" denotes "point meson loops".) This is the form which will be used in the calculation of the complete radiative correction in Sec. 7.

To obtain the radiative correction within a model with a pointlike pion, we now assume $m^2 = \lambda^2$ and calculate the n-point functions. The result is

$$\begin{aligned}\frac{\delta\mathcal{M}_{PML}}{\mathcal{M}_0}(m_l^2, m_M^2, \lambda^2) &= \frac{\alpha}{4\pi}\Bigg\{-\frac{3}{2}\Delta + \frac{3}{2}\ln\frac{m_M^2}{\mu^2} - 4 - 4\left[\frac{1+r_l^2}{1-r_l^2}\ln r_l + 1\right]\ln\frac{\lambda}{m_M} \\ &\quad + 2\frac{1+r_l^2}{1-r_l^2}(\ln r_l)^2 + \left(5 - \frac{4r_l^2}{1-r_l^2}\right)\ln r_l\Bigg\} \qquad (42)\end{aligned}$$

where

$$r_l := \frac{m_l}{m_M} \qquad (43)$$

and μ is the mass scale of dimensional regularization.

By interference with the Born amplitude, the virtual correction of the decay rate is of course

$$\frac{\delta\Gamma_{PML}}{\Gamma_0} = 2\frac{\delta\mathcal{M}_{PML}}{\mathcal{M}_0} \qquad (44)$$

To these virtual corrections the integrated decay rate for internal bremsstrahlung must be added. We divide this rate into "soft" ($E_\gamma \leq E_0$) and "hard" ($E_\gamma \geq E_0$) corrections,

$$\frac{\delta\Gamma_{IB}(\tau\to M\nu_\tau\gamma)}{\Gamma_0(\tau\to M\nu_\tau)} = \frac{\delta\Gamma^\tau_{soft}}{\Gamma^\tau_0} + \frac{\delta\Gamma^\tau_{hard}}{\Gamma^\tau_0} \qquad (45)$$

where E_0 is assumed to be small, ie. $x_0 := (m_\tau/2)E_0 \ll 1$. For the τ decay the results are

$$\begin{aligned}\frac{\delta\Gamma^\tau_{soft}(x\leq x_0)}{\Gamma^\tau_0} &= \frac{\alpha}{2\pi}\Bigg\{4\left(\frac{1+r_M^2}{1-r_M^2}\ln r_M + 1\right)\left(\ln\frac{\lambda}{m_M} - \ln x_0\right) + 2\frac{1+r_M^2}{1-r_M^2}(\ln r_M)^2 \\ &\quad + \frac{2-6r_M^2}{1-r_M^2}\ln r_M + 2 - \frac{1}{3}\pi^2 + 2\frac{1+r_M^2}{1-r_M^2}\mathrm{Li}_2(r_M^2) - \frac{2}{3}\frac{r_M^2}{1-r_M^2}\pi^2 \\ &\quad + 4\frac{1+r_M^2}{1-r_M^2}\ln r_M\ln(1-r_M^2) + O(x_0)\Bigg\} \qquad (46)\end{aligned}$$

and

$$\begin{aligned}\frac{\delta\Gamma^\tau_{hard}(x\geq x_0)}{\Gamma^\tau_0} &= \frac{\alpha}{2\pi}\Bigg\{4\left(\frac{1+r_M^2}{1-r_M^2}\ln r_M + 1\right)\ln x_0 + \frac{25}{4} - \frac{1}{3}\pi^2 + \frac{4-2r_M^2+r_M^4}{(1-r_M^2)^2}\ln r_M \\ &\quad + \left(\frac{3}{2} - \frac{2}{3}\pi^2\right)\frac{r_M^2}{1-r_M^2} - 4\ln(1-r_M^2) + 2\frac{1+r_M^2}{1-r_M^2}\mathrm{Li}_2(r_M^2) \\ &\quad + O(x_0)\Bigg\} \qquad (47)\end{aligned}$$

where

$$r_M := \frac{m_M}{m_\tau} \qquad (48)$$

For the meson decay the sum of soft and hard bremsstrahlung is

$$\frac{\delta\Gamma_{IB}(M \to l\nu_l\gamma)}{\Gamma_0(M \to l\nu_l)} = \frac{\delta\Gamma^M_{soft}(x \le x_0)}{\Gamma^M_0} + \frac{\delta\Gamma^M_{hard}(x \ge x_0)}{\Gamma^M_0} \tag{49}$$

$$\begin{aligned} &= \frac{\alpha}{2\pi}\Bigg\{4\left(\frac{1+r_l^2}{1-r_l^2}\ln r_l + 1\right)\ln\frac{\lambda}{m_M} - 2\frac{1+r_l^2}{1-r_l^2}(\ln r_l)^2 \\ &\quad - 4\left(\frac{1+r_l^2}{1-r_l^2}\ln r_l + 1\right)\ln(1-r_l^2) + \frac{1-6r_l^2+2r_l^4}{(1-r_l^2)^2}\ln r_l \\ &\quad - 4\frac{1+r_l^2}{1-r_l^2}\mathrm{Li}_2(1-r_l^2) + \frac{27}{4} - \frac{3}{2}\frac{r_l^2}{1-r_l^2}\Bigg\} \end{aligned} \tag{50}$$

In these formulae, Li_2 denotes the dilogarithmic function

$$\mathrm{Li}_2(x) = -\int_0^x dt\,\frac{\ln(1-t)}{t} \tag{51}$$

Now we add up virtual, soft and hard photonic corrections as obtained in the point meson model. Writing the radiatively corrected rates as

$$\frac{\Gamma(\text{channel})}{\Gamma^0(\text{channel})} = 1 + \frac{\delta\Gamma}{\Gamma^0}\big(\text{channel}\big) \tag{52}$$

we obtain

$$\frac{\delta\Gamma}{\Gamma^0}\Big(M \to l\nu_l(\gamma)\Big) = \frac{\alpha}{2\pi}\Bigg\{-\frac{3}{2}\Delta + \frac{3}{2}\ln\frac{m_M^2}{\mu^2} + 6\ln r_l + \frac{11}{4} - \frac{2}{3}\pi^2 + f(r_l)\Bigg\} \tag{53}$$

and

$$\frac{\delta\Gamma}{\Gamma^0}\Big(\tau \to M\nu_\tau(\gamma)\Big) = \frac{\alpha}{2\pi}\Bigg\{-\frac{3}{2}\Delta + \frac{3}{2}\ln\frac{m_\tau^2}{\mu^2} + \frac{17}{4} - \frac{2}{3}\pi^2 + g(r_M)\Bigg\} \tag{54}$$

where

$$\begin{aligned} f(r_l) &= 4\left(\frac{1+r_l^2}{1-r_l^2}\ln r_l - 1\right)\ln(1-r_l^2) - \frac{r_l^2(8-5r_l^2)}{(1-r_l^2)^2}\ln r_l \\ &\quad + 4\frac{1+r_l^2}{1-r_l^2}\mathrm{Li}_2(r_l^2) - \frac{r_l^2}{1-r_l^2}\left(\frac{3}{2} + \frac{4}{3}\pi^2\right) \end{aligned} \tag{55}$$

and

$$\begin{aligned} g(r_M) &= 4\left(\frac{1+r_M^2}{1-r_M^2}\ln r_M - 1\right)\ln(1-r_M^2) - \frac{r_M^2(2-5r_M^2)}{(1-r_M^2)^2}\ln r_M \\ &\quad + 4\frac{1+r_M^2}{1-r_M^2}\mathrm{Li}_2(r_M^2) + \left(\frac{3}{2} - \frac{4}{3}\pi^2\right)\frac{r_M^2}{1-r_M^2} \end{aligned} \tag{56}$$

We have written the corrections in such way that $f(0) = g(0) = 0$, so from Eqn. (53) we get the well-known lepton mass singularity of the total radiative correction to pion decay as $3\alpha/\pi \ln(m_l)$, whereas in the radiative correction to the tau decay the meson mass singularities cancel according to Eqn. (54). Our results are a nice application of the Kinoshita-Lee-Naunenberg theorem, which states that mass singularities cancel in inclusive decay rates [2, 19]. In the case of the pion decay the Born amplitude is proportional to m_l and therefore a logarithm $\ln m_l$ is allowed in the radiative correction, while for the tau decay the Born amplitude is not proportional to m_π, and therefore a logarithm $\ln m_\pi$ is forbidden in the radiative correction.

The ultra-violet divergences of the corrections to the tau and the meson decays are equal and they cancel in the ratio $R_{\tau/M}$ defined by

$$R_{\tau/M} = \frac{\Gamma(\tau \to M\nu_\tau(\gamma))}{\Gamma(M \to \mu\nu_\mu(\gamma))} = \frac{1}{2}\frac{m_\tau^3}{m_M m_\mu^2}\frac{\left(1 - m_M^2/m_\tau^2\right)^2}{\left(1 - m_\mu^2/m_M^2\right)^2}\left(1 + \delta R_{\tau/M}\right) \tag{57}$$

with the finite radiative correction

$$\left(\delta R_{\tau/M}\right)_{PM} = \frac{\alpha}{2\pi}\left\{\frac{3}{2}\ln\frac{m_\tau^2}{m_M^2} - 6\ln\frac{m_\mu}{m_M} + \frac{3}{2} + g\left(\frac{m_M}{m_\tau}\right) - f\left(\frac{m_\mu}{m_M}\right)\right\} \tag{58}$$

So within this model with an effective pointlike meson ("PM"), we end up with the results [3]

$$\begin{aligned}\left(\delta R_{\tau/\pi}\right)_{PM} &= +1.05\% \\ \left(\delta R_{\tau/K}\right)_{PM} &= +1.67\%\end{aligned} \tag{59}$$

Note that this result differs from the Marciano-Sirlin estimate quoted in the introduction,

$$\begin{aligned}\delta R_{\tau/\pi}^{M.S.} &= -0.98\% \\ \delta R_{\tau/K}^{M.S.} &= -0.53\%\end{aligned} \tag{60}$$

This is mainly due to the fact that the point meson and the short distance corrections do not have the same UV divergences.

From Eqn. (53) the radiative correction to the ratio of the electronic and the muonic decay modes of the pion

$$R_{e/\mu} = \frac{\Gamma(\pi \to e\nu_e(\gamma))}{\Gamma(\pi \to \mu\nu_\mu(\gamma))} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2\left(1 + \delta R_{e/\mu}\right) \tag{61}$$

can also be calculated in the same way and we obtain

$$\left(\delta R_{e/\mu}\right)_{PM} = \frac{\alpha}{2\pi}\left\{6\ln\frac{m_e}{m_\mu} + f\left(\frac{m_e}{m_\pi}\right) - f\left(\frac{m_\mu}{m_\pi}\right)\right\} = -3.93\% \tag{62}$$

a result derived long ago by Kinoshita [2].

5 The Leading Short Distance Logarithm

In Ref. [4] the author shows that for any semileptonic weak process, the leading $O(\alpha)$ correction in the limit of a large Z boson mass, $m_Z^2 \to \infty$, is

$$\mathcal{M}_0 \to \left[1 + \frac{\alpha}{\pi} \ln \frac{m_Z}{\mu}\right] \mathcal{M}_0 \tag{63}$$

where $\mathcal{M}_0$ denotes the Born amplitude and μ is an unspecified mass scale characteristic of the process.

We will now rederive this result and then show that we are able to fix the scale μ in terms of the lepton mass m_l and the scale μ_{cut}. Consider the short distance corrections to the amplitude $\mathcal{A}_0$ for the initial weak process $\tau \to \nu_\tau \bar{u} d$

$$\mathcal{A}_0 = -i \frac{G_F \cos\theta_C}{\sqrt{2}} [\bar{u}_d \gamma^\mu \gamma_- u_u][\bar{u}_\nu \gamma_\mu \gamma_- u_\tau] \tag{64}$$

The six Feynman diagrams which give the radiative corrections to this short distance amplitude are shown in Fig. 5. In order to find the term of the order $O(\alpha \ln m_Z)$, we have to calculate the UV divergence of the short distance correction

$$\delta\mathcal{A} = \delta\mathcal{A}_a + \delta\mathcal{A}_b + \ldots + \delta\mathcal{A}_f \tag{65}$$

The calculation can be simplified by using the Landau gauge [20]. In this gauge the amplitudes corresponding to external line renormalization, $\delta\mathcal{A}_a \ldots \delta\mathcal{A}_c$, are UV finite, as are $\delta\mathcal{A}_e$ and $\delta\mathcal{A}_f$, where the photon loop connects an ingoing with an outgoing fermion line. In the Landau gauge, the only UV divergent amplitude is $\delta\mathcal{A}_d$:

$$\begin{aligned}
\delta\mathcal{A}_d &= e^2 Q_u Q_\tau \frac{G_F}{\sqrt{2}} \mu^{D-4} \int \frac{d^D k}{(2\pi)^D} \frac{[\bar{u}_d \gamma^\mu \gamma_- \not{k} \gamma^\alpha u_u][\bar{u}_\mu \gamma_\mu \gamma_- \not{k} \gamma^\beta \bar{u}_\tau]}{(k^2)^3} \left[g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right] + \ldots \\
&= \frac{\alpha}{2\pi} \Delta\, \mathcal{A}_0 + \ldots
\end{aligned} \tag{66}$$

where the dots indicate terms which are UV finite.

Now replacing Δ by the UV cut-off $\ln m_Z^2$, we obtain

$$\delta\mathcal{A} = \frac{\alpha}{\pi} \ln \frac{m_Z}{\mu}\, \mathcal{A}_0 + \ldots \tag{67}$$

where μ is some unspecified characteristic mass scale which must be introduced by dimensional arguments. This is Sirlin's result.

If we replace the photon propagator $1/k^2$ by $1/(k^2 - \mu_{cut}^2)$ and if we do not neglect the lepton mass, we obtain

$$\delta\mathcal{A} = \frac{\alpha}{\pi} \frac{1}{m_l^2 - \mu_{cut}^2} \left(m_l^2 \ln \frac{m_Z}{m_l} - \mu_{cut}^2 \ln \frac{m_Z}{\mu_{cut}}\right) \mathcal{A}_0 + \ldots \tag{68}$$

We will now apply this result to the radiative correction to $R_{\tau/\pi}$. In addition to the short distance correction which contains the above logarithm, there are the long distance corrections associated with the integration over $k_E^2 = 0 \ldots \mu_{cut}^2$, which have to be added. Of course these long distance corrections have to be computed using the effective pointlike pion (and hadronic resonances) and give rise to other logarithms which in fact turn out to be more important in $R_{\tau/\pi}$ that the short distance ones.

So we will now combine the long and the short distance corrections. First we have to discuss which scale is a good choice for μ_{cut}. The effective point meson is a good approximation only if all momentum transfers squared are small compared to m_ρ^2, and so for the long distance part one should require $\mu_{cut}^2 \ll m_\rho^2$. The short distance part, on the other hand, uses asymptotically free quarks which one would believe in only for $\mu_{cut}^2 > (1 \ldots 2\ \mathrm{GeV})^2$. A compromise between these non-overlapping regions would be $\mu_{cut}^2 = m_\rho^2$.

The long distance corrections are to be integrated over $k_E^2 = 0 \ldots \mu_{cut}^2$, which is taken into account by using an UV cut-off $\mu_{cut} = m_\rho$ for the point meson results of Sec. 4 (ie. by the replacement $\Delta \to \ln(m_\rho^2/\mu^2)$ in Eqns. (53) and (54)). Thus we obtain

$$\frac{\delta\Gamma}{\Gamma^0}\Big(\pi \to l\nu_l(\gamma)\Big) = \underbrace{\overbrace{\frac{2\alpha}{\pi}\ln\frac{m_Z}{m_\rho}}^{\text{short dist.}}}_{2.22\%} + \underbrace{\overbrace{\frac{\alpha}{2\pi}\left\{\frac{3}{2}\ln\frac{m_\pi^2}{m_\rho^2} + 6\ln r_l + \frac{11}{4} - \frac{2}{3}\pi^2 + f(r_l)\right\}}^{\text{long dist.}}}_{-1.02\%} \tag{69}$$

and

$$\begin{aligned}\frac{\delta\Gamma}{\Gamma^0}\Big(\tau \to \pi\nu_\tau(\gamma)\Big) &= \underbrace{\overbrace{\frac{2\alpha}{\pi}\frac{1}{m_\tau^2 - m_\rho^2}\left(m_\tau^2\ln\frac{m_Z}{m_\tau} - m_\rho^2\ln\frac{m_Z}{m_\rho}\right)}^{\text{short dist.}}}_{1.74\%} \\ &\quad + \underbrace{\overbrace{\frac{\alpha}{2\pi}\left\{\frac{3}{2}\ln\frac{m_\tau^2}{m_\rho^2} + \frac{17}{4} - \frac{2}{3}\pi^2 + g(r_\pi)\right\}}^{\text{long dist.}}}_{0.03\%}\end{aligned} \tag{70}$$

This modifies the point meson results in the following way:

$$\begin{aligned}\delta R_{\tau/\pi} &= \Big(\delta R_{\tau/\pi}\Big)_{PM} + \underbrace{\frac{2\alpha}{\pi}\frac{m_\tau^2}{m_\tau^2 - m_\rho^2}\ln\frac{m_\rho}{m_\tau}}_{-0.48\%} \\ &= 0.57\%\end{aligned} \tag{71}$$

Similarly

$$\begin{aligned}\delta R_{\tau/K} &= \Big(\delta R_{\tau/\pi}\Big)_{PM} + \underbrace{\frac{2\alpha}{\pi}\frac{m_\tau^2}{m_\tau^2 - m_{K^\star}}\ln\frac{m_{K^\star}}{m_\tau}}_{-0.43\%} \\ &= 1.24\%\end{aligned} \tag{72}$$

The prediction for $\delta R_{\tau/\pi}$ obtained in this way by matching the leading short distance logarithms with the effective point pion correction, $\delta R_{\tau/\pi} = 0.57\%$, differs strongly from the first Marciano-Sirlin estimate [5], $\delta R_{\tau/\pi} = -0.98\%$, which is based on the short distance logarithms only. Note, however, that it is numerically close to the predictions in [6, 8], see Eqns. (10,11) above. The estimate in [6] (and up to very small additional corrections, the value in [8] as well) includes long and short distance corrections as in Eqn. (69) for the pion decay. For the tau decay, the authors only include the short distance correction, estimated using Eqn. (63) with $\mu = m_\tau$, ie. they estimate the distance correction to be $(2\alpha/\pi)\ln(m_Z/m_\tau) = 1.83\%$, which has to be compared to the 1.74% calculated in Eqn. (70).

And so, what is really missing in [6, 8], as compared to our estimate in Eqn. (71), are the long distance corrections to the tau decay. In [6, 8] the authors only estimated their possible size, resulting in a $\pm 1\%$ uncertainty. As we have calculated, the long distance corrections in the model with an effective point meson happen to be extremely small (0.03%), which explains why our estimate in Eqn. (71) and the estimates in [6, 8] are quite similar numerically.

However, these estimates still cannot be considered as safe for the following reasons:

1. The value $\mu_{cut} = m_\rho$ is too large for the point meson approximation in the long distance part and too small for the assumption of almost free quarks in the short distance part. Therefore the range of validity in the long distance part should be extended to $1 \ldots 2$ GeV by including vector meson resonances and then $\mu_{cut} = 1 \ldots 2$ GeV will be a good value to use. Indeed we will show below that the vector meson effects in the loops change the final result considerably.

2. For the long distance corrections to the tau decay, the UV cut-off μ_{cut} is smaller than m_τ. Therefore terms proportional to m_τ^2/μ_{cut}^2 are missing in Eqn. (70).

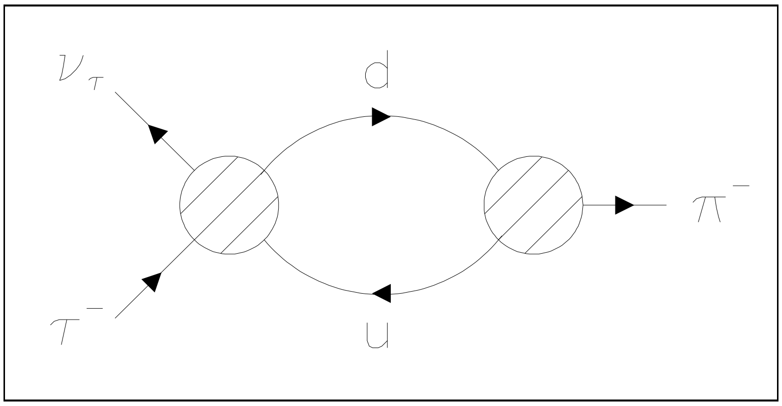
3. The large scale m_Z appearing in the short distance logarithm cancels in the ratio $R_{\tau/\pi}$, leaving logarithms of comparable scales ($\ln m_\rho$ and $\ln m_\tau$). Therefore it is not obvious that non-logarithmic contributions in the short distance correction can be neglected.

So in the following sections we will improve the calculation of the long distance corrections by properly using the regulated photon propagator of Eqn. (35) and by including vector meson resonances, and that of of the short distance corrections by going beyond the leading logarithm.

6 Short Distance Beyond Leading Logarithm

In this section we will present a complete calculation of the short distance corrections beyond the logarithm which is leading in the limit $m_Z^2 \to \infty$. This leading logarithmic contribution to the correction $\delta\mathcal{A}$ of the short distance amplitude $\mathcal{A}_0(\tau \to \nu_\tau \bar{u} d)$ could be written as a factor times the Born amplitude, $\delta\mathcal{A} = C \times \mathcal{A}_0 + \ldots$. Then the correction $\delta\mathcal{M}$ to the exclusive rate $\mathcal{M}_0(\tau \to \pi\nu_\tau)$ involves the same logarithm, $\delta\mathcal{M} = C \times \mathcal{M}_0 + \ldots$

Figure 4: Decay $\tau \to \pi\nu_\tau$ via an intermediate quark-antiquark state. The bubble on the left hand side is to be replaced by the short distance diagrams $\mathcal{A}_0$ and $\delta\mathcal{A}_i$, cf. Fig. 5



The situation, however, is more involved for the complete result for $\delta\mathcal{A}$. Firstly, in general it can not be written as a number times $\mathcal{A}_0$, but rather it involves other operators, and secondly $\delta\mathcal{A}$ will depend on the relative momentum of the two quarks. The first problem can be solved by projection on the $J^P = 0^-$ state of the two quarks, and the second problem by use of the parton model.

Let the short distance amplitude for the decay

$$\tau(s) \to \nu_\tau(q)\bar{u}(\frac{1}{2}p - l)d(\frac{1}{2}p + l) \tag{73}$$

of the tau into neutrino and an quark-antiquark pair with relative momentum $2l$ be given by $\mathcal{A}(l)$ (and similarly for the decay into $\bar{u}s$, for the time being we consider the case of non-strange decay into the pion only). In a frame, where the pion is moving with infinite momentum, l is proportional to p,

$$l = \frac{u}{2}p \tag{74}$$

and $\mathcal{A} = \mathcal{A}(u)$. Then the amplitude of the exclusive tau decay $\tau \to \pi\nu_\tau$ is (see Fig. 4)

$$\mathcal{M}(\tau^- \to \pi^-\nu_\tau) = \mathcal{T}(\mathcal{A}) \tag{75}$$

where

$$\mathcal{T}(\mathcal{A}) = -i\frac{3\sqrt{2}}{8}\int_{-1}^{+1} du\, \frac{\Phi_\pi(u)}{m_q}(-)\sum\left[\mathcal{A}(u)\bar{u}_u\left(\frac{1-u}{2}p\right)\gamma_5 u_d\left(\frac{1+u}{2}p\right)\right] \tag{76}$$

Here $\Phi_\pi(u) = \Phi_\pi(-u)$ is a symmetric parton distribution function (The numerical factor is just conventional. Our treatment follows closely that of Refs. [21, 22]). The sum $\sum$ is over the spins of the quarks. For the quark mass m_q we assume isospin (and in the case of the kaon, $SU(3)$-flavour) symmetry,

$$m_q = m_u = m_d = m_s \tag{77}$$

The limit $m_q \to 0$ is implied. Note that all relevant operators in $\mathcal{A}$ involve an odd number of Dirac matrices between the $\bar{u}_d$ and the u_u Dirac spinors, so the sum is proportional to m_q, and this m_q cancel the m_q in the denominator.

In the case of the Born amplitude $\mathcal{M}_0$, $\mathcal{A}_0$ is

$$\mathcal{A}_0 = -i\frac{G_F \cos\theta_c}{\sqrt{2}}[\bar{u}_d\gamma^\mu\gamma_- u_u][\bar{u}_\nu\gamma_\mu\gamma_- u_\tau] \tag{78}$$

and so

$$\begin{aligned}
&\mathcal{M}_0(\tau^- \to \pi^-\nu_\tau) = \mathcal{T}(\mathcal{A}_0) \\
&= -\frac{3}{8}G_F\cos\theta_c\int_{-1}^{+1}du\,\frac{\Phi_\pi(u)}{m_q}(-)\mathrm{tr}\left[\gamma^\mu\gamma_-\left(\frac{1-u}{2}\not{p}+m_q\right)\gamma_5\left(\frac{1+u}{2}\not{p}+m_q\right)\right] \\
&\qquad\times\left[\bar{u}_\nu\gamma_\mu\gamma_- u_\tau\right] \\
&= -\frac{3}{2}G_F\cos\theta_c\int_{-1}^{+1}du\Phi_\pi(u)\left[\bar{u}_\nu\not{p}\gamma_- u_\tau\right] \\
&= -G_F\cos\theta_c f_\pi\left[\bar{u}_\nu\not{p}\gamma_- u_\tau\right]
\end{aligned} \tag{79}$$

where the last line follows from the definition of f_π. So we have rederived the sum rule [23]

$$\frac{3}{2}\int_{-1}^{+1}du\,\phi_\pi(u) = f_\pi \tag{80}$$

After these preparations, we can now calculate the full short distance corrections $\delta\mathcal{A}_i$ for $i = a\ldots f$ given by the Feynman diagrams in Fig. 5, using a massive photon propagator with $1/k^2 \to 1/(k^2-\mu_{cut}^2)$, Feynman gauge, and neglecting the quark masses. According to Eqn. (75), the corrections $\delta\mathcal{A}_i$ induce corrections $\delta\mathcal{M}_i(\tau \to M\nu_\tau)$ to the rate for the tau decay

$$\delta\mathcal{M}_i(\tau \to M\nu_\tau) = \mathcal{T}(\delta\mathcal{A}_i) \qquad\qquad (i = a\ldots f) \tag{81}$$

(and analogously for the pion decay). The results depend on the unknown distribution function $\Phi_M(u)$ and are given in the form of integrals over u

$$\frac{\delta\mathcal{M}_i}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) = \frac{3}{2f_M}\int_{-1}^{+1}du\,\phi_M(u)w_i(u, m_l^2, m_M^2, \mu_{cut}^2) \tag{82}$$

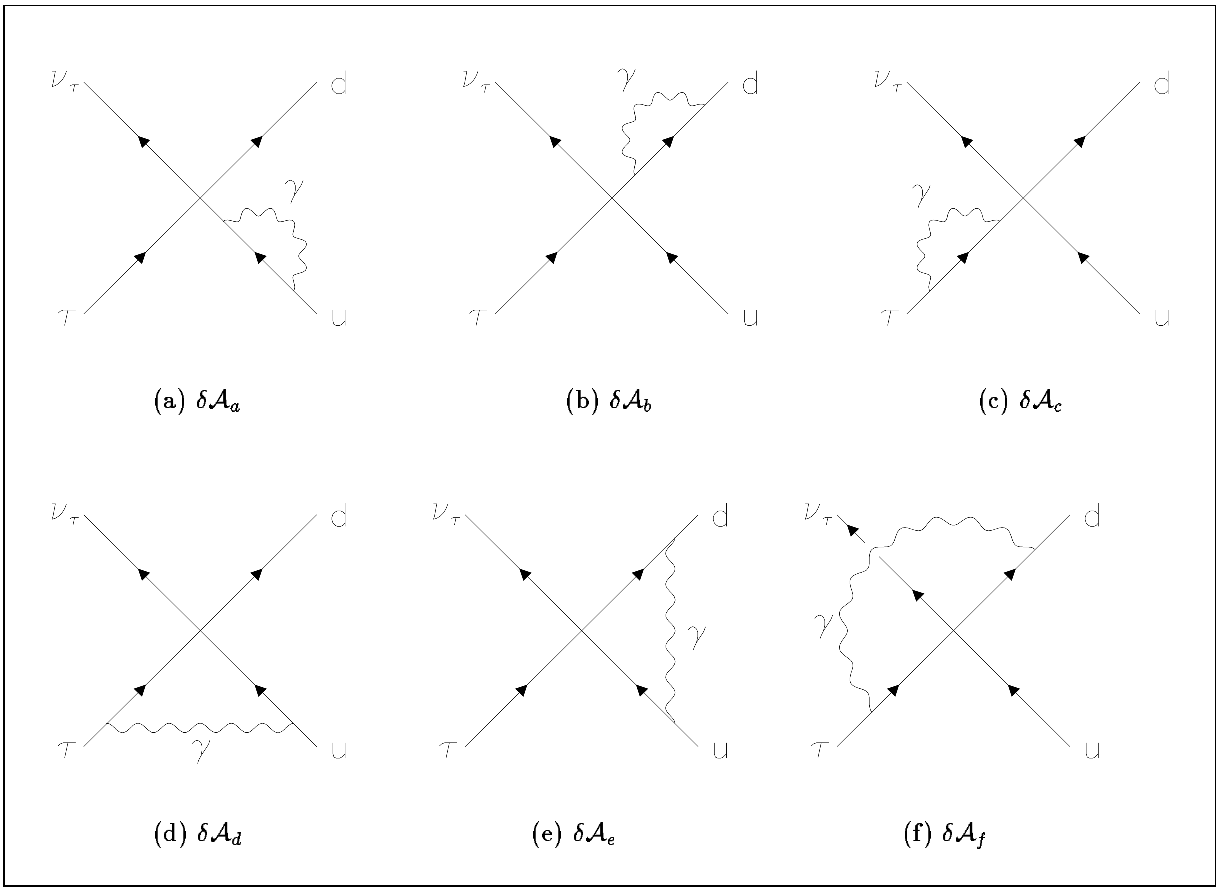
with weight functions $w_i(u)$. Define the total short distance correction amplitude

$$\frac{\delta\mathcal{M}_{sd}}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) = \sum_{i=a}^{f}\frac{\delta\mathcal{M}_i}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) \tag{83}$$

and the sum of the weight functions

$$w(u, m_l^2, m_M^2, \mu_{cut}^2) = \sum_{i=a}^{f} w_i(u, m_l^2, m_M^2, \mu_{cut}^2) \tag{84}$$

Figure 5: Short distance diagrams



(a) $\delta\mathcal{A}_a$ (b) $\delta\mathcal{A}_b$ (c) $\delta\mathcal{A}_c$

(d) $\delta\mathcal{A}_d$ (e) $\delta\mathcal{A}_e$ (f) $\delta\mathcal{A}_f$

Then the short distance correction $(\delta R_{\tau/M})_{sd}$ is given by

$$
\begin{aligned}
(\delta R_{\tau/M})_{sd} &= 2\left[\frac{\delta\mathcal{M}_{sd}}{\mathcal{M}_0}(m_\tau^2, m_m^2, \mu_{cut}^2) - \frac{\delta\mathcal{M}_{sd}}{\mathcal{M}_0}(m_\mu^2, m_m^2, \mu_{cut}^2)\right] \\
&= \frac{3}{f_M}\int_{-1}^{1} du\,\phi_M(u)[w(u, m_\tau^2, m_M^2, \mu_{cut}^2) - w(u, m_\mu^2, m_M^2, \mu_{cut}^2)] \qquad (85)
\end{aligned}
$$

Note that w_a, w_b and w_e are identical for tau and pion decay, ie. they do not depend on the lepton mass. Using furthermore the symmetry of the distribution function ϕ_M under $u \to -u$, we can write

$$
(\delta R_{\tau/M})_{sd} = \frac{3}{2f_M}\int_{-1}^{1} du\,\phi_M(u) r_{\tau/M}(u) \qquad (86)
$$

with

$$
r_{\tau/M}(u) := 2\left[\bar{w}(u, m_\tau^2, m_M^2, \mu_{cut}^2) - \bar{w}(u, m_\mu^2, m_M^2, \mu_{cut}^2)\right] \qquad (87)
$$

where $\bar{w}$ is the symmetrized sum of the weightfunctions w_c, w_d and w_f:

$$\begin{aligned}\bar{w}(u,m_l^2,m_M^2,\mu_{cut}^2) \;:=\; & \frac{1}{2}\Big[w_c(u,m_l^2,m_M^2,\mu_{cut}^2)+w_d(u,m_l^2,m_M^2,\mu_{cut}^2)\\ & +w_f(u,m_l^2,m_M^2,\mu_{cut}^2)-w_c(-u,m_l^2,m_M^2,\mu_{cut}^2)\\ & -w_d(-u,m_l^2,m_M^2,\mu_{cut}^2)-w_f(-u,m_l^2,m_M^2,\mu_{cut}^2)\Big] \qquad (88)\end{aligned}$$

Similarly the short distance correction to $R_{e/\mu}$ is

$$(\delta R_{e/\mu})_{sd} = \frac{3}{2f_M}\int_{-1}^{1} du\, \phi_M(u) r_{e/\mu}(u) \qquad (89)$$

with

$$r_{e/\mu}(u) := 2\left[\bar{w}(u,m_e^2,m_\pi^2,\mu_{cut}^2)-\bar{w}(u,m_\mu^2,m_\pi^2,\mu_{cut}^2)\right] \qquad (90)$$

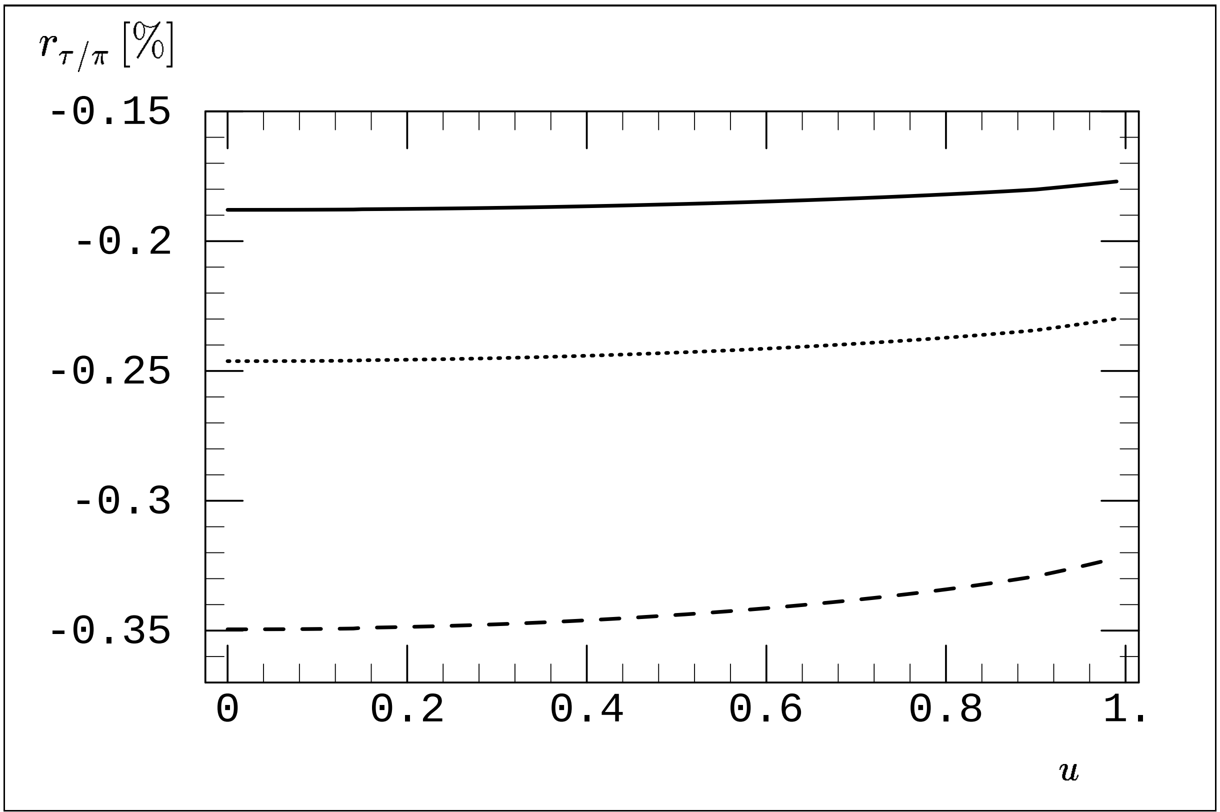
The relevant weight functions are given by

$$\begin{aligned}& w_c(u,m_l^2,m_M^2,\mu_{cut}^2) =\\ &\quad \frac{\alpha}{8\pi}\left\{1-2B_0-2B_1-4m_l^2[B_1'-B_0']\right\}\\[1ex] & w_d(u,m_l^2,m_M^2,\mu_{cut}^2) =\\ &\quad \frac{\alpha}{6\pi}\Big\{16C_{00}^d+(1+u)^2m_M^2C_{11}^d+4m_l^2C_{22}^d+2(1+u)(m_M^2+m_l^2)C_{12}^d\\ &\qquad +[(1+u)^2m_M^2+(1+u)m_l^2]C_1^d+[(1+u)m_M^2+(3+u)m_l^2]C_2^d\\ &\qquad +(1+u)m_l^2C_0^d-2\Big\}\\[1ex] & w_f(u,m_l^2,m_M^2,\mu_{cut}^2) =\\ &\quad \frac{\alpha}{12\pi}\Big\{4C_{00}^f+(1-u)^2m_M^2C_{11}^f+2(m_M^2+m_l^2)C_{22}^f+(1-u)(3m_M^2+m_l^2)C_{12}^f\\ &\qquad +[(1-u)^2m_M^2+(1-u)m_l^2]C_1^f+[(1-u)m_M^2+(3-u)m_l^2]C_2^f\\ &\qquad +(1-u)m_l^2C_0^f\Big\} \qquad (91)\end{aligned}$$

where

$$\begin{aligned}B_{\dots}^{(\prime)} &= B_{\dots}^{(\prime)}(m_l^2,\mu_{cut},m_l)\\ C_{\dots}^d &= C_{\dots}\left(\frac{(1+u)^2}{4}m_M^2,\frac{1-u}{2}m_l^2-\frac{1-u^2}{4}m_M^2,m_l^2,\mu_{cut},0,m_l\right)\\ C_{\dots}^f &= C_{\dots}\left(\frac{(1-u)^2}{4}m_M^2,\frac{1+u}{2}m_l^2-\frac{1-u^2}{4}m_M^2,m_l^2,\mu_{cut},0,m_l\right) \qquad (92)\end{aligned}$$

Figure 6: The function $r_{\tau/\pi}(u)$ for $\mu_{cut} = 1$ GeV (dashed), $\mu_{cut} = 1.5$ GeV (dotted) and $\mu_{cut} = 2$ GeV (solid)



In Figs. 6 – 8 we have plotted the symmetric functions $r_{\tau/\pi}(u)$, $r_{\tau/K}(u)$ and $r_{e/\mu}(u)$ for three different values of μ_{cut}. It turns out that these functions change only very little while varying u from $u = 0$, where the distribution function ϕ_π is presumably peaked, to $u = 1$. Therefore we can approximate the weight functions by their values at $u = 0$:

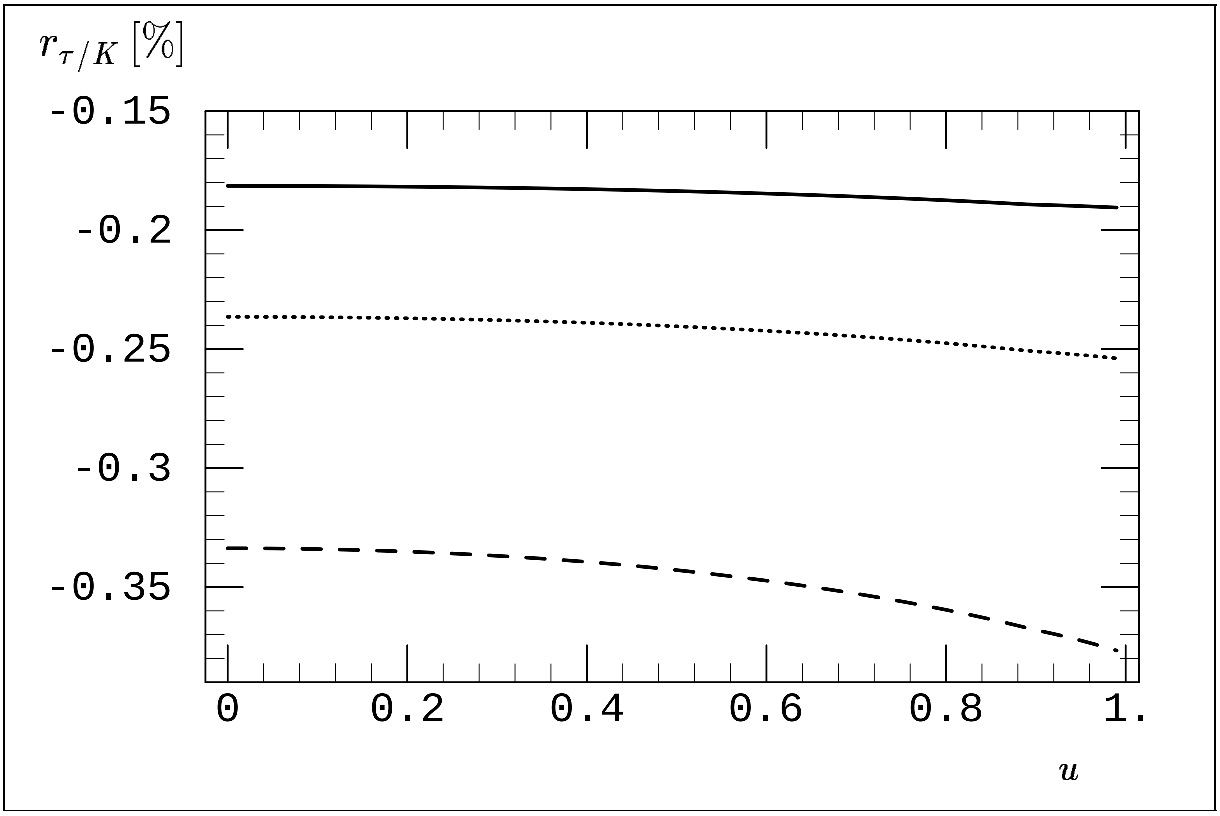
$$\begin{aligned} \frac{\delta\mathcal{M}_i}{\Gamma_0}(m_l^2, m_M^2, \mu_{cut}^2) &\approx \frac{3}{2f_M} w_i(0, m_l^2, m_M^2, \mu_{cut}^2) \int_{-1}^{+1} du\, \phi_M(u) \\ &= w_i(0, m_l^2, m_M^2, \mu_{cut}^2) \end{aligned} \tag{93}$$

and then

$$\begin{aligned} (\delta R_{\tau/M})_{sd} &= \frac{3}{2f_M} r_{\tau/M}(0) \int_{-1}^{1} du\, \phi_M(u) \\ &= r_{\tau/M}(0) \end{aligned} \tag{94}$$

It is clear from Figs. 6 – 8, that this approximation induces an uncertainty which is well below $\begin{smallmatrix}+0.02\% \\ -0.00\%\end{smallmatrix}$ for $R_{\tau/\pi}$, well below $\begin{smallmatrix}+0.00\% \\ -0.02\%\end{smallmatrix}$ for $R_{\tau/K}$ and well below $\begin{smallmatrix}+0.003\% \\ -0.000\%\end{smallmatrix}$ for $R_{e/\mu}$.

Figure 7: The function $r_{\tau/K}(u)$ for $\mu_{cut} = 1$ GeV (dashed), $\mu_{cut} = 1.5$ GeV (dotted) and $\mu_{cut} = 2$ GeV (solid)



Note that we can make these statements without any specific assumptions on the form of the distribution functions ϕ_π and ϕ_K.

In Fig. 9 we compare $(\delta R_{\tau/\pi})_{sd}$ as a function of μ_{cut} as given by Eqn. (94) with the estimate from the leading logarithms

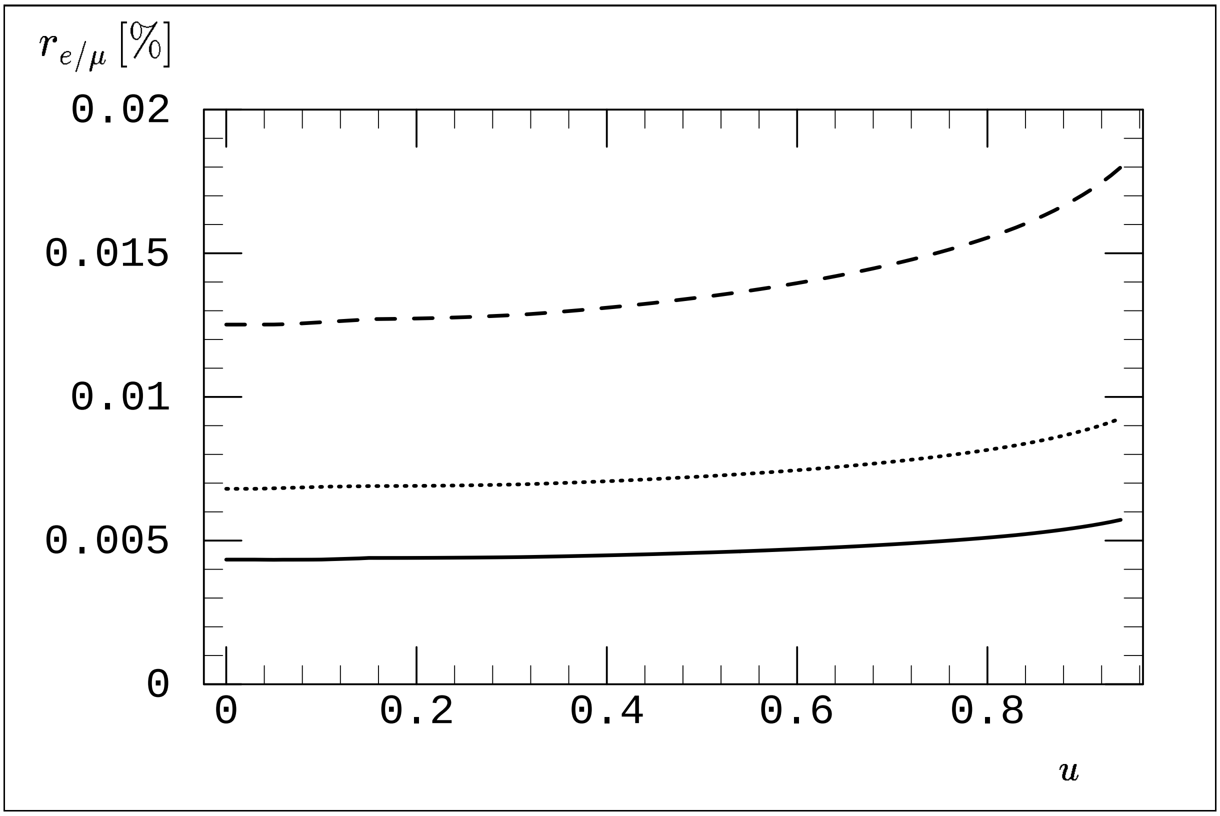
$$(\delta R_{\tau/\pi})_{sd} \approx \frac{2\alpha}{\pi} \frac{m_\tau^2}{m_\tau^2 - \mu_{cut}^2} \ln \frac{\mu_{cut}}{m_\tau} \tag{95}$$

We find that Eqn. (95) gives an excellent approximation to Eqn. (94) and that the non-leading contributions to $(\delta R_{\tau/\pi})_{sd}$ are very small.

7 Hadronic Structure Dependent Loops

For the real photon emission we have calculated the hadronic structure dependent effects in [10], cf. Sec. 2. In the long distance virtual corrections there are two different effects associated with hadronic structure. On the one hand, the photon emitted in the

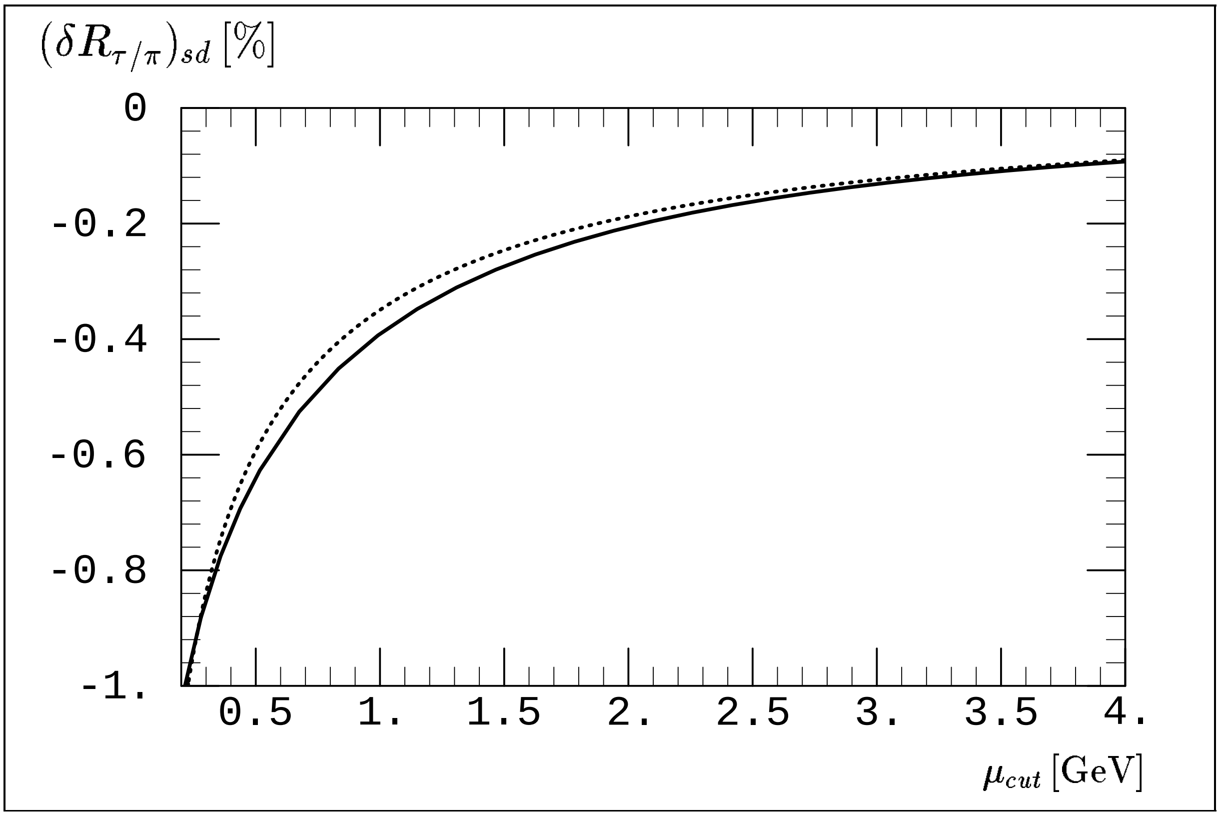
Figure 8: The function $r_{e/\mu}(u)$ for $\mu_{cut} = 1$ GeV (dashed), $\mu_{cut} = 1.5$ GeV (dotted) and $\mu_{cut} = 2$ GeV (solid)



radiative decay by hadronic structure dependent radiation (SD) as a real photon could be reabsorbed either by the lepton or by the pion (kaon), see Fig. 10. If in the radiative decay ($k^2 = 0$) there is a SD amplitude proportional to F_V and F_A , the corresponding hadronic "structure dependent loops" (SDL) must also be there for sufficiently small k^2 (long distance). Actually for the last two diagrams, where the photon couples to the meson, the respective corrections to tau and to meson decay will be identical and therefore they will cancel in $R_{\tau/\pi}$. So for simplicity we will not consider them.

On the other hand, according to the notion of vector meson dominance, the photon does not couple directly to the pion (kaon) but rather through a ρ intermediate state, see Fig. 12. Similarly in the diagrams proportional to F_V and F_A the photon might couple through vector meson dominance as indicated in Fig. 14. In the remaining of this article these modifications of the photon couplings will *only* be called "vector meson dominance" of the respective diagrams, either of the point meson loops (PML) or of the hadronic structure dependent loops (SDL). The name "hadronic structure dependent" will be used only for amplitudes such as $\delta\mathcal{M}_6$ and $\delta\mathcal{M}_7$ which are proportional to the form factors F_V and F_A. This is of course just a naming convention; the vector meson

Figure 9: Short distance correction $(\delta R_{\tau/\pi})_{sd}$: complete result according to Eqn. (94) (dotted) and estimate based on the leading logarithms according to Eqn. (95) (solid)



dominance of the photon coupling is also an effect of hadronic structure.

We will discuss the modifications due to vector meson dominance of the photon couplings later and start with the calculation of $\delta\mathcal{M}_6$ and $\delta\mathcal{M}_7$ using the generalized photon propagator with mass m^2

$$\frac{1}{k^2-\lambda^2} \to \frac{1}{k^2-m^2} \tag{96}$$

and no vector meson dominance in the coupling of the photon to the vector meson and the pion (see Figs. 10 (a) and (b)).

The result for $\delta\mathcal{M}_6$ is

$$\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_M^2, M_V^2, m^2) = \frac{\alpha}{2\pi}\frac{F_V^M(0)M_V^2}{\sqrt{2}m_M f_M}\Big\{3C_{00} + \frac{1}{2}(m_l^2 - m_M^2)(C_{22}+C_{12})\Big\} \tag{97}$$

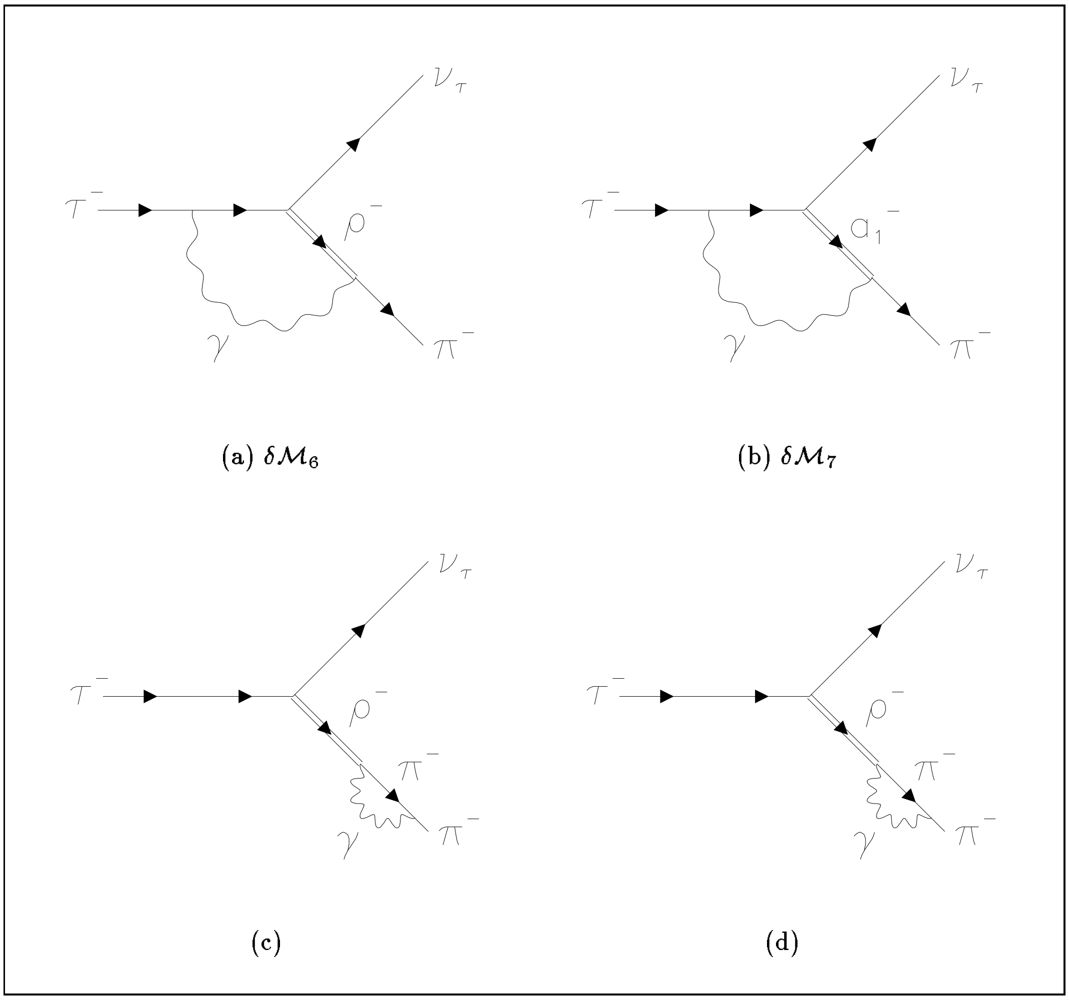
where

$$C_{ij} = C_{ij}(m_M^2, 0, m_l^2, m, M_V, m_l) \tag{98}$$

M_V denotes a – possibly complex — vector meson mass

$$M_V^2 = m_V^2 - i m_V \Gamma_V \tag{99}$$

Figure 10: Diagrams for hadronic structure dependent corrections



A comment on the treatment of the meson propagators for the ρ, ρ', a_1, ... particles is in order. In the calculation of the radiative tau decay we used sophisticated Breit-Wigner resonance factors with energy dependent widths. However, the difference between fixed and energy dependent widths is very small after integration over the spectrum. In the virtual corrections one has to integrate over all possible loop momenta anyway. For the virtual corrections, we will therefore only use Breit-Wigner resonance factors with fixed widths

$$\mathrm{BW}_V(k^2) = \frac{M_V^2}{M_V^2 - k^2} = \frac{m_V^2 - im_V\Gamma_V}{m_V^2 - k^2 - im_V\Gamma_V} \tag{100}$$

or even with real masses $M_V^2 = m_V^2$.

Similarly the amplitude $\delta\mathcal{M}_7$ is equal to

$$\frac{\delta\mathcal{M}_7}{\mathcal{M}_0}(m_l^2, m_M^2, M_A^2, m^2) \quad = \quad \frac{\alpha}{4\pi}\frac{F_A^M(0)M_A^2}{\sqrt{2}m_M f_M}\Big\{6C_{00} + 3m_M^2 C_{11} + (m_l^2 + 2m_M^2)C_{22}$$

$$+ (2m_l^2 + 4m_M^2)C_{12} + (m_l^2 - m_M^2)C_1 - 1\Big\} \tag{101}$$

with

$$C_{\dots} = C_{\dots}(m_M^2, 0, m_l^2, m, M_A, m_l) \tag{102}$$

In the case of the meson being the pion, our standard parameterization of the form factor $F_V^{(\pi)}(t)$ in the radiative decay was a tripole dominated by resonances ρ, ρ' and ρ'' with relative strengths σ and ρ (see Sec. 2). The same parameterization will be used here for the structure dependent loop $\delta\mathcal{M}_6$, resulting in

$$\begin{aligned}
\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_\pi^2, m^2) &= \frac{1}{1+\sigma+\rho}\Big\{\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_\pi^2, M_\rho^2, m^2) \\
&\quad + \sigma\,\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_\pi^2, M_{\rho'}^2, m^2) \\
&\quad + \rho\,\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_\pi^2, M_{\rho''}^2, m^2)\Big\} \\
\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2, m_\pi^2, m^2) &= \frac{\delta\mathcal{M}_7}{\mathcal{M}_0}(m_l^2, m_K^2, M_{a_1}^2, m^2)
\end{aligned} \tag{103}$$

and for the kaon we define

$$\begin{aligned}
\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_K^2, m^2) &= \frac{1}{1+\sigma_K+\rho_K}\Big\{\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_K^2, M_{K^\star}^2, m^2) \\
&\quad + \sigma_K\,\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_K^2, M_{K^{\star\prime}}^2, m^2) \\
&\quad + \rho_K\,\frac{\delta\mathcal{M}_6}{\mathcal{M}_0}(m_l^2, m_K^2, M_{K^{\star\prime\prime}}^2, m^2)\Big\} \\
\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2, m_K^2, m^2) &= \frac{\delta\mathcal{M}_7}{\mathcal{M}_0}(m_l^2, m_K^2, M_{K_1}^2, m^2)
\end{aligned} \tag{104}$$

Here again we use the convention that capital letters for masses indicate possibly complex masses,

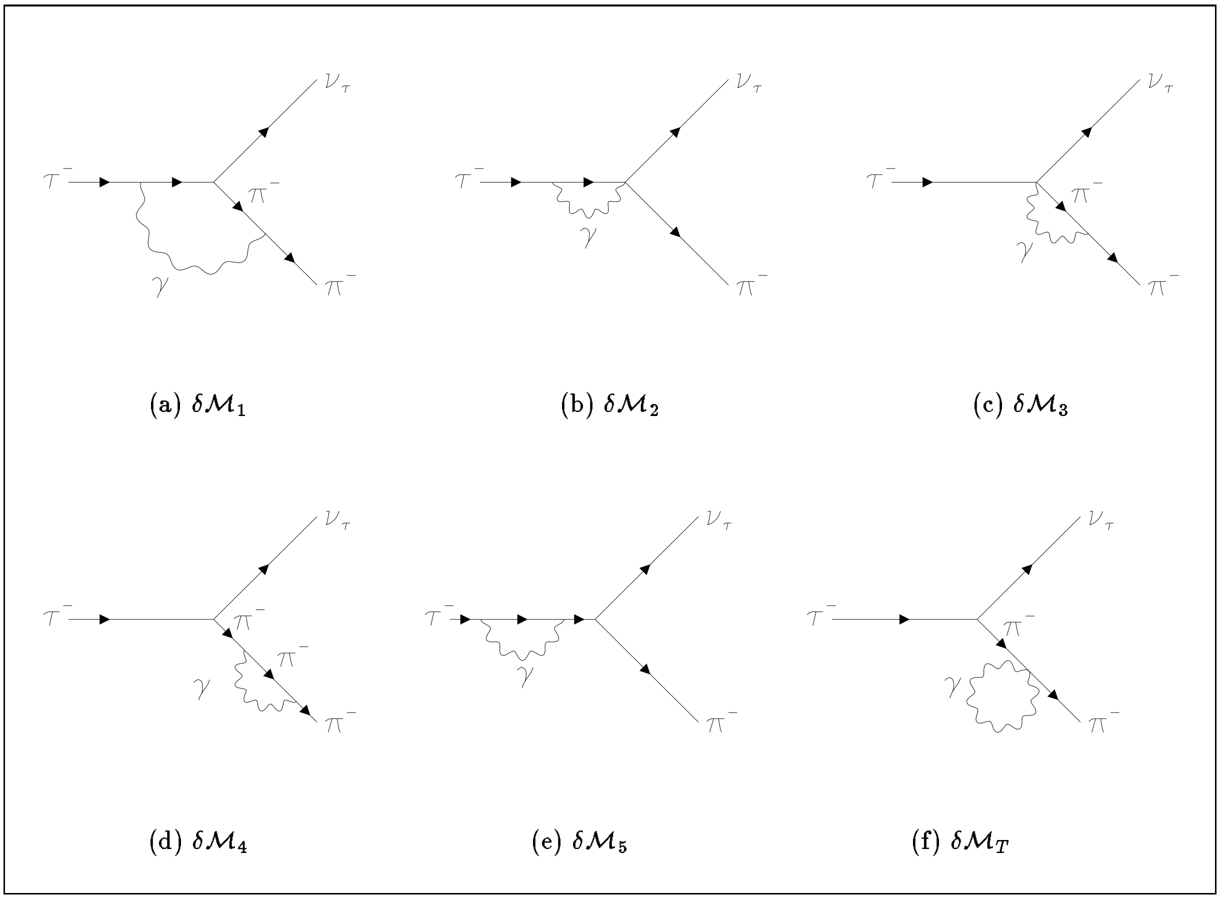
$$M_{a_1}^2 = m_{a_1}^2 - i m_{a_1}\Gamma_{a_1} \qquad M_{K_1}^2 = m_{K_1}^2 - i m_{K_1}\Gamma_{K_1} \tag{105}$$

and so on. In terms of the rate the corrections are then given by

$$\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_l^2, m_M^2, m^2) = 2\mathrm{Re}\left\{\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, m^2) + \frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2, m_M^2, m^2)\right\} \tag{106}$$

Now we will discuss diagram by diagram whether the coupling to the photon should be modified by vector meson dominance. Consider the effective point meson graphs in Fig. 11. In $\delta\mathcal{M}_1$, the coupling $\gamma\pi\pi$ is determined by the electromagnetic form factor of the pion, which is well know to be dominated by the ρ vector meson (see Fig. 12). The diagrams $\delta\mathcal{M}_3$ and $\delta\mathcal{M}_4$ cancel in the ratio $R_{\tau/\pi}$, so we will not include vector

Figure 11: Point pion diagrams



(a) $\delta\mathcal{M}_1$ (b) $\delta\mathcal{M}_2$ (c) $\delta\mathcal{M}_3$

(d) $\delta\mathcal{M}_4$ (e) $\delta\mathcal{M}_5$ (f) $\delta\mathcal{M}_T$

Figure 12: Vector meson dominance of coupling of the photon to the pion

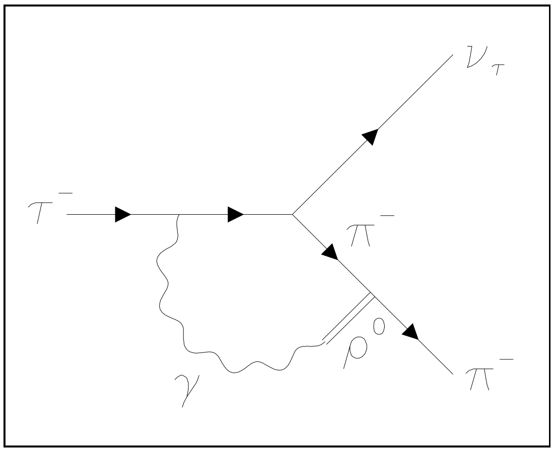
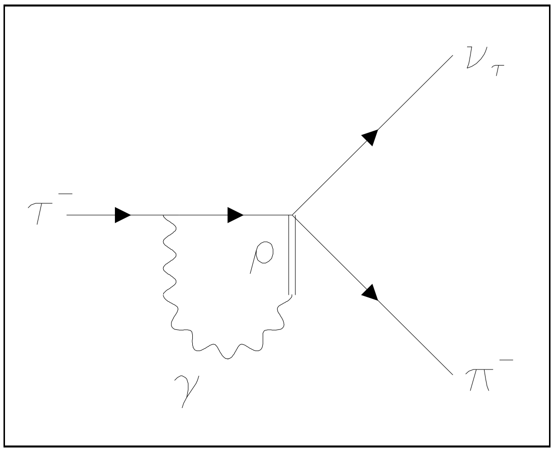


Figure 13: Vector meson dominance in the seagull coupling



meson dominance here. In diagram $\delta\mathcal{M}_5$ the photon couples to the tau only, so the only diagram which remains to be considered apart of $\delta\mathcal{M}_1$ is the diagram $\delta\mathcal{M}_2$ with the seagull coupling of the photon to the weak interaction vertex. Here it not clear at all whether this graph should be multiplied by a rho Breit Wigner (compare Fig. 13). However, care must be taken to insure gauge invariance. The individual diagrams are not gauge invariant, but their sum is. So the modification of $\delta\mathcal{M}_2$ must be made in such a way that the sum of the diagrams is gauge invariant. This imposition of gauge invariance determines that the correct modification of the diagram $\delta\mathcal{M}_2$ is given neither by the multiplication with 1 (ie. no vector meson dominance, VMD) nor by the multiplication with BW_ρ (complete VMD), but by the multiplication with

$$2\mathrm{BW}_\rho(k^2) - 1 \tag{107}$$

We will call this below the "seagull type VMD".

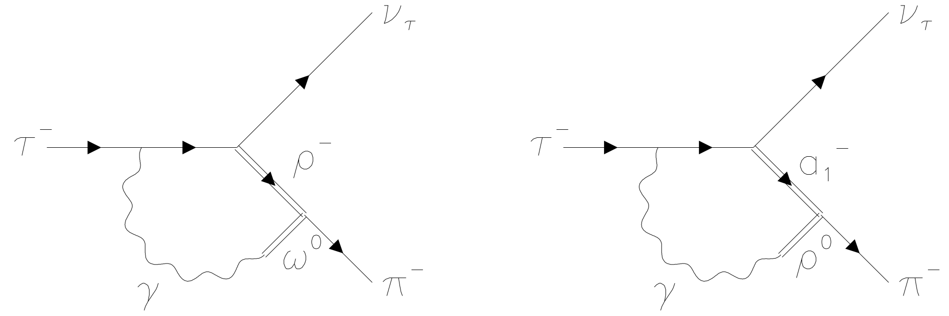
Next consider the corrections $\delta\mathcal{M}_6$ and $\delta\mathcal{M}_7$. In these diagrams the coupling of the photon could be dominated by the ω and the ρ mesons, respectively (see Fig. 14). We do not have any experimental information on whether or not these couplings are actually dominated by vector mesons. (An experimental test could be made by measuring the e^+e^- invariant mass spectra in the decays $\rho \to \pi e^+e^-$ and $a_1 \to \pi e^+e^-$, respectively.) By extrapolation from the experience with other hadronic couplings of the photon one could expect vector meson dominance here as well, but in order to be unprejudiced we will below consider both possibilities, complete VMD and no VMD.

Now according to Sec. 3, in the long distance diagrams a regularized photon propagator

$$\frac{1}{k^2 - \lambda^2} \to \frac{1}{k^2 - \lambda^2}\frac{\mu_{cut}^2}{\mu_{cut}^2 - k^2} \tag{108}$$

is to be used. If the photon additionally couples via vector meson dominance with a

Figure 14: Vector meson dominance of the photon coupling in the hadronic structure dependent diagrams



(a) $\delta\mathcal{M}_6'$ (b) $\delta\mathcal{M}_7'$

vector resonance mass M_R, the simple photon propagator has to be replaced by

$$\frac{1}{k^2-\lambda^2} \to \frac{1}{k^2-\lambda^2}\frac{\mu_{cut}^2}{\mu_{cut}^2-k^2}\frac{M_R^2}{M_R^2-k^2} \tag{109}$$

Assume a long distance diagram has been calculated with the photon propagator

$$\frac{1}{k^2-m^2} \tag{110}$$

and without vector meson dominance coupling of the photon, with the result $G(m^2)$. Then the following replacements have to be made in order to get the correct answer:

- long distance diagram without VMD

$$G(m^2) \to G(\lambda^2) - G(\mu_{cut}^2) \tag{111}$$

- long distance diagram with usual VMD

$$G(m^2) \to G(\lambda^2) - G(\mu_{cut}^2) + \frac{\mu_{cut}^2}{\mu_{cut}-M_R^2}\left[G(\mu_{cut}^2) - G(M_R^2)\right] \tag{112}$$

- long distance diagram with seagull type VMD

$$G(m^2) \to G(\lambda^2) - G(\mu_{cut}^2) + \frac{2\mu_{cut}^2}{\mu_{cut}-M_R^2}\left[G(\mu_{cut}^2) - G(M_R^2)\right] \tag{113}$$

8 Complete Radiative Correction

We write the complete radiative correction as

$$\delta R_{\tau/M} = (\delta R_{\tau/M})_{CPM} + (\delta R_{\tau/M})_{VMD(PML)} + (\delta R_{\tau/M})_{HSD} + (\delta R_{\tau/M})_{VMD(HSD)} + (\delta R_{\tau/M})_{sd} \tag{114}$$

using the naming conventions

- "CPM" = cut point meson, ie. the long distance correction due to an effective point meson, including real and virtual photonic correction, where the loops have been calculated with the regulated photon propagator with the cut-off scale μ^2_{cut}
- "VMD(PML)" = vector meson dominance of the point meson loops, ie. the correction of the CPM result due to the vector meson dominance of the meson electromagnetic form factor,
- "HSD" = hadronic structure dependent, ie. the correction due to the diagrams proportional to F_V and F_A, including real and virtual corrections,
- "VMD(SDL)" = vector meson dominance of the structure dependent loops, ie. the modification of the HSD result due to vector meson dominance coupling of the photon in these diagrams, and finally
- "sd", the short distance correction.

For the integrated rates of the real photon emission we divide into pure internal bremsstrahlung (IB) and the rest (SD + INT), viz. the sum of pure structure dependent radiation and of the interference between the internal bremsstrahlung and the structure dependent radiation.

And so

$$\begin{aligned}(\delta R_{\tau/M})_{CPM} &= \frac{\Gamma_{IB}(\tau\to M\nu_\tau\gamma)}{\Gamma_0(\tau\to M\nu_\tau)} - \frac{\Gamma_{IB}(M\to\mu\nu_\mu\gamma)}{\Gamma_0(M\to\mu\nu_\mu)} \\ &+\frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\tau^2, m_M^2, \lambda^2) - \frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\tau^2, m_M^2, \mu_{cut}^2) \\ &-\frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\mu^2, m_M^2, \lambda^2) + \frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\mu^2, m_M^2, \mu_{cut}^2)\end{aligned} \tag{115}$$

and

$$\begin{aligned}(\delta R_{\tau/M})_{HSD} &= \frac{\Gamma_{SD+INT}(\tau\to M\nu_\tau\gamma)}{\Gamma_0(\tau\to M\nu_\tau)} - \frac{\Gamma_{SD+INT}(M\to\mu\nu_\mu\gamma)}{\Gamma_0(M\to\mu\nu_\mu)} \\ &+\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\tau^2, m_M^2, \lambda^2) - \frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\tau^2, m_M^2, \mu_{cut}^2) \\ &-\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\mu^2, m_M^2, \lambda^2) + \frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\mu^2, m_M^2, \mu_{cut}^2)\end{aligned} \tag{116}$$

For the point meson loops the vector meson dominance is taken into account by

$$(R_{\tau/M})_{VMD(PML)} = \frac{\Gamma_{VMD(PML)}}{\Gamma_0}(m_\tau^2, m_M^2) - \frac{\Gamma_{VMD(PML)}}{\Gamma_0}(m_\mu^2, m_M^2) \tag{117}$$

where according to the last section

$$\begin{aligned}\frac{\delta\Gamma_{VMD(PML)}}{\Gamma_0}(m_l^2, m_M^2) = \frac{2}{1+\sigma_1+\rho_1}\mathrm{Re}\Bigg\{ \\
\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_\rho^2}\Bigg[\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) + 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) \\
-\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, M_\rho^2) - 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, M_\rho^2)\Bigg] \\
+\sigma_1\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_{\rho'}^2}\Bigg[\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) + 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) \\
-\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\rho'}^2) - 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\rho'}^2)\Bigg] \\
+\rho_1\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_{\rho''}^2}\Bigg[\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) + 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) \\
-\frac{\delta\mathcal{M}_1}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\rho''}^2) - 2\frac{\delta\mathcal{M}_2}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\rho''}^2)\Bigg]\Bigg\}\end{aligned} \tag{118}$$

where the parameters σ_1 and ρ_1 describe the relative contribution of the ρ' and the ρ'' in the electromagnetic form factor of the meson (compare σ and ρ in the vector form factor F_V). Note that for the case of the kaon, $m_M = m_K$, this assumes complete $U(3)$ flavour symmetry in the vector meson sector, $M_\rho = M_\omega = M_\phi$. Otherwise the relative contributions of the ρ^0, the ω and the Φ in the electromagnetic form factor of the kaon would have to be considered.

The vector meson dominance in the "hadronic structure dependent loops" (SDL) is implemented by

$$(R_{\tau/M})_{VMD(SDL)} = \frac{\Gamma_{VMD(SDL)}}{\Gamma_0}(m_\tau^2, m_M^2) - \frac{\Gamma_{VMD(SDL)}}{\Gamma_0}(m_\mu^2, m_M^2) \tag{119}$$

with

$$\begin{aligned}\frac{\delta\Gamma_{VMD(SDL)}}{\Gamma_0}(m_l^2, m_M^2) = \frac{2f_2}{1+\sigma_2+\rho_2}\mathrm{Re}\Bigg\{ \\
\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_\omega^2}\Bigg[\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) - \frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, M_\omega^2)\Bigg] \\
+\sigma_2\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_{\omega'}^2}\Bigg[\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) - \frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\omega'}^2)\Bigg] \\
+\rho_2\frac{\mu_{cut}^2}{\mu_{cut}^2 - M_{\omega''}^2}\Bigg[\frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, \mu_{cut}^2) - \frac{\delta\mathcal{M}_V}{\mathcal{M}_0}(m_l^2, m_M^2, M_{\omega''}^2)\Bigg]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{2f_3}{1+\sigma_3+\rho_3}\mathrm{Re}\Bigg\{ \\
&\quad\frac{\mu_{cut}^2}{\mu_{cut}^2-M_\rho^2}\left[\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,\mu_{cut}^2)-\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,M_\rho^2)\right] \\
&\quad+\sigma_3\frac{\mu_{cut}^2}{\mu_{cut}^2-M_{\rho'}^2}\left[\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,\mu_{cut}^2)-\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,M_{\rho'}^2)\right] \\
&\quad+\rho_3\frac{\mu_{cut}^2}{\mu_{cut}^2-M_{\rho''}^2}\left[\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,\mu_{cut}^2)-\frac{\delta\mathcal{M}_A}{\mathcal{M}_0}(m_l^2,m_M^2,M_{\rho''}^2)\right]\Bigg\} \qquad (120)
\end{aligned}
$$

where f_2 and f_3 are flags which are either one or zero, determining whether or not the photon coupling to the vector resonance and the pion or to the axial resonance and the pion are dominated by ω and ρ type resonances, respectively. The parameters σ_2 and ρ_2 for the ω type ones and σ_3 and ρ_3 for the ρ type ones describe the relative contributions of higher radial excitations.

From our results we can also calculate the correction $\delta R_{e/\mu}$ to the ratio of the electronic and the muonic decay modes of the pion:

$$
\begin{aligned}
\delta R_{e/\mu} &= (\delta R_{e/\mu})_{CPM}+(\delta R_{e/\mu})_{VMD(PML)}+(\delta R_{e/\mu})_{HSD} \\
&\quad+(\delta R_{e/\mu})_{VMD(HSD)}+(\delta R_{e/\mu})_{sd} \qquad (121)
\end{aligned}
$$

where $(\delta R_{e/\mu})_{CPM}$, $\ldots$, $(\delta R_{e/\mu})_{sd}$ are defined and calculated in a completely analogous way.

9 Numerical Results

We will now evaluate the formulae of the last section numerically and so get our final results. For the calculation of the standard loop integrals we have used the numerical programs FF [24] and AA/FF [25].

Uncertainties of the final result come from two different sources: on the one hand from uncertainties in the hadronics and on the other hand from uncertainties in the matching of long and short distances. We estimate their sizes by varying the various parameters involved, ie. $F_A^\pi(0)$, relative contributions of higher radial resonances and others for the hadronics and by varying μ_{cut} for the matching. We obtain a central value with an error estimate in the following way. We discuss the various parameters and find for each one of them a central value and some reasonable range over which we will vary them. Taking all parameters at their central values, we obtain the central value for the total radiative correction. We then vary the various parameters and determine which choices give the smallest and the largest radiative correction, respectively. Then by simultaneously taking those values which result in the largest (smallest) correction, we obtain an upper (lower) limit for the radiative correction.

For the parameters of the hadronics, we obtain in this way three parameter sets (I)–(III), we are given is the tables 1, 2 and 3, corresponding to the central values (I) and to the lower (II) and upper limits (III) on the total radiative correction.

Table 1: Parameter sets for $R_{\tau/\pi}$

Parameter	I	II	III
$F_A(0) =$	0.0116	0.0100	0.0132
Γ_{a_1} [MeV] =	400	600	250
$\sigma =$	0.136	0.0584	0.000
$\rho =$	-0.051	0.0	0.000
widths in real radiation: F_V	energy dependent	energy dependent	fixed
widths in real radiation: F_A	energy dependent	fixed	energy dependent
$\sigma_1 =$	-0.1	-0.1	0
$\rho_1 =$	-0.04	-0.04	0
$f_2 = f_3 =$	1	0	1
$\sigma_2 = \sigma_3 =$	0	—	-0.1
$\rho_2 = \rho_3 =$	0	—	-0.04

For the scale μ_{cut} the range $\mu_{cut} = 1 \ldots 2$ GeV is reasonable and we will use $\mu_{cut} = 1.5$ GeV as an intermediate standard value.

If all parameters have their standard values we find the total radiative correction

$$R_{\tau/\pi} = 0.16\% \tag{122}$$

Because of all the ambiguities of the non-perturbative strong interactions in the $O(1$ GeV$)$ regime, it is interesting to consider the result which is obtained if in the long distance part only the point meson is used, without vector meson dominance and without hadronic structure dependent loops. This point pion contribution is fixed by the low energy theorems of QCD and therefore free from any ambiguities. However, in this case the result depends very strongly on the value of the scale μ_{cut}. This is shown in Fig. 15 where we show the correction $\delta R_{\tau/\pi}$ in variation with the scale μ_{cut} both for our complete calculation, using standard parameters, and for the calculation where in the long distance part only the point meson is taken into account. In the last case the dependence on μ_{cut} is very strong. As has been discussed before, the pure point meson is reliable only for $\mu_{cut}^2 \ll m_\rho^2$, but the short distance correction only for $\mu_{cut} \gtrsim (1 \ldots 2)$ GeV. So here a very large range for μ_{cut} has to be considered. For a small μ_{cut} the correction $\delta R_{\tau/\pi}$ could even become negative, and at $\mu_{cut} = 2$ GeV the correction is about $\delta R_{\tau/\pi} = 0.7\%$ and still rising strongly.

This large dependence on μ_{cut} results from the incomplete treatment of the long distance part and hints to the necessicity of improving the model. The inclusion of VMD and of the structure dependent loops decreases the dependence on μ_{cut} very much, as can be seen from the other curve in Fig. 15. Above $\mu_{cut} = 2$ GeV the curve becomes almost completely flat, and the variation in the (relevant) range $\mu_{cut} = (1 \ldots 2)$ GeV is smaller than $\pm 0.05\%$. In Tab. 4 we show the different contributions which add up to the total radiative correction, using $\mu_{cut} = 1.5$ GeV. The individual contributions involve moderately large logarithms such as $\ln(m_\tau/m_\pi)$ or $\ln(m_\rho/m_\tau)$. However they

Table 2: Parameter sets for $R_{\tau/K}$

Parameter	I	II	III
$F_A^{(K)}(0) =$	0.0525	0.0425	0.0625
Γ_{K_1} [MeV] $=$	90	110	70
$\sigma_K =$	0.000	0.136	0.000
$\rho_K =$	0.000	-0.051	0.000
widths in real radiation	fixed	fixed	fixed
$\sigma_1 =$	0	-0.1	0
$\rho_1 =$	0	-0.04	0
$f_2 = f_3 =$	1	0	1
$\sigma_2 = \sigma_3 =$	0	—	-0.1
$\rho_2 = \rho_3 =$	0	—	-0.04

have opposite signs, such that most of the corrections cancel and only a very small total radiative correction of $\delta R_{\tau/\pi} = +0.16\%$ is left.

In Fig. 18 we show the long and short distance corrections in variation with μ_{cut}. We display separately the correction due to the effective point meson with vector meson dominance coupling in the photon coupling

$$(\delta R_{\tau/M})_{CPM} + (\delta R_{\tau/M})_{VMD(PML)} \tag{123}$$

the hadronic structure dependent correction with vector meson dominance

$$(\delta R_{\tau/M})_{HSD} + (\delta R_{\tau/M})_{VMD(HSD)} \tag{124}$$

and the short distance correction

$$(\delta R_{\tau/M})_{sd} \tag{125}$$

It can be seen clearly that the long and the short distance corrections vary in the opposite way with μ_{cut}, such that the dependence of the total radiative correction on μ_{cut} is significantly smaller than that of the individual corrections separately. This is a sensible result. If our long and short distance corrections would exactly describe the real world within an overlapping region of μ_{cut}, the sum of the two would be independent of μ_{cut} within this overlap region. The small remaining dependence of our final result on μ_{cut} within the range $\mu_{cut} = 1.0 \ldots 2.0$ GeV indicates that our model is not unreasonable.

In Fig. 16 we show $\delta R_{\tau/\pi}$ in variation with μ_{cut} for three different choices for the parameters of the resonance physics, viz. for the standard set (I) and for the set (II) which gives the smallest correction and for the set (III), which gives the largest correction (see Tab. 1). All the three curves have been obtained using real vector meson masses (narrow width approximation) in the virtual corrections, but we have compared with the results obtained with complex vector meson masses, and in all cases the difference is extremely small and completely negligible. While the two curves for (I) and (II) are close together, the curve for (III) lies significantly higher. The large difference is due to the question whether or not the photon in the hadronic structure dependent loops couples

Table 3: Parameter sets for $R_{e/\mu}$

Parameter	I	II	III
$F_A(0) =$	0.0116	0.0100	0.0132
Γ_{a_1} [MeV] =	400	600	250
$\sigma =$	0.136	0.0584	0.000
$\rho =$	-0.051	0.0	0.000
widths in real radiation: F_V	energy dependent	energy dependent	fixed
widths in real radiation: F_A	energy dependent	fixed	energy dependent
$\sigma_1 =$	-0.1	0	-0.1
$\rho_1 =$	-0.04	0	-0.04
$f_2 = f_3 =$	1	1	0
$\sigma_2 = \sigma_3 =$	0	-0.1	—
$\rho_2 = \rho_3 =$	0	-0.04	—

Figure 15: Radiative correction to $R_{\tau/\pi}$: Complete prediction using the standard parameter set (solid) and prediction for short distance plus point pion only, ie. $(\delta R_{\tau/\pi})_{CPM} + (\delta R_{\tau/\pi})_{sd}$ (dashed)

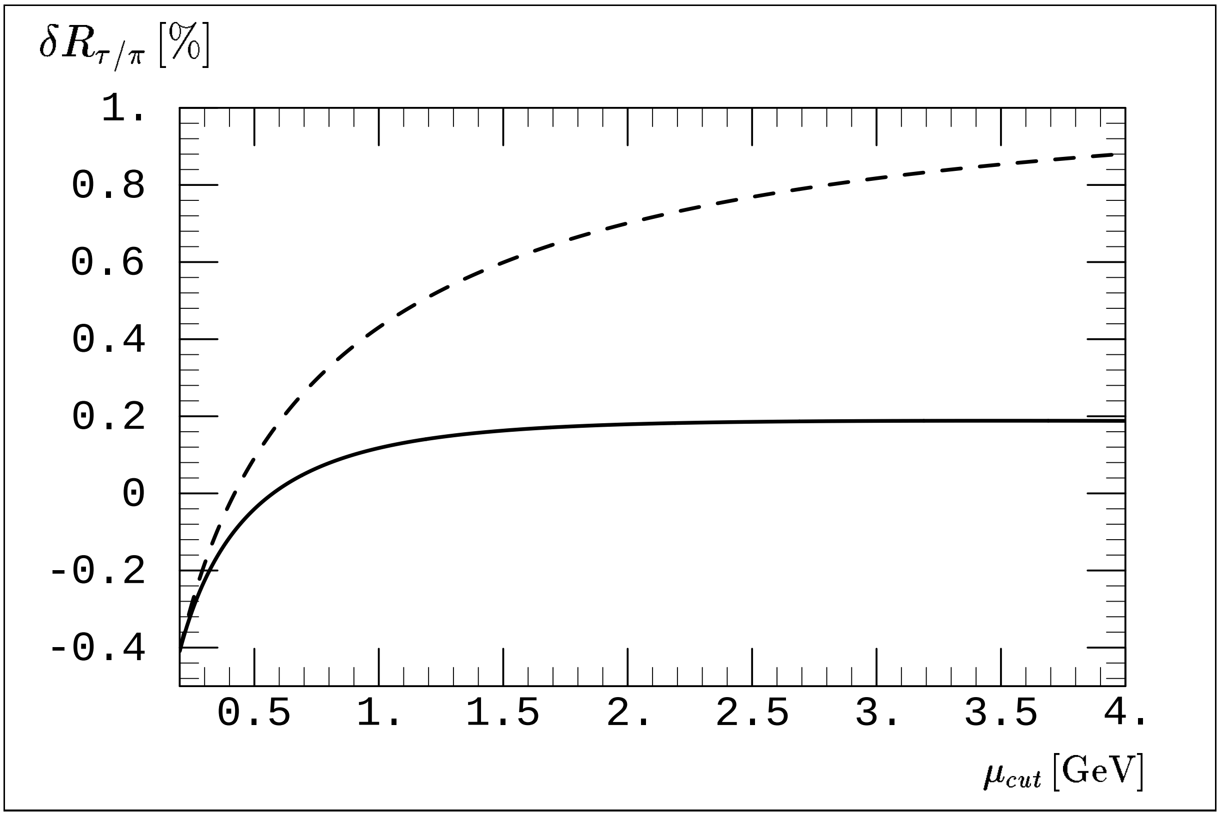
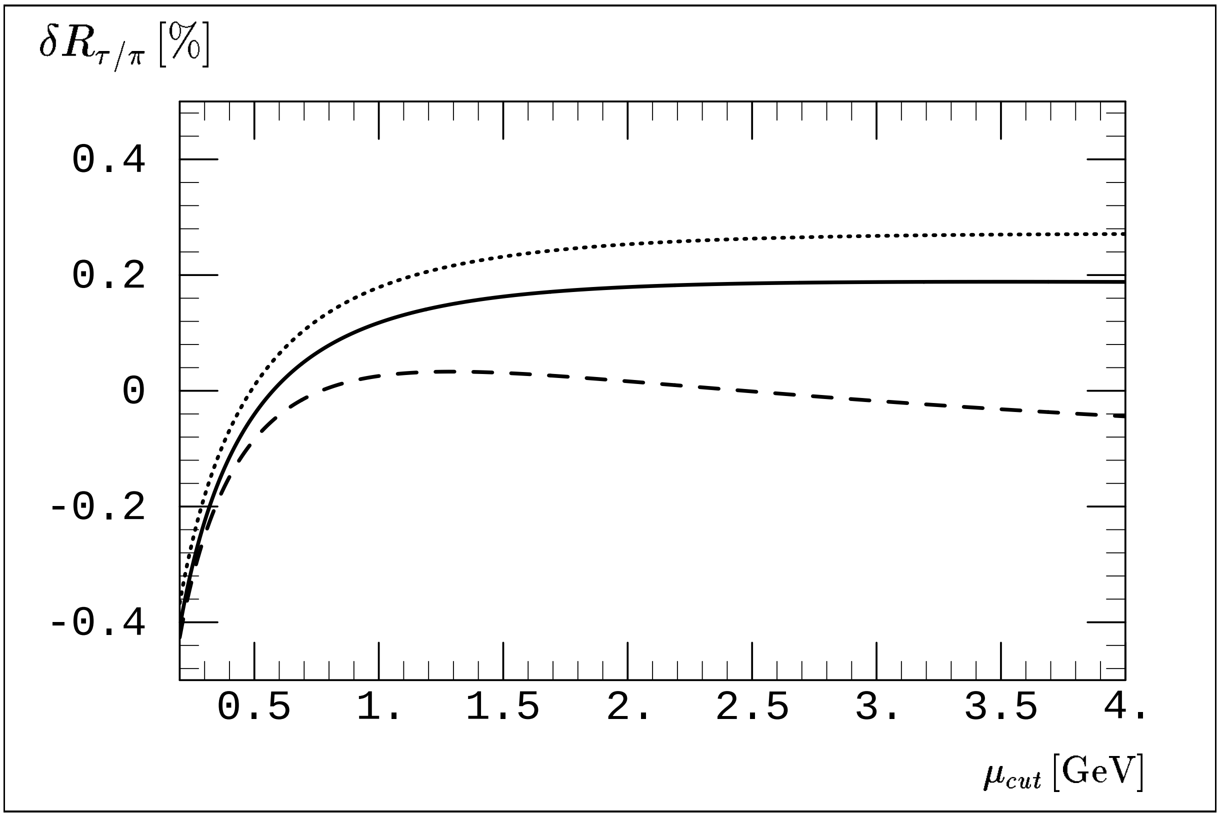


Table 4: The different contributions adding up to the total radiative correction $\delta R_{\tau/\pi}$ for $\mu_{cut} = 1.5$ GeV

effective point pion	$(\delta R_{\tau/\pi})_{PM}$	+1.05 %
cutting off the point pion loops at μ_{cut}	$-\frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\tau^2, m_\pi^2, \mu_{cut}^2)$ $+\frac{\delta\Gamma_{PML}}{\Gamma_0}(m_\mu^2, m_\pi^2, \mu_{cut}^2)$	−0.21%
vector meson dominance of the pion electromagnetic form factor	$(R_{\tau/\pi})_{VMD(PML)}$	−0.38%
Structure dependent radiation and SD-IB interference (real photon emission)	$\frac{\Gamma_{SD+INT}(\tau \to \pi\nu_\tau\gamma)}{\Gamma_0(\tau \to \pi\nu_\tau)}$ $-\frac{\Gamma_{SD+INT}(\pi \to \mu\nu_\mu\gamma)}{\Gamma_0(\pi \to \mu\nu_\mu)}$	+0.05%
hadronic structure dependent loops, cut off at μ_{cut}	$\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\tau^2, m_\pi^2, 0)$ $-\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\tau^2, m_\pi^2, \mu_{cut}^2)$ $-\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\mu^2, m_\pi^2, 0)$ $+\frac{\delta\Gamma_{SDL}}{\Gamma_0}(m_\mu^2, m_\pi^2, \mu_{cut}^2)$	−0.24%
vector meson dominance of photon couplings in the hadronic structure dependent loops	$(\delta R_{\tau/\pi})_{VMD(SDL)}$	+0.13%
short distance contribution for $k^2 > \mu_{cut}^2$	$(\delta R_{\tau/\pi})_{sd}$	−0.25%
sum	$\delta R_{\tau/\pi}$	+0.16%

Figure 16: The total radiative correction $\delta R_{\tau/\pi}$ for different choices for the parameters of the structure dependent correction: Standard choice (I) (solid), choice (II) (dashed) and choice (III) (dotted)



to the mesons via vector meson dominance, all other parameter variations have a much smaller influence.

As our final result on $\delta R_{\tau/\pi}$ we get from the parameter sets (I) — (III) and from varying $\mu_{cut} = 1.0 \ldots 2.0$ GeV:

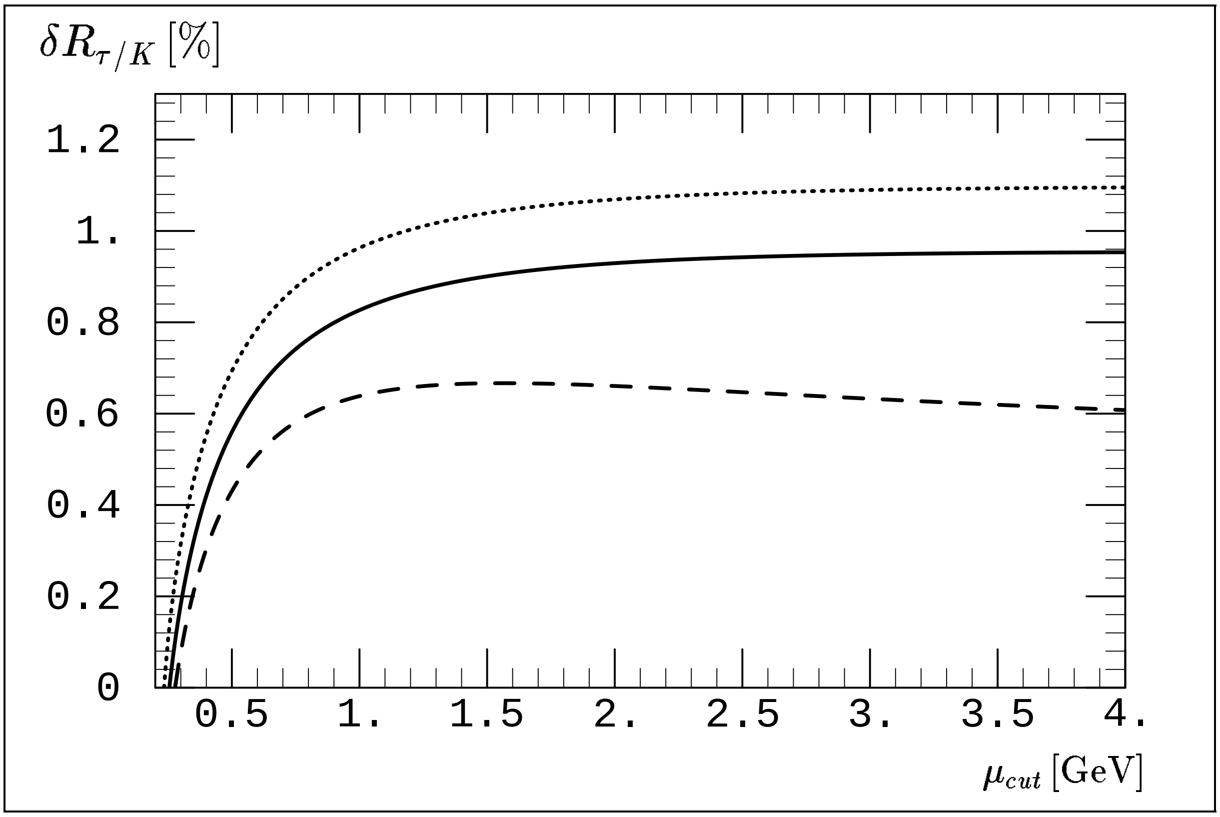
$$\boxed{\delta R_{\tau/\pi} = \left(0.16^{+0.09}_{-0.14}\right)\%} \tag{126}$$

In Fig. 17 we show $\delta R_{\tau/K}$ in variation with μ_{cut} for the three parameter sets (I)–(III) which are defined in Tab. 2. Our final result for the correction to $R_{\tau/K}$ is

$$\boxed{\delta R_{\tau/K} = \left(0.90^{+0.17}_{-0.26}\right)\%} \tag{127}$$

where the central value is for the parameter set (I) and with $\mu_{cut} = 1.5$ GeV, and the lower and upper limits are from the sets (II) and (III) and $\mu_{cut} = 1.0$ GeV, 2.0 GeV, respectively.

Figure 17: The total radiative correction $R_{\tau/K}$ for different choices for the parameters in the structure dependent correction: Standard choice (I) (solid), choice (II) (dashed) and choice (III) (dotted)

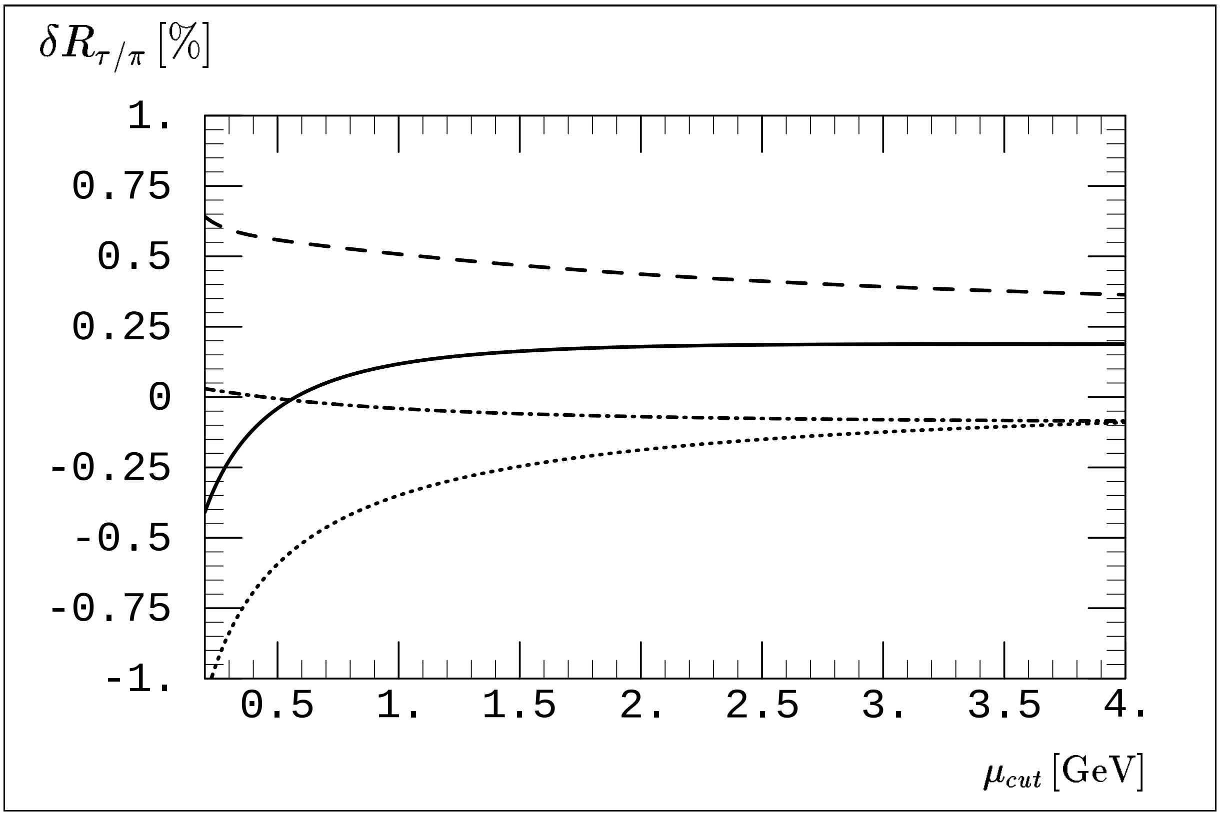


For the normalized branching ratios this results in

$$\begin{aligned}
\frac{\mathrm{BR}_\pi}{\mathrm{BR}_e} &= 0.6129 \pm \overbrace{0.0007}^{expt} \overbrace{{}^{+0.0005}_{-0.0009}}^{theo} = 0.6129^{+0.0009}_{-0.0011} \\
\frac{\mathrm{BR}_K}{\mathrm{BR}_e} &= 0.0406 \pm \overbrace{0.0002}^{expt} \overbrace{{}^{+0.0000}_{-0.0001}}^{theo} = 0.0406 \pm 0.0002 \\
\Rightarrow \frac{\mathrm{BR}_{\pi+K}}{\mathrm{BR}_e} &= 0.6535 \pm \overbrace{0.0007}^{expt} \overbrace{{}^{+0.0005}_{-0.0010}}^{theo} = 0.6535^{+0.0009}_{-0.0012} \qquad (128)
\end{aligned}$$

The first errors given (called "experimental") are due to the uncertainties in the lifetimes of the mesons and in the tau mass m_τ, and the second errors (called "theoretical") are due to the uncertainties in the radiative correction. This final result deviates from the

Figure 18: The different contributions which add up to the total correction: The point pion correction, supplemented with vector meson dominance in the photon coupling (dashed), the hadronic structure dependent correction with vector meson dominance in the photon coupling (dash-dotted), short distance correction (dotted) and their sum, the total correction (solid)



experimental result [26] given in the introduction

$$\left(\frac{\mathrm{BR}_{\pi+K}}{\mathrm{BR}_e}\right)_{exp} = \frac{(11.99 \pm 0.25)\%}{(17.76 \pm 0.15)\%} = 0.675 \pm 0.015 \tag{129}$$

by 1.4 standard deviations. The agreement between theory and experiment is not significantly enhanced by the inclusion of the $O(\alpha)$ corrections to the decay rate for $\tau \to \pi\nu_\tau$. Thus if the standard model is correct, either the current experimental number for the branching ratio $\mathrm{BR}_{\pi+K}$ is slightly too large or the one for BR_e slightly too small, or both.

Some comments on the reliability of our matching procedure are in order. First it is important to note that indeed the dependence of the long and short distance corrections separately on μ_{cut} is considerably larger than that of their sum, as has been discussed above in connection with Fig. 18. Second we have performed the matching in a certain way be splitting the photon propagator into a long and a short distance part according to Eqn. (35). This corresponds to a soft transition from long to short distances. Another way to perform the matching would be a sharp transition from long to short distances at μ_{cut}, by integrating the long distances from $k_E^2 = 0 \ldots \mu_{cut}^2$ and the short distances from $k_E^2 = \mu_{cut}^2 \ldots m_Z^2$. In principle the results obtained with this method might differ somewhat from our results. However, the uncertainty of our final result for $\delta R_{\tau/\pi}$ is dominanted by the hadronic uncertainties and not by the matching uncertainties. Varying the parameters of the hadronics and the matching scale μ_{cut} separately, we obtain

$$\delta R_{\tau/\pi} = \left(0.16 \; \overbrace{\begin{matrix} +0.02 \\ -0.05 \end{matrix}}^{matching} \; \overbrace{\begin{matrix} +0.07 \\ -0.13 \end{matrix}}^{hadronics} \right) \% \tag{130}$$

which clearly displays the dominance of the hadronic uncertainties. Therefore we think that the precise procedure for performing the matching is not essential. Still it might be an interesting task to repeat the calculation with a matching based on a sharp transition at μ_{cut}.

Now we will use our results to predict for the pion decay the ratio $R_{e/\mu}$ of the electronic and muonic modes. In Fig. 19 we show $\delta R_{e/\mu}$ in variation with μ_{cut} for three parameter sets (I)—(III) defined in Tab. 3. We find for the total radiative correction

$$\delta R_{e/\mu} = (-3.79 \pm 0.01)\,\% \tag{131}$$

But note that this error, which is due to the uncertainties in μ_{cut} and in the hadronic resonance physics, is smaller than the one which is to be expected because of the neglect higher order corrections of $O(\alpha^2)$, as we will explain below.

The value in Eqn. (131) results in

$$\begin{aligned} R_{e/\mu} &= R_{e/\mu}^{(0)}\Big(1 + \delta R_{e/\mu}\Big) \\ &= \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \Big(1 + \delta R_{e/\mu}\Big) \\ &= 1.2835 \cdot 10^{-4}\Big(1 + (-3.79 \pm 0.01)\,\%\Big) \\ &= (1.2349 \pm 0.0001) \cdot 10^{-4} \end{aligned} \tag{132}$$

where the error given includes only the uncertainties resulting from the hadronic structure dependent corrections and the matching scale μ_{cut} and misses the comparable uncertainty resulting from the neglect of the $O(\alpha)$ corrections.

This result is to be compared with the recent result given by Marciano and Sirlin [8]:

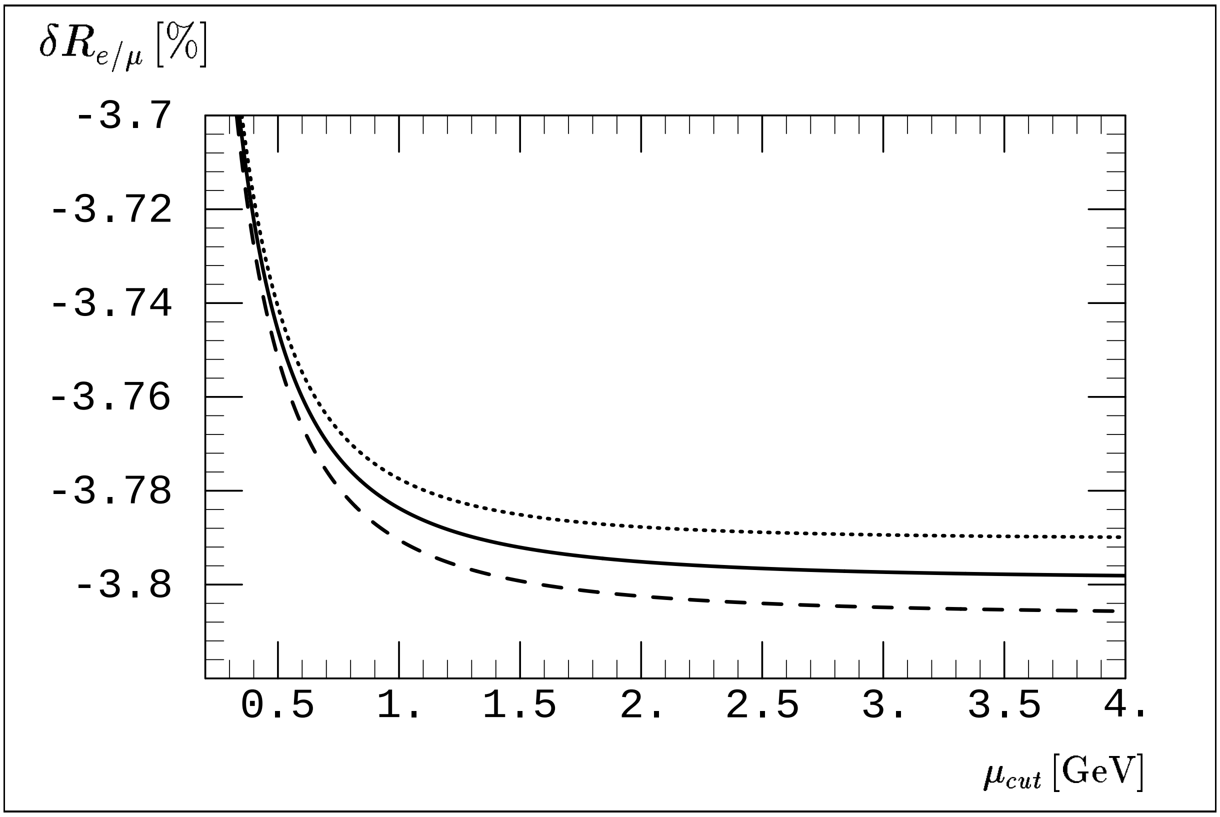
$$R_{e/\mu} = (1.2352 \pm 0.0005) \cdot 10^{-4} \tag{133}$$

which is equivalent to

$$\delta R_{e/\mu} = (-3.76 \pm 0.04)\% \tag{134}$$

Their calculation is rather similar to ours but differs in some details. In the short distance corrections they use the leading logarithms only, supplemented by leading QCD

Figure 19: The total radiative correction $R_{e/\mu}$ for different choices for the parameters in the structure dependent correction: Standard choice (I) (solid), choice (II) (dashed) and choice (III) (dotted)



corrections and furthermore by summing up the leading logarithms to all orders in α via the renormalization group. But since the short distance corrections cancel almost completely in $R_{e/\mu}$, these differences are not important. For the long distance corrections they also consider both the effective point pion field and hadronic structure dependent corrections. However, for the latter they do not use a specific model but rather use a theorem given by Terent'ev [27] on the leading lepton mass dependent hadronic structure effects of the order $\frac{\alpha}{\pi}\frac{m_l^2}{m_\rho^2}\ln\frac{m_\rho^2}{m_l^2}$. The possible size of the remaining hadronic structure dependent corrections, which they do not calculate, represents the main source of the uncertainty in their $R_{e/\mu}$ as quoted above. The authors also consider the effect of corrections of higher order in α. Since the correction $3(\alpha/\pi)\ln(m_\mu/m_e) \approx -3.7\%$, which results from the lepton mass singularities, dominates the $O(\alpha)$ correction, one expects its higher order counterparts to similarly dominate their respective orders. Summing all such logarithms via the renormalization group gives the enhancement factor

$$\frac{\left(1-\frac{2\alpha}{3\pi}\ln\frac{m_\mu}{m_e}\right)^{9/2}}{1-\frac{3\alpha}{\pi}\ln\frac{m_\mu}{m_e}} = 1.00055 \tag{135}$$

which multiplies $R_{e/\mu}$. This implies that the $O(\alpha^2)$ correction to $R_{e/\mu}$ is of the order of 0.05%.

If we multiply our result with this enhancement factor (see Eqn. (135)), we get

$$R_{e/\mu} = (1.2356 \pm 0.0001) \cdot 10^{-4} \tag{136}$$

or

$$\delta R_{e/\mu} = (-3.74 \pm 0.01)\,\% \tag{137}$$

which agrees with the Marciano-Sirlin result within their error bars.

10 Summary and Conclusions

We have calculated the radiative corrections to the decays $\tau \to M\nu_\tau$ and to $M \to l\nu_l$ ($M = \pi$ or K, $l = e$ or μ). The central issue of this paper was the treatment of the strong interaction. The amplitude of the radiative decays with the emission of a real photon can be divided into the amplitude for internal bremsstrahlung (IB) and the amplitude for structure dependent radiation (SD). In the virtual corrections one has to integrate over the momentum k of the virtual photon, and therefore all the three energy regimes of the strong interaction have to be taken into account. For small k^2 we have contributions from the long distance correction, which consists of the point meson contribution and of the hadronic structure dependent part. The short distance corrections, which we have calculated using the parton model, contribute in the large k^2 region.

Our final result for the radiative correction $\delta R_{\tau/\pi}$ to the ratio $\Gamma(\tau \to \pi\nu_\tau(\gamma))/\Gamma(\pi \to \mu\nu_\mu(\gamma))$ is

$$\delta R_{\tau/\pi} = \left(0.16^{+0.09}_{-0.14}\right)\%$$

and for the ratio $\Gamma(\tau \to K\nu_\tau(\gamma))/\Gamma(K \to \mu\nu_\mu(\gamma))$ we obtain

$$\delta R_{\tau/K} = \left(0.90^{+0.17}_{-0.26}\right)\%$$

Note that these numbers are calculated by summing up virtual, soft and hard photonic corrections.

We can translate the radiative corrections into predictions for the branching ratios

$$\begin{aligned} \mathrm{BR}(\tau \to \pi\nu_\tau(\gamma) &= (11.10 \pm 0.02)\% \times \left(\frac{\tau_\tau}{295.7\ \mathrm{fs}}\right) \\ \mathrm{BR}(\tau \to K\nu_\tau(\gamma) &= (0.737 \pm 0.005)\% \times \left(\frac{\tau_\tau}{295.7\ \mathrm{fs}}\right) \\ \mathrm{BR}(\tau \to h\nu_\tau(\gamma) &= (11.84 \pm 0.02)\% \times \left(\frac{\tau_\tau}{295.7\ \mathrm{fs}}\right) \end{aligned} \tag{138}$$

where h denotes the inclusive sum of pions and kaons.

For the ratio $R_{e/\mu}$ of the electronic and muonic decay modes of the pion, we obtain a radiative correction of

$$\delta R_{e/\mu} = (-3.74 \pm 0.01)\,\% \tag{139}$$

resulting in

$$R_{e/\mu} = (1.2356 \pm 0.0001) \cdot 10^{-4} \tag{140}$$

Acknowledgement

One of us (M.F.) would like to thank J.H. Kühn for helpfull discussions.

Appendix

Except for the tau mass we use the standard particle data of [12] for masses and widths. In the case of the tau mass, we use

$$m_\tau = (1777.1 \pm 0.5)\ \mathrm{MeV} \tag{141}$$

In this paper we use the notation of Bjorken and Drell [28], especially

$$\epsilon_{0123} = +1 \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{142}$$

and the e is the charge of the electron, $e < 0$. The standard loop integrals are defined by

$$\begin{aligned}
&\frac{i}{16\pi^2} A_0(m_0) \\
&\qquad = \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m_0^2} \\
&\frac{i}{16\pi^2} [B_0 \mid B_\mu \mid B_{\mu\nu}](p_1, m_0, m_1) \\
&\qquad = \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{[1 \mid k_\mu \mid k_\mu k_\nu]}{(k^2 - m_0^2)[(k+p_1)^2 - m_1^2]} \\
&\frac{i}{16\pi^2} [C_0 \mid C_\mu \mid C_{\mu\nu}](p_1, p_2, m_0, m_1, m_2) \\
&\qquad = \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{[1 \mid k_\mu \mid k_\mu k_\nu]}{(k^2 - m_0^2)[(k+p_1)^2 - m_1^2][(k+p_2)^2 - m_2^2]}
\end{aligned} \tag{143}$$

The scalar functions depend on invariant combinations of the momenta only,

$$\begin{aligned}
B_0(p_1, m_0, m_1) &\equiv B_0(p_1^2, m_0, m_1) \\
C_0(p_1, p_2, m_0, m_1, m_2) &\equiv C_0(p_1^2, (p_1 - p_2)^2, p_2^2, m_0, m_1, m_2)
\end{aligned} \tag{144}$$

and the vector and tensor integrals are decomposed covariantly in the form

$$\begin{aligned} B^{\mu}(p_1,m_0,m_1) &= B_1 p_1^{\mu} \\ B^{\mu\nu}(p_1,m_0,m_1) &= B_{00} g^{\mu\nu} + B_{11} p_1^{\mu} p_1^{\nu} \\ C^{\mu}(p_1,p_2,m_0,m_1,m_2) &= C_1 p_1^{\mu} + C_2 p_2^{\mu} \\ C^{\mu\nu}(p_1,p_2,m_0,m_1,m_2) &= C_{00} g^{\mu\nu} + C_{11} p_1^{\mu} p_1^{\nu} + C_{22} p_2^{\mu} p_2^{\nu} + C_{12} \left(p_1^{\mu} p_2^{\nu} + p_2^{\mu} p_1^{\nu}\right) \end{aligned} \quad (145)$$

where

$$\begin{aligned} B_i &= B_i(p_1^2, m_0, m_1) \\ B_{ij} &= B_{ij}(p_1^2, m_0, m_1) \\ C_i &= C_i(p_1^2, (p_1-p_2)^2, p_2^2, m_0, m_1, m_2) \\ C_{ij} &= C_{ij}(p_1^2, (p_1-p_2)^2, p_2^2, m_0, m_1, m_2) \end{aligned} \quad (146)$$

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