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DISTRIBUTIONS OF LEPTONS IN DECAYS OF POLARISED HEAVY QUARKS *

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Abstract

Analytic formulae are given for QCD corrections to the lepton spectra in decays of polarised up and down type heavy quarks. These formulae are much simpler than the published ones for the corrections to the spectra of charged leptons originating from the decays of unpolarised quarks and polarised up type quarks. Distributions of leptons in semileptonic Λ_c and Λ_b decays can be used as spin analysers for the corresponding heavy quarks. Thus our results can be applied to the decays of polarised charm and bottom quarks. For the charged leptons the corrections to the asymmetries are found to be small in charm decays whereas for bottom decays they exhibit a non-trivial dependence on the energy of the charged lepton. Short life-time enables polarisation studies for the top quark. Our results are directly applicable for processes involving polarised top quarks.

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1 Introduction

Inclusive semileptonic decays of heavy flavours play important role in present day particle physics. In near future with increasing statistics at LEP and good prospects for B-factories quantitative description of these processes may offer the most interesting tests of the standard quantum theory of particles. At the high energy frontier semileptonic decays of the top quark will be instrumental in establishing its properties [1, 2].

Recent theoretical developments, discovery of Heavy Quark Symmetries [3, 4] and Effective Theory [5] as well as subsequent applications of an expansion in inverse powers of the heavy quark mass m_Q [6], have led to a consistent treatment of semileptonic decays of charmed and beautiful hadrons. In the limit $m_Q \to \infty$ the inclusive decays of heavy flavour hadrons are described by the decays of the corresponding heavy quarks. For finite quark masses there arise non-perturbative corrections due to the heavy quark motion inside the heavy hadrons [7]. A systematic approach to these corrections based on $1/m_Q$ expansion is described in [6] and other articles cited in [4]. In the present article we consider only perturbative QCD corrections to the decays of heavy quarks. Non-perturbative effects have to be treated according to the recipies provided by the simple model of [7] or by the more formal approach of [6]. On the other hand inclusion of perturbative corrections is necessary for a quantitative description of semileptonic decays because these corrections are of the order of 20 percent.

Interesting new opportunities are provided by decays of polarised heavy quarks. It is well known [1, 2] that the heavy quarks produced in Z^0 decays are polarised. According to the Standard Model the degree of longitudinal polarisation is fairly large, amounting to $\langle P_b \rangle = -0.94$ for b and $\langle P_c \rangle = -0.68$ for c quarks [1]. The polarisations depend weakly on the production angle. QCD corrections to the production cross section have been calculated recently [8]. Unfortunately, this original high polarisation is to large extend washed out during hadronisation into mesons. At the time being only charmed and beautiful Λ baryons seem to offer a practical method to measure the polarisation of the corresponding heavy quarks. This method proposed by Bjorken [9] has been successfully applied to both semileptonic [10] and hadronic [11] decays of Lambda baryons. The polarisation transfer from a heavy quark Q to the corresponding Λ_Q baryon is 100% [12], at least in the limit $m_Q \to \infty$. In view of the growing sample of Λ_c and Λ_b baryons produced at LEP there arises an opportunity to measure the polarisation of the c and b quarks originating from the decays of the Z bosons. The angular distributions of charged leptons [13, 14] and neutrinos [15] from semileptonic decays of Λ_c and Λ_b can be used as spin analysers for the decaying heavy quarks. Polarisation studies are much more direct for the top quark which is a short lived object and decays before hadronisation takes place.

Angular distributions of leptons from semileptonic decays of the top quark can provide a very interesting information on the space-time structure of the corresponding electroweak current [16].

First order QCD corrections to the energy spectra of charged leptons in the decays of c and b quarks have been calculated analytically in ref.[17] and [18]. The results of a later analytical calculation presented in ref.[19] disagree with those of [17] and [18] and agree with a Monte Carlo for charm and bottom [20] as well as with numerical results for top [21]. In [21] the formulae from [20] were used and the accuracy achieved was good enough to observe discrepancies with [17] and [18]. The formulae for the virtual corrections used in [20] and [21] were taken from the classic articles on muon decays [22].

The formulae given in the present article agree with the ones given in [19] but are much simpler. They agree also with those given in [23] for the joint angular and energy charged lepton distribution in top quark decays. In the latter case simplification is even more striking. Some numerical results for polarised charm and bottom quarks have been already presented in [15].

In section 2 the formulae are given for the QCD corrected distributions of leptons originating from weak decays of polarised heavy quarks: subsection 2.1 contains our main result – the triple differential angular and energy distributions of leptons, and in subsections 2.2 and 2.3 the double differential distributions are described. The corresponding formulae for a massless quark in the final state are presented in section 3. In section 4 QCD corrections to the angular distributions in charm and bottom decays are presented. For the sake of completeness we summarize our calculations based on [19] and [23] in Appendix A.

2 Angular and energy distributions

In this section we give the formulae [1] for the distributions of leptons from the weak decays of a polarised heavy quark Q of mass m_1 and the weak isospin $I_3 = \pm 1/2$. The mass of the quark q originating from the decay is denoted by m_2, and $\epsilon = m_2/m_1$. These formulae are written in the Q rest frame. However, the two variables which we use, namely the scaled energy $x = 2\ell^0/m_1 = 2Q\cdot\ell/m_1^2$, where ℓ is the four-momentum of the charged lepton, and the scaled effective mass squared of the leptons $y = (\ell+\nu)^2/m_1^2$ are Lorentz invariants. For the massless lepton case considered in the present article also the third variable $\cos\theta$, can be related to the scalar product $\ell\cdot s$, where $s^\mu = (0,\vec{s})$ is the spin four-vector of the decaying quark and θ denotes the angle between $\vec{s}$ and the direction of the charged lepton. $S \equiv |\vec{s}| = 1$

[1] Fortran77 version of the formulae given in this article is available upon request from jezabek@chopin.ifj.edu.pl

corresponds to fully polarised, $S = 0$ to unpolarised decaying quarks.

The distribution of the neutrino for $I_3 = \pm 1/2$ is given by the formulae describing the distribution of the charged lepton from the decay of Q with the weak isospin $I_3 = \mp 1/2$.

2.1 Triple differential distributions

The QCD corrected triple differential distribution for the semileptonic decay of the polarised quark with $I_3 = \pm 1/2$ can be written in the following way

$$\begin{aligned}\frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\cos\theta} = \; & \frac{G_F^2 m_1^5}{32\pi^3}\frac{|V_{CKM}|^2}{(1-y/\bar{y})^2+\gamma^2}\Bigg(\mathrm{F}_0^{\pm}(x,y) + S\cos\theta\,\mathrm{J}_0^{\pm}(x,y) \\ & -\frac{2\alpha_s}{3\pi}\Big[\mathrm{F}_1^{\pm}(x,y) + S\cos\theta\,\mathrm{J}_1^{\pm}(x,y)\Big]\Bigg) \qquad (1)\end{aligned}$$

V_{CKM} denotes the element of the Cabibbo–Kobayashi–Maskawa quark mixing matrix corresponding to the decay channel under consideration. The effects of W propagator are included as a factor $1/[(1-y/\bar{y})^2+\gamma^2]$, where $\bar{y} = m_{\mathrm{w}}^2/m_1^2$ and $\gamma = \Gamma_{\mathrm{w}}/m_{\mathrm{w}}$; c.f. [19]. These effects are important for the top quark [19, 24]. For charm and bottom quarks m_1 is much smaller than the mass of W boson and the four-fermion limit can be employed in which the factor mentioned above is replaced by 1.

The Dalitz variables x and y fulfil kinematic constraints:

$$\begin{aligned} 0 \leq \;\; & x \;\; \leq x_m = 1-\epsilon^2 \\ 0 \leq \;\; & y \;\; \leq y_m = x(x_m - x)/(1-x) \end{aligned} \qquad (2)$$

or

$$\begin{aligned} 0 \leq y \leq (1-\epsilon)^2 \\ w_- \leq x \leq w_+ \end{aligned} \qquad (3)$$

The functions $w_\pm$ and other useful kinematic functions are defined as follows:

$$\begin{aligned} p_0 &= (1-y+\epsilon^2)/2 \\ p_3 &= \sqrt{p_0^2-\epsilon^2} \\ p_\pm &= p_0 \pm p_3 = 1 - w_\mp \\ Y_p &= \tfrac{1}{2}\ln(p_+/p_-) = \ln(p_+/\epsilon) \\ Y_w &= \tfrac{1}{2}\ln(w_+/w_-) = \ln(w_+/\sqrt{y}) \\ z_m &= (1-x)(1-y/x) \end{aligned} \qquad (4)$$

The functions $\mathrm{F}_0^{\pm}(x,y)$ and $\mathrm{J}_0^{\pm}(x,y)$ corresponding to Born approximation read:

$$
\begin{aligned}
\mathrm{F}_0^{+}(x,y) &= x(x_m - x) && (5)\\
\mathrm{J}_0^{+}(x,y) &= \mathrm{F}_0^{+}(x,y) && (6)\\
\mathrm{F}_0^{-}(x,y) &= (x-y)(x_m - x + y) && (7)\\
\mathrm{J}_0^{-}(x,y) &= (x-y)(x_m - x + y - 2y/x) && (8)
\end{aligned}
$$

The first order QCD corrections and the corresponding functions $\mathrm{F}_1^{\pm}(x,y)$ have been calculated in [19] whereas $\mathrm{J}_1^{+}(x,y)$ has been given in [23]. In the course of the present work we rederived these old results and from them we obtained the much simpler expressions given in this article. The result for $\mathrm{J}_1^{-}(x,y)$ is new. The formulae for $\mathrm{F}_1^{\pm}(x,y)$ and $\mathrm{J}_1^{+}(x,y)$ read[2]:

$$
\mathrm{F}_1^{\pm}(x,y) = \mathrm{F}_0^{\pm}(x,y)\,\Phi_0 + \sum_{n=1}^{5} A_n^{\pm}\,\Phi_n + A_6^{\pm} \tag{9}
$$

$$
\mathrm{J}_1^{\pm}(x,y) = \mathrm{J}_0^{\pm}(x,y)\,\Phi_0 + \sum_{n=1}^{5} B_n^{\pm}\Phi_n + B_6^{\pm} \tag{10}
$$

where

$$
\begin{aligned}
\Phi_0 &= \frac{2p_0}{p_3}\left[\mathrm{Li}_2\left(1-\frac{1-x}{p_+}\right) + \mathrm{Li}_2\left(1-\frac{1-y/x}{p_+}\right) - \mathrm{Li}_2\left(1-\frac{1-x}{p_-}\right)\right.\\
&\qquad \left. -\mathrm{Li}_2\left(1-\frac{1-y/x}{p_-}\right) + \mathrm{Li}_2\left(w_-\right) - \mathrm{Li}_2\left(w_+\right) + 4Y_p\ln\epsilon\right]\\
&\quad +4\left(1-\frac{p_0}{p_3}Y_p\right)\ln\left(z_m-\epsilon^2\right) - 4\ln z_m\\
\Phi_1 &= \mathrm{Li}_2\left(w_-\right) + \mathrm{Li}_2\left(w_+\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y/x)\\
\Phi_2 &= Y_p/p_3\\
\Phi_3 &= \tfrac{1}{2}\ln\epsilon\\
\Phi_4 &= \tfrac{1}{2}\ln(1-x)\\
\Phi_5 &= \tfrac{1}{2}\ln(1-y/x)
\end{aligned} \tag{11}
$$

[2]Throughout this article polylogarithms are defined as real functions. In particular:

$$
\mathrm{Li}_2(x) = -\int_0^x \mathrm{d}y\ln|1-y|/y
$$

$$
\mathrm{Li}_3(x) = \int_0^x \mathrm{d}y\,\mathrm{Li}_2(y)/y
$$

$$
\begin{aligned}
A_1^+ &= x+\epsilon^2 x \\
A_2^+ &= -p_3^2\,(3+2x+y+\epsilon^2)-2\epsilon^2 y \\
A_3^+ &= -(1-y)(3+2x+y)+2\epsilon^2(1+5x)+\epsilon^4 \\
A_4^+ &= 5-5x-\epsilon^2(4+5x)-\epsilon^4 \\
A_5^+ &= -2xy+9x-4x^2-2y-y^2-7\epsilon^2 x \\
A_6^+ &= (-4xy+4y-y^2+y^2/x)/2+\epsilon^2[x+y-x/(1-y/x)]/2
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
B_1^+ &= -x+\epsilon^2 x \\
B_2^+ &= p_3^2\,[\,(5-2x-y-2y/x-2/x)+\epsilon^2(-1+2/x)\,]\,+\epsilon^2(-xy-x \\
&\quad +2y-y/x-y^2/x)+\epsilon^4(x+y/x) \\
B_3^+ &= (1-y)(5-2x-y-2y/x-2/x)+2\epsilon^2(-3+7x+2/x) \\
&\quad +\epsilon^4(1-2/x) \\
B_4^+ &= -3+x+2/x+\epsilon^2(4-9x-4/x)+\epsilon^4(-1+2/x) \\
B_5^+ &= -2xy+3x-4x^2+6y-y^2-2y^2/x-7\epsilon^2 x \\
B_6^+ &= (2-2x^2+2y-3y^2-2y/x+3y^2/x)/2+\epsilon^2[-4+x+y+2y/x \\
&\quad -x/(1-y/x)]/2+\epsilon^4
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
A_1^- &= -2xy+x+y+y^2+\epsilon^2(x-y) \\
A_2^- &= p_3^2\,(-5+2x-3y+\epsilon^2)-2\epsilon^2 y \\
A_3^- &= (1-y)(-5+2x-3y)+6\epsilon^2(1+x-2y)-\epsilon^4 \\
A_4^- &= 5+4xy-5x+3y-5y^2-2y^2/x+\epsilon^2(-4-5x+11y)-\epsilon^4 \\
A_5^- &= 6xy+9x-4x^2-11y-2y^2+2y^2/x+7\epsilon^2(-x+y) \\
A_6^- &= (3xy+2y-3y^2-2y^2/x)/2+\epsilon^2[3y-y/(1-x)]/2
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
B_1^- &= -2xy-x-5y+y^2-2y^2/x+\epsilon^2(x-y) \\
B_2^- &= p_3^2\,[\,(3+10x+y+10y/x-2/x)+\epsilon^2(1+2/x)\,]+\epsilon^2(xy+x-2y \\
&\quad +y/x+y^2/x)-\epsilon^4(x+y/x) \\
B_3^- &= (1-y)(3+10x+y+10y/x-2/x)+2\epsilon^2(-1+5x-4y-2y/x \\
&\quad +2/x)-\epsilon^4(1+2/x) \\
B_4^- &= -3+12xy+x-7y-y^2-12y/x+8y^2/x+2/x+\epsilon^2(4-9x \\
&\quad +7y-4/x)+\epsilon^4(-1+2/x) \\
B_5^- &= 6xy-9x-4x^2-y-2y^2+10y^2/x+7\epsilon^2(-x+y) \\
B_6^- &= (2-5xy-2x^2+2y+7y^2-2y/x-2y^2/x)/2+\epsilon^2[-4-9y+2y/x \\
&\quad +y/(1-x)]/2+\epsilon^4
\end{aligned}
\tag{15}
$$

2.2 Double differential $x-\theta$ distributions

For the top quark in the narrow W width limit $\gamma \to 0$ the mass squared of the leptons is fixed at value $y = (m_{\rm w}/m_t)^2$. Thus, in this limit the triple differential distributions of subsection 2.1 reduce to the double differential $x-\theta$ distributions. In the four-fermion limit ($m_{\rm w} \to \infty$) the double differential $x-\theta$ distributions can be calculated from the following formulae:

$$\frac{\mathrm{d}\Gamma^\pm}{\mathrm{d}x\,\mathrm{d}\cos\theta} = \int_0^{y_m} \mathrm{d}y\, \frac{\mathrm{d}\Gamma^\pm}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\cos\theta} = \frac{G_F^2 m_1^5}{32\pi^3} |V_{CKM}|^2 \Bigg(f_0^\pm(x) + S\cos\theta\, j_0^\pm(x) - \frac{2\alpha_s}{3\pi}\left[f_1^\pm(x) + S\cos\theta\, j_1^\pm(x)\right]\Bigg) \quad (16)$$

where

$$\begin{aligned}
f_0^+(x) &= x^2(x_m-x)^2/(1-x) && (17)\\
j_0^+(x) &= f_0^+(x,y) && (18)\\
f_0^-(x) &= \tfrac{1}{6}x^2(x_m-x)^2[\,3-2x+\epsilon^2+2\epsilon^2/(1-x)\,]/(1-x)^2 && (19)\\
j_0^-(x) &= \tfrac{1}{6}x^2(x_m-x)^2[\,1-2x+\epsilon^2-2\epsilon^2/(1-x)\,]/(1-x)^2 && (20)
\end{aligned}$$

and

$$\begin{aligned}
f_1^\pm(x) &= \int_0^{y_m} \mathrm{d}y\, \mathrm{F}_1^\pm(x,y) && (21)\\
j_1^\pm(x) &= \int_0^{y_m} \mathrm{d}y\, \mathrm{J}_1^\pm(x,y) && (22)
\end{aligned}$$

The integrals in eqs.(21)-(22) can be calculated numerically. In the massless limit $\epsilon \to 0$ analytic formulae for $f_1^\pm(x)$ and $f_1^\pm(x)$ have been derived. These formulae are listed in subsection 3.2. The results for charm and bottom quarks have been already published in [15]. For the polarisation independent parts of the distributions they are in perfect agreement with the results of [19]. Thus, they are in conflict with the results of [17] and [18]; see section 5 in [19] where a detailed comparison is given and cross checks descibed.
A convenient way to present the results is to express the double differential distributions as products of the energy distributions and the asymmetry functions, c.f. eqs.(5)-(7) in [15],

$$\frac{\mathrm{d}\Gamma^\pm}{\mathrm{d}x\,\mathrm{d}\cos\theta} = \frac{\mathrm{d}\Gamma^\pm}{\mathrm{d}x}\,[\,1 \pm \alpha_\pm(x)\cos\theta\,]\,/2 \quad (23)$$

where

$$\alpha_\pm(x) = \pm S\,\left[j_0^\pm(x) - \tfrac{2\alpha_s}{3\pi} j_1^\pm(x)\right] / \left[f_0^\pm(x) - \tfrac{2\alpha_s}{3\pi} f_1^\pm(x)\right] \quad (24)$$

The functions $\alpha_\pm(x)$ for charm and bottom quarks have been given in [15].

2.3 Double differential $y - \theta$ distributions

We follow the normalisation convention of [25] and write the double differential $y - \theta$ distributions in the following way:

$$
\begin{aligned}
\frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}y\,\mathrm{d}\cos\theta} &= \int_{w_-}^{w_+} \mathrm{d}x\; \frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\cos\theta} = \frac{G_F^2 m_1^5}{192\pi^3}\,\frac{|V_{CKM}|^2}{(1-y/\bar{y})^2+\gamma^2} \\
&\times\frac{1}{2}\left(\mathcal{F}_0(y) + S\cos\theta\,\mathcal{J}_0^{\pm}(y) - \frac{2\alpha_s}{3\pi}\left[\mathcal{F}_1(y) + S\cos\theta\,\mathcal{J}_1^{\pm}(y)\right]\right) \qquad (25)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}_0(y) &= 4p_3\left[(1-\epsilon^2)^2 + y(1+\epsilon^2) - 2y^2\right] &(26)\\
\mathcal{J}_0^{+}(y) &= \mathcal{F}_0(y) &(27)\\
\mathcal{J}_0^{-}(y) &= \mathcal{F}_0(y) + 48y\,(yY_w - p_3) &(28)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_1(y) &= 12\int_{w_-}^{w_+}\mathrm{d}x\;\mathrm{F}_1^{\pm}(x,y) &(29)\\
\mathcal{J}_1^{\pm}(y) &= 12\int_{w_-}^{w_+}\mathrm{d}x\;\mathrm{J}_1^{\pm}(x,y) &(30)
\end{aligned}
$$

Eq.(29) provides a very non-trivial cross check of this calculation. It has been verified both analytically and numerically that for $\mathrm{F}_1^{\pm}(x,y)$ given by eq.(9) the integrals in (29) are equal. Moreover, $\mathcal{F}_1(y)$ can be calculated in an easier way, see formula (2.6) in [25], if the phase space integrals over the momenta of leptons are performed first. This cross check is also fulfilled. It has been shown, see sect.5 of [19], that when $\mathcal{F}_1(y)$ is analytically continued it agrees with the analytic results of [26] for QCD corrections to the width of W boson. All these tests indicate that our formulae for $\mathrm{F}_1^{\pm}(x,y)$ are correct. Since all the calculations for both $\mathrm{F}_1^{\pm}(x,y)$ and $\mathrm{J}_1^{\pm}(x,y)$ were performed in parallel using the same algebraic computer programs we believe that the same is true also for $\mathrm{J}_1^{\pm}(x,y)$.

Analytic expression for $\mathcal{F}_1(y)$ has been first given in [25]. For the sake of completeness we rewrite this expression using the notation of the present article and some identities for dilogarithms. Our formulae for $\mathcal{F}_1(y)$ and $\mathcal{J}_1^{\pm}(y)$ read:

$$
\begin{aligned}
\mathcal{F}_1(y) &= \mathcal{F}_0(y)\Psi_0 + \mathcal{A}_1\Psi_1 + \mathcal{A}_2 Y_w + \mathcal{A}_3 Y_p + \mathcal{A}_4 p_3\ln\epsilon + \mathcal{A}_5 p_3 &(31)\\
\mathcal{J}_1^{+}(y) &= \mathcal{J}_0^{+}(y)\,\Psi_0 + \mathcal{B}_1^{+}\Psi_1 + \mathcal{B}_2^{+}Y_w + \mathcal{B}_3^{+}Y_p + \mathcal{B}_4^{+}p_3\ln\epsilon + \mathcal{B}_5^{+}p_3 &\\
&\quad + \mathcal{B}_6^{+}Y_wY_p + \mathcal{B}_7^{+}Y_w\ln\epsilon &(32)\\
\mathcal{J}_1^{-}(y) &= \mathcal{J}_0^{-}(y)\,\Psi_0 + \mathcal{B}_1^{-}\Psi_1 + \mathcal{B}_2^{-}Y_w + \mathcal{B}_3^{-}Y_p + \mathcal{B}_4^{-}p_3\ln\epsilon + \mathcal{B}_5^{-}p_3 &\\
&\quad + \mathcal{B}_6^{-}Y_wY_p + \mathcal{B}_7^{-}Y_w\ln\epsilon + 48y^2\left(\mathcal{B}_8^{-} + \mathcal{B}_9^{-}p_0/p_3\right) &(33)
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_0 &= 2\left[\,4\mathrm{Li}_2(2p_3/p_+) - 4Y_p\ln(2p_3/p_+) - \ln p_-\ln w_+ + \ln p_+\ln w_-\,\right]p_0/p_3 \\
&\quad +8\ln(2p_3) - 2\ln y \qquad (34) \\
\Psi_1 &= 4\,\mathrm{Li}_2(w_-) \;-\; 4\,\mathrm{Li}_2(w_+) \qquad (35)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_1 &= \mathcal{F}_0(y)\,p_0/p_3 \\
\mathcal{A}_2 &= -8(1-\epsilon^2)\left[1+y-4y^2-\epsilon^2(2-y)+\epsilon^4\right] \\
\mathcal{A}_3 &= -2\left[3+6y-21y^2+12y^3-\epsilon^2(1+12y+5y^2)+\epsilon^4(11+2y)-\epsilon^6\right] \\
\mathcal{A}_4 &= -12\left[1+3y-4y^2-\epsilon^2(4-y)+3\epsilon^4\right] \\
\mathcal{A}_5 &= -2\left[5+9y-6y^2-\epsilon^2(22-9y)+5\epsilon^4\right] \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1^+ &= 5-9y^2+4y^3-\epsilon^2(8-8y+6y^2)+\epsilon^4+2\epsilon^6 \\
\mathcal{B}_2^+ &= -8\left[1+4y-y^2-\epsilon^2(3+3y-4y^2)+\epsilon^4(3-y)-\epsilon^6\right] \\
\mathcal{B}_3^+ &= -2\left[3(1-y)^2(1+4y)+\epsilon^2(11-5y^2)-\epsilon^4(13-2y)-\epsilon^6\right] \\
\mathcal{B}_4^+ &= -12\left[(1-y)(1+4y)-\epsilon^2(4-y)+3\epsilon^4\right] \\
\mathcal{B}_5^+ &= 2\left[15-y+2y^2-\epsilon^2(12+7y)-3\epsilon^4\right] \\
\mathcal{B}_6^+ &= -24\left(1+y-\epsilon^2\right)\left(2p_3+y\epsilon^2/p_3\right) \\
\mathcal{B}_7^+ &= -24\left(1-y^2-2\epsilon^2+\epsilon^4\right) \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1^- &= 5-42y+45y^2+4y^3-2\epsilon^2(4+8y+3y^2)+\epsilon^4+2\epsilon^6 \\
\mathcal{B}_2^- &= 8\left[-1+8y+10y^2+\epsilon^2(3-9y-4y^2)-\epsilon^4(3-y)+\epsilon^6\right] \\
\mathcal{B}_3^- &= 2\left[-3(1-y)(3-23y-4y^2)+\epsilon^2(1+24y+5y^2)+\epsilon^4(7-2y)+\epsilon^6\right] \\
\mathcal{B}_4^- &= -12\left[3-23y-4y^2-\epsilon^2(6-y)+3\epsilon^4\right] \\
\mathcal{B}_5^- &= -2\left[-15+37y-2y^2+\epsilon^2(12+7y)+3\epsilon^4\right] \\
\mathcal{B}_6^- &= -24\left[2p_3(1-5y-\epsilon^2)\;-\;\epsilon^2(1+y-\epsilon^2)y/p_3\right] \\
\mathcal{B}_7^- &= -24\left[(1-y)(1-5y)-2\epsilon^2(1-y)+\epsilon^4\right] \\
\mathcal{B}_8^- &= 4\left[\mathrm{Li}_2\left(\tfrac{w_-p_-}{w_+p_+}\right)-\mathrm{Li}_2\left(p_-/p_+\right)\right]+\mathrm{Li}_3(w_+)-\mathrm{Li}_3(w_-)+8Y_p\ln(w_+) \\
&\quad -Y_w\left[\mathrm{Li}_2(w_-)+\mathrm{Li}_2(w_+)+8\ln(2p_3)-2\ln y-4\ln(w_+)\right] \\
\mathcal{B}_9^- &= 4\left[\mathrm{Li}_3(1)+\mathrm{Li}_3\left(\tfrac{w_-p_-}{w_+p_+}\right)-\mathrm{Li}_3(p_-/p_+)-\mathrm{Li}_3(w_-/w_+)\right] \\
&\quad +2Y_w\left[\mathrm{Li}_2(w_+)-\mathrm{Li}_2(w_-)-4\mathrm{Li}_2(2p_3/p_+)+4Y_p\ln(2p_3/p_+)\right. \\
&\qquad \left.+2Y_w\ln p_+-4Y_p\ln w_+\right] \;+\; 4Y_p\left[\mathrm{Li}_2\left(\tfrac{w_-p_-}{w_+p_+}\right)-\mathrm{Li}_2(p_-/p_+)\right] \qquad (38)
\end{aligned}
$$

3 Massless case

3.1 Triple differential distributions

For $m_2 = 0$ the functions in eq.(1) simplify considerably. The limit $\epsilon \to 0$ can be easily obtained for $\mathrm{F}_1^\pm(x,y)$ and $\mathrm{J}_1^\pm(x,y)$. In this limit

$$
\begin{aligned}
\mathrm{F}_0^+(x,y) &= x(1-x) && (39)\\
\mathrm{J}_0^+(x,y) &= \mathrm{F}_0^+(x,y) && (40)\\
\mathrm{F}_0^-(x,y) &= (x-y)(1-x+y) && (41)\\
\mathrm{J}_0^-(x,y) &= (x-y)(1-x+y-2y/x) && (42)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{F}_1^+(x,y) = {} & \mathrm{F}_0^+\,\Phi_0 + x\Phi_1 - (3+2x+y)\Phi_{2\diamond 3} + 5(1-x)\Phi_4 + (-2xy \\
& +9x - 4x^2 - 2y - y^2)\Phi_5 + y(4-4x-y+y/x)/2 \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{F}_1^-(x,y) = {} & \mathrm{F}_0^-\,\Phi_0 + (-2xy + x + y + y^2)\Phi_1 + (-5+2x-3y)\Phi_{2\diamond 3} \\
& +(5+4xy-5x+3y-5y^2-2y^2/x)\Phi_4 + (6xy+9x-4x^2 \\
& -11y - 2y^2 + 2y^2/x)\Phi_5 + y(2+3x-3y-2y/x)/2 \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{J}_1^+(x,y) = {} & \mathrm{J}_0^+\,\Phi_0 - x\Phi_1 + (5-2x-y-2y/x-2/x)\Phi_{2\diamond 3} + (-3+x \\
& +2/x)\Phi_4 + (-2xy+3x-4x^2+6y-y^2-2y^2/x)\Phi_5 \\
& +(2-2x^2+2y-3y^2-2y/x+3y^2/x)/2 \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{J}_1^-(x,y) = {} & \mathrm{J}_0^-\,\Phi_0 + (-2xy - x - 5y + y^2 - 2y^2/x)\Phi_1 + (3+10x+y \\
& +10y/x - 2/x)\Phi_{2\diamond 3} + (-3+12xy+x-7y-y^2-12y/x \\
& +8y^2/x + 2/x)\Phi_4 + (6xy - 9x - 4x^2 - y - 2y^2 + 10y^2/x)\Phi_5 \\
& +(2-5xy-2x^2+2y+7y^2-2y/x-2y^2/x)/2 \qquad (46)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_0 &= \tfrac{\pi^2}{3} + 2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(y/x) + \ln^2\left(\tfrac{1-y/x}{1-x}\right)\\
\Phi_1 &= \tfrac{\pi^2}{6} + \mathrm{Li}_2(y) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y/x)\\
\Phi_{2\diamond 3} &= \tfrac{1}{2}(1-y)\ln(1-y)\\
\Phi_4 &= \tfrac{1}{2}\ln(1-x)\\
\Phi_5 &= \tfrac{1}{2}\ln(1-y/x) && (47)
\end{aligned}
$$

Eqs.(43)-(45) agree with those given in the literature; c.f. eqs.(3.9) and (4.9) in [19] and eq.(3.3) in [23].

3.2 Massless limit for $x - \theta$ distributions

For $\epsilon = 0$ explicit analytic formulae can be derived for the double differential $x - \theta$ distributions. The functions $f_0^{\pm}(x)$, $j_0^{\pm}(x)$, $f_1^{\pm}(x)$ and $j_1^{\pm}(x)$ which appear in eq.(16) are given by the following expressions, cf. eqs.(3.10) and (4.10) in [19]:

$$f_0^+(x) = x^2(1-x) \tag{48}$$
$$f_0^-(x) = x^2(3-2x)/6 \tag{49}$$
$$j_0^+(x) = f_0^+(x) \tag{50}$$
$$j_0^-(x) = x^2(1-2x)/6 \tag{51}$$
$$\begin{aligned} f_1^+(x) &= f_0^+(x)\phi_0(x) + (1-x)\left[\tfrac{1}{6}(5+8x+8x^2)\ln(1-x)\right. \\ &\quad \left. + \tfrac{1}{12}x(10-19x)\right] \end{aligned} \tag{52}$$
$$\begin{aligned} f_1^-(x) &= f_0^-(x)\phi_0(x) + \tfrac{1}{36}(41-36x+42x^2-16x^3)\ln(1-x) \\ &\quad + \tfrac{1}{72}x(82-153x+86x^2) \end{aligned} \tag{53}$$
$$\begin{aligned} j_1^+(x) &= j_0^+(x)\phi_0(x) + (1-x)\left[\tfrac{1}{6}(-3+12x+8x^2+4/x)\ln(1-x)\right. \\ &\quad \left. + \tfrac{1}{12}(8-2x-15x^2)\right] \end{aligned} \tag{54}$$
$$\begin{aligned} j_1^-(x) &= j_0^-(x)\phi_0(x) + \tfrac{1}{36}(11-36x+14x^2-16x^3-4/x)\ln(1-x) \\ &\quad + \tfrac{1}{72}(-8+18x-103x^2+78x^3) \end{aligned} \tag{55}$$
$$\phi_0(x) = 2\mathrm{Li}_2(x) + 2\pi^2/3 + \ln^2(1-x) \tag{56}$$

3.3 Massless limit for $y - \theta$ distributions

For $\epsilon \to 0$ the functions in eq.(25) read, cf. eqs.(3.1)-(3.2) in [25] :

$$\mathcal{F}_0(y) = 2(1-y)^2(1+2y) \tag{57}$$
$$\mathcal{J}_0^+(y) = \mathcal{F}_0(y) \tag{58}$$
$$\mathcal{J}_0^-(y) = 2(1-y)(1-11y-2y^2) - 24y^2\ln y \tag{59}$$
$$\begin{aligned} \mathcal{F}_1(y) &= \mathcal{F}_0\left[\tfrac{2}{3}\pi^2 + 4\mathrm{Li}_2(y) + 2\ln y\ln(1-y)\right] - (1-y)(5+9y-6y^2) \\ &\quad + 4y(1-y-2y^2)\ln y + 2(1-y)^2(5+4y)\ln(1-y) \end{aligned} \tag{60}$$
$$\begin{aligned} \mathcal{J}_1^+(y) &= -\tfrac{2}{3}\pi^2(1-y)(1+y+4y^2) + (1-y)(15-y+2y^2) \\ &\quad + 4(1-y)(5+5y-4y^2)\,\mathrm{Li}_2(y) + 8(1-y)(2+2y-y^2)\ln(1-y)\ln y \\ &\quad + 8y(2+y-y^2)\ln y + 2(1-y)^2(5+4y)\ln(1-y) \end{aligned} \tag{61}$$
$$\begin{aligned} \mathcal{J}_1^-(y) &= -\tfrac{2}{3}\pi^2(1+6y+9y^2-4y^3) + (1-y)(15-37y+2y^2) \\ &\quad + 240y^2\left[\,\mathrm{Li}_3(1) - \mathrm{Li}_3(y)\,\right] + 4\left[5-42y+45y^2+4y^3+18y^2\ln y\right]\mathrm{Li}_2(y) \\ &\quad + 8(1-y)(2-13y-y^2)\ln(1-y)\ln y + 4y\left[4-(19+\pi^2)y-2y^2\right]\ln y \\ &\quad - 2(1-y)(1+19y+4y^2)\ln(1-y) \end{aligned} \tag{62}$$

4 Angular distributions

Angular distributions are related to the double differential $y - \theta$ distributions discussed in subsection 2.3

$$\frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}\cos\theta} = \int_0^{(1-\epsilon)^2} \mathrm{d}y\, \frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}y\,\mathrm{d}\cos\theta} \tag{63}$$

with $\mathrm{d}\Gamma^{\pm}/\mathrm{d}y\,\mathrm{d}\cos\theta$ given in eq.(25).
For the top quark and the narrow W width the double differential distributions are proportional to Dirac delta function $\delta\left(y - m_{\mathrm{w}}^2/m_t^2\right)$ and the integral in (63) is trivial. For the charm and for the bottom quarks the four-fermion approximation can be used.

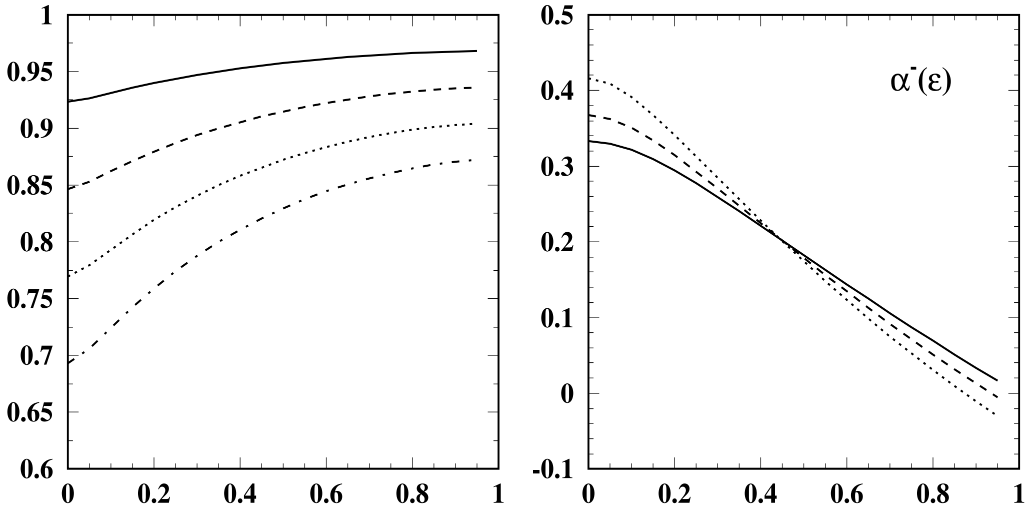

Figure 1: a) QCD reduction of the total decay rate: the ratios $\mathcal{R}(\epsilon)/\mathcal{R}_0(\epsilon)$ as functions of ϵ for $\alpha_s = 0.1, 0.2, 0.3$ and 0.4 – solid, dashed, dotted and dash-dotted lines; b) QCD correction to the angular asymmetry $\alpha^-(\epsilon)$ for $\alpha_s = 0, 0.2$ and 0.4 – solid, dashed and dotted lines, respectively.

In order to illustrate the size of the radiative corrections discussed in the present article we employ the four-fermion limit and write the resulting angular distributions in the following way:

$$\frac{\mathrm{d}\Gamma^{\pm}}{\mathrm{d}\cos\theta} = \Gamma_0\,\mathcal{R}(\epsilon)\,\Big[1 \pm \alpha^{\pm}(\epsilon)\,\cos\theta\Big] \tag{64}$$

where $\Gamma_0 = G_F^2 m_1^5 |V_{CKM}|^2/384\pi^3$. In Born approximation for completely polarised quarks ($S = 1$)

$$\mathcal{R}_0(\epsilon) = 1 - 8\epsilon^2 + 8\epsilon^6 - \epsilon^8 - 24\epsilon^4 \ln\epsilon \tag{65}$$

$$\alpha_0^+(\epsilon) = 1 \tag{66}$$

$$\alpha_0^-(\epsilon) = \tfrac{1}{3}\,\Big[1 - 12\epsilon^2 - 36\epsilon^4 + 44\epsilon^6 + 3\epsilon^8 \;-\; 24\epsilon^4\,(3 + 2\epsilon^2)\ln\epsilon\Big]\,/\,\mathcal{R}_0(\epsilon) \tag{67}$$

QCD corrections decrease the total decay rate. In Fig.1a the ratios $\mathcal{R}(\epsilon)/\mathcal{R}_0(\epsilon)$ are shown as functions of ϵ for $\alpha_s = 0.1, 0.2, 0.3$ and 0.4 as the solid, dashed, dotted and dash-dotted lines, respectively. The corrections to eq.(66) are of the order of one percent or smaller. In Fig.1b the function $\alpha^-(\epsilon)$ is shown for $\alpha_s = 0, 0.2$ and 0.4 as the solid, dashed and dotted lines, respectively.

A Derivations and formulae

In this Appendix we sketch our derivation of eq.(1). The method has been described in our earlier papers [19] and [23]. Many formulae presented here have been already given in these articles. We rewrite them using the notation adopted in the present article. Some formulae are simplified and misprints corrected. In the following Q, q, ℓ, ν and G denote the scaled four-momenta of the decaying quark and the decay products: quark, charged lepton, neutrino and gluon. The common scaling factor is $1/m_1$, and $Q^2 = 1$, $q^2 = \epsilon^2$, $\ell^2 = \nu^2 = G^2 = 0$. Thus, the gluon is massless. However, at intermediate steps we introduce the scaled gluon mass λ_G in order to regularize those expressions which are infrared divergent. We introduce also $P = q + G$, $z = P^2$, s – the spin four-vector of the decaying quark and $R = Q - s$. Some formulae are given only for the weak isospin $I_3 = 1/2$ of the decaying quark. This implies that the analogous formulae for $I_3 = -1/2$ can be obtained by replacing ℓ with ν and vice versa.

A.1 Differential decay rate

The QCD corrected differential rate is given by the following formula

$$\mathrm{d}\Gamma^{\pm} = \mathrm{d}\Gamma_0^{\pm} + \mathrm{d}\Gamma_{1,3}^{\pm} + \mathrm{d}\Gamma_{1,4}^{\pm} \tag{68}$$

where

$$\mathrm{d}\Gamma_0^{\pm} = G_F^2 m_1^5 \, |V_{CKM}|^2 \, \mathcal{M}_{0,3}^{\pm} \mathrm{d}\mathcal{R}_3(Q; q, \ell, \nu)/\pi^5 \tag{69}$$

is Born approximation,

$$\mathrm{d}\Gamma_{1,3}^{\pm} = \tfrac{2}{3}\, \alpha_s \, G_F^2 m_1^5 \, |V_{CKM}|^2 \, \mathcal{M}_{1,3}^{\pm} \, \mathrm{d}\mathcal{R}_3(Q; q, \ell, \nu)/\pi^6 \tag{70}$$

describes the virtual gluon contribution and

$$\mathrm{d}\Gamma_{1,4}^{\pm} = \tfrac{2}{3}\, \alpha_s \, G_F^2 m_1^5 \, |V_{CKM}|^2 \, \mathcal{M}_{1,4}^{\pm} \mathrm{d}\mathcal{R}_4(Q; q, G, \ell, \nu)/\pi^7 \tag{71}$$

comes from real gluon emission. Lorentz invariant n-body phase space is defined as

$$\mathrm{d}\mathcal{R}_n(P; p_1, p_2, \ldots, p_n) = \delta^{(4)}(P - \boldsymbol{\Sigma} p_i) \prod_i \frac{\mathrm{d}^3\mathbf{p}_i}{2p_{0,i}} \tag{72}$$

A.2 Three-body contributions

In Born approximation the rates for the decays into three-body final states are proportional to the expressions

$$\begin{aligned} \mathcal{M}^+_{0,3} &= q\cdot\nu\, R\cdot\ell \,/\, \left[(1-y/\bar{y})^2+\gamma^2\right] \\ \mathcal{M}^-_{0,3} &= q\cdot\ell\, R\cdot\nu \,/\, \left[(1-y/\bar{y})^2+\gamma^2\right] \end{aligned} \tag{73}$$

The formulae for $\mathrm{F}^\pm_0(x,y)$ and $\mathrm{J}^\pm_0(x,y)$ can be easily derived from (69) and (73) when Dalitz parametrization of the three-body phase space

$$\mathrm{d}\mathcal{R}_3(Q;q,\ell,\nu) = \tfrac{1}{32}\mathrm{d}x\mathrm{d}y\mathrm{d}\alpha\mathrm{d}(\cos\theta)\mathrm{d}\beta \tag{74}$$

is employed. Integration over Euler angle β is trivial, and non-trivial integrals over α read, see Appendix B in [23],

$$\begin{aligned} \int \mathrm{d}\alpha\; s\cdot q &= 2\pi S\cos\theta[(1+y-q^2)/2-y/x] \\ \int \mathrm{d}\alpha\; s\cdot \nu &= 2\pi S\cos\theta[y/x-(1+y-q^2-x)/2] \end{aligned} \tag{75}$$

As a final step the scalar products in $\mathcal{M}^\pm_{0,3}$ are expressed in terms of the variables x, y and $\cos\theta$

$$\begin{array}{rclcrcl} Q\cdot s &=& 0 & \qquad & \ell\cdot s &=& -xS\cos\theta/2 \\ Q\cdot q &=& (1+q^2-y)/2 & & \ell\cdot q &=& (x-y)/2 \\ Q\cdot \nu &=& (1-q^2-x+y)/2 & & \ell\cdot\nu &=& y/2 \\ Q\cdot\ell &=& x/2 & & \nu\cdot q &=& (1-q^2-x)/2 \end{array} \tag{76}$$

and eqs.(5)-(8) follow immediately.

The same steps are performed for the three-body radiative corrections which are given by the formula (70), where

$$\begin{aligned} \mathcal{M}^+_{1,3} = & -\left[(1-y/\bar{y})^2+\gamma^2\right]^{-1} \left[\, q\cdot\nu\, R\cdot\ell\, H_0 \;+\; \epsilon^2\, Q\cdot\nu\, R\cdot\ell\, H_+ \right. \\ & + q\cdot\nu\,(q\cdot\ell \;+\; R\cdot\ell\, Q\cdot q \;-\; Q\cdot\ell\, R\cdot q)\; H_- \\ & \left. + \tfrac{1}{2}\epsilon^2\,(\nu\cdot\ell \;+\; Q\cdot\nu\, R\cdot\ell \;-\; R\cdot\nu\, Q\cdot\ell)\,(H_+ + H_-)\,\right] \end{aligned} \tag{77}$$

$$\begin{aligned} H_0 = & \; 4\,(1-p_0Y_p/p_3)\ln\lambda_G + (2p_0/p_3)\left[\,\mathrm{Li}_2\left(1-\tfrac{p_-w_-}{p_+w_+}\right)\right. \\ & \left. -\mathrm{Li}_2\left(1-\tfrac{w_-}{w_+}\right) - Y_p(Y_p+1) + 2(\ln\epsilon + Y_p)(Y_w+Y_p)\,\right] \\ & +[2p_3Y_p + (1-\epsilon^2-2y)\ln\epsilon]/y + 4 \end{aligned} \tag{78}$$

$$H_\pm = \tfrac{1}{2}\left[1\pm(1-\epsilon^2)/y\right]Y_p/p_3 \pm \tfrac{1}{y}\ln\epsilon \tag{79}$$

A.3 Four-body contributions

Neglecting those terms whose contributions to the decay rates are $\mathcal{O}(\lambda_G \ln \lambda_G)$ or smaller (i.e. vanishing in the limit $\lambda_G \to 0$) one can cast the contribution of real radiation into the following expression

$$\mathcal{M}^+_{1,4} = \frac{1}{(1-y/\bar{y})^2+\gamma^2}\left[\frac{\mathcal{B}^+_1}{(Q\cdot G)^2} - \frac{\mathcal{B}^+_2}{Q\cdot G\; P\cdot G} + \frac{\mathcal{B}^+_3}{(P\cdot G)^2}\right] \quad (80)$$

where

$$\begin{aligned}
\mathcal{B}^+_1 &= q\cdot\nu\,[\,R\cdot\ell\,(Q\cdot G-1) + G\cdot\ell\,R\cdot Q - Q\cdot\ell\,R\cdot G + G\cdot\ell\,R\cdot G\,]\\
\mathcal{B}^+_2 &= q\cdot\nu\,[\,G\cdot\ell\,R\cdot q - q\cdot\ell\,R\cdot G + R\cdot\ell\,(q\cdot G - Q\cdot G - 2\,q\cdot Q\,)\,]\\
&\quad + R\cdot\ell\,(Q\cdot\nu\,q\cdot G - G\cdot\nu\,q\cdot Q)\\
\mathcal{B}^+_3 &= R\cdot\ell\,(G\cdot\nu\,q\cdot G - q^2\,P\cdot\nu)
\end{aligned} \quad (81)$$

The four-body phase space is decomposed as follows

$$\mathrm{d}\mathcal{R}_4(Q;q,G,\ell,\nu) = \mathrm{d}z\mathrm{d}\mathcal{R}_3(Q;P,\ell,\nu)\mathrm{d}\mathcal{R}_2(P;q,G) \quad (82)$$

After q is substituted by $P-G$ integration of $\mathcal{M}^\pm_{1,4}$ over $\mathrm{d}\mathcal{R}_2(P;q,G)$ is performed. Due to Lorentz invariance all the integrals which appear in this calculation can be reduced to the scalar integrals

$$I_n = \int \mathrm{d}\mathcal{R}_2(P;q,G)(Q\cdot G)^n \quad (83)$$

Explicit formulae are listed in Appendix C of [23] for the reduction of tensor integrals $\int \mathrm{d}\mathcal{R}_2(P;q,G)(Q\cdot G)^n G^\alpha$, $\int \mathrm{d}\mathcal{R}_2(P;q,G)(Q\cdot G)^n G^\alpha G^\beta$ and $\int \mathrm{d}\mathcal{R}_2(P;q,G)(Q\cdot G)^n G^\alpha G^\beta G^\gamma$ to the scalar ones.
In the next step Dalitz parametrization is employed for the three-body phase space $\mathrm{d}\mathcal{R}_3(Q;P,\ell,\nu)$ in the same way as described in the previous subsection, but now for q replaced with P in eqs.(74)-(76). In particular $q^2=\epsilon^2$ is replaced with $P^2=z$.
After integrations over Euler angles α and β the contribution of real radiation is splitted into two pieces: the infrared divergent part of the form

$$\mathrm{Const}\;[\mathrm{F}^\pm_0(x,y) + S\cos\theta\,\mathrm{J}^\pm_0(x,y)]\;I_{div}$$

where

$$I_{div} = I_{-2} - (1-y+\epsilon^2)\,I_{-1}/(P\cdot G) + \epsilon^2\,I_0/(P\cdot G)^2 \quad (84)$$

and the rest which is infrared finite.
For $\lambda_G \ll 1$ the infrared divergent part can be integrated over z and the result is proportional to

$$\frac{1}{\pi}\int_{(\epsilon+\lambda_G)}^{z_m} \mathrm{d}z\, I_{div} = 4\left(1-\frac{p_0}{p_3}Y_p\right)\ln\left(\frac{z_m-\epsilon^2}{\epsilon\,\lambda_G}\right) - 2\ln\left(z_m/\epsilon^2\right) - 2$$
$$+\frac{2p_0}{p_3}\left[\mathrm{Li}_2\left(1-\frac{1-x}{p_+}\right)+\mathrm{Li}_2\left(1-\frac{1-y/x}{p_+}\right)-\mathrm{Li}_2\left(1-\frac{1-x}{p_-}\right)\right.$$
$$\left.-\mathrm{Li}_2\left(1-\frac{1-y/x}{p_-}\right)-\mathrm{Li}_2\left(1-\frac{p_-}{p_+}\right)-Y_p(Y_p+1)\right] \tag{85}$$

When this result is added to the contribution of virtual corrections the infrared divergent terms $\sim \ln\lambda_G$ cancel out. The sum is simplified using the following identity

$$\mathrm{Li}_2\left(1-\tfrac{p_-w_-}{p_+w_+}\right)-\mathrm{Li}_2\left(1-\tfrac{w_-}{w_+}\right)-\mathrm{Li}_2\left(1-\tfrac{p_-}{p_+}\right) =$$
$$\mathrm{Li}_2\left(w_-\right)-\mathrm{Li}_2\left(w_+\right)-2Y_w\ln p_+ \tag{86}$$

For the infrared finite part the limit $\lambda_G \to 0$ is performed. The formulae for the scalar integrals I_n simplify considerably in this limit and read $I_{-2} = \pi/(P\cdot G)$, $\qquad I_{-1} = \pi\mathcal{Y}_p(z)/\sqrt{-\Delta}$,

$$I_n = \frac{\pi(PG)^{n+1}[\;(Q\cdot P+\sqrt{-\Delta})^{n+1}-(Q\cdot P-\sqrt{-\Delta})^{n+1}\;]}{2(n+1)(P^2)^{n+1}\sqrt{-\Delta}} \quad (n\geq 0) \tag{87}$$

where[3]

$$\begin{aligned}
\Delta &= Q^2P^2-(Q\cdot P)^2 = -\lambda(z)/4\\
\lambda(z) &= z^2-2(1+y)z+(1-y)^2\\
\mathcal{Y}_p(z) &= \tfrac{1}{2}\ln\left(\tfrac{1-y+z+\sqrt{\lambda(z)}}{1-y+z-\sqrt{\lambda(z)}}\right)\\
P\cdot G &= (z-\epsilon^2)/2
\end{aligned} \tag{88}$$

Integration of the infrared finite part over z is tedious. It would be difficult to accomplish this task without FORM [27]. All the integrals which appear can be reduced to integrals $\int \mathrm{d}z\, z^n/\lambda^m(z)$ and $\int \mathrm{d}z\, z^n\,\mathcal{Y}_p(z)/\lambda^{m+1/2}(z)$. Thus, they can be derived from the recursion relations given in Appendix B of [19].

A.4 Formulae for $\mathrm{F}_1^{\pm}(x,y)$ and $\mathrm{J}_1^{\pm}(x,y)$

After some algebra mentioned in preceding subsections one derives the following formulae for $\mathrm{F}_1^{\pm}(x,y)$:

$$\mathrm{F}_1^{\pm}(x,y) = \mathcal{H}_1^{\pm}(x,y)+\mathcal{H}_2^{\pm}(x,y,\epsilon^2)-\mathcal{H}_2^{\pm}(x,y,z_m) \tag{89}$$

[3]In the present calculation only $n\leq 1$ is needed. Thus, instead of eq.(87) one can use: $I_n = \pi(P\cdot G)^{n+1}(Q\cdot P)^n/(P^2)^{n+1}$ for $(n=0,1)$.

where the first term in r.h.s. is the sum of the three-body virtual corrections and the infrared divergent four-body ones. The two other terms originate from the infrared finite four-body piece. For $\mathrm{J}_1^{\pm}(x,y)$ one has:

$$\mathrm{J}_1^{\pm}(x,y) = \mathcal{K}_1^{\pm}(x,y) + \mathcal{K}_2^{\pm}(x,y,\epsilon^2) - \mathcal{K}_2^{\pm}(x,y,z_m) \tag{90}$$

The results of [19] and [23] are presented in this way; see eqs.(3.2)-(3.6) and (4.5)-(4.8) in [19], and (3.1),(A.1)-(A.15) in [23].
Both $\mathcal{H}_2^{\pm}(x,y,z)$ and $\mathcal{K}_2^{\pm}(x,y,z)$ are given by lenghty expressions:

$$\begin{aligned}\mathcal{H}_2^{\pm}(x,y,z) = (a_1^{\pm} + z a_2^{\pm})\,\mathcal{Y}_p(z)/\lambda^{3/2}(z) \;+\; (a_3^{\pm} + z a_4^{\pm})\,\mathcal{Y}_p(z)/\sqrt{\lambda(z)} \\ + a_5^{\pm}\,\mathcal{Y}_p(z)\sqrt{\lambda(z)} + (a_6^{\pm} + z a_7^{\pm})/\lambda(z) \\ + a_8^{\pm}\, z + a_9^{\pm}\, z^{-1} + a_{10}^{\pm}\, \ln z \;+\; a_{11}^{\pm}\,[\mathrm{Li}_2(\mathcal{W}_+(z)) + \mathrm{Li}_2(\mathcal{W}_-(z))]\end{aligned} \tag{91}$$

$$\begin{aligned}\mathcal{K}_2^{\pm}(x,y,z) = (b_1^{\pm} + z b_2^{\pm})\,\mathcal{Y}_p(z)/\lambda^{5/2}(z) + \;(b_3^{\pm} + z b_4^{\pm})\,\mathcal{Y}_p(z)/\lambda^{3/2}(z) \\ + (b_5^{\pm} + z b_6^{\pm})\,\mathcal{Y}_p(z)/\sqrt{\lambda(z)} + \;b_7^{\pm}\,\mathcal{Y}_p(z)\sqrt{\lambda(z)} \\ +(b_8^{\pm} + z b_9^{\pm})/\lambda^2(z) + (b_{10}^{\pm} + z b_{11}^{\pm})/\lambda(z) \\ + b_{12}^{\pm}\, z + b_{13}^{\pm}\, z^{-1} + b_{14}^{\pm}\, \ln z \;+\; b_{15}^{\pm}\,[\mathrm{Li}_2(\mathcal{W}_+(z)) + \mathrm{Li}_2(\mathcal{W}_-(z))]\end{aligned} \tag{92}$$

where $a_j^{\pm}$ and $a_j^{\pm}$ are rational functions of x, y and ϵ^2, and

$$\mathcal{W}_{\pm}(z) = \left[\; 1 + y - z \pm \sqrt{\lambda(z)}\; \right]/2 \tag{93}$$

These complicated expressions simplify dramatically for $z = \epsilon^2$ and $z = z_m$. At these special points

$$\begin{aligned}&\mathcal{Y}_p(\epsilon^2) = Y_p \\ &\sqrt{\lambda(\epsilon^2)} = 2p_0 \\ &\mathcal{W}_{\pm}(\epsilon^2) = w_{\pm} \\ &\lambda(z_m) = (x - y/x)^2 \\ &\mathcal{Y}_p(z_m)/\sqrt{\lambda(z_m)} = \tfrac{1}{2(x-y/x)} \ln\left(\tfrac{1-y/x}{1-x}\right) \\ &\mathrm{Li}_2(\mathcal{W}_+(z_m)) + \mathrm{Li}_2(\mathcal{W}_-(z_m)) = \mathrm{Li}_2(x) + \mathrm{Li}_2(y/x)\end{aligned} \tag{94}$$

and after some algebra eqs.(9) and (10) are derived.

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