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K^0 Decay into Three Photons *

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Abstract

The decays $K_{L,S} \to 3\gamma$ are not forbidden by any selection rules or symmetry principles. However, gauge invariance and Bose statistics dictate that every photon pair in these transitions has at least two units of angular momentum. This gives rise to an extraordinary suppression. Using a simple model, we obtain the branching ratios $B(K_L \to 3\gamma) \sim 3 \cdot 10^{-19}$, $B(K_S \to 3\gamma) \sim 5 \cdot 10^{-22}$.

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* *in celebration of the S.N. Bose birth centenary*

The decays $K_L \to 2\gamma$ and $K_S \to 2\gamma$ have been observed with branching ratios of $5.7 \cdot 10^{-4}$ and $2.4 \cdot 10^{-6}$, respectively. What is the expected rate of the decays $K_{L,S} \to 3\gamma$?

First of all, one should observe that both $K_L \to 3\gamma$ and $K_S \to 3\gamma$ are possible without violating CP invariance or any general symmetry principle. Since the 3γ system has $C = -1$, the decay $K_L \to 3\gamma$ can proceed via the C–violating, P–violating part of the $|\Delta S| = 1$ nonleptonic weak interaction, while $K_S \to 3\gamma$ proceeds via the C–conserving, P–conserving part. Naively, one would imagine that these decays would occur at rates that are roughly a factor α_{em} times the two–photon decay rates.

This naive expectation, however, disregards the constraints of gauge invariance and Bose statistics. Gauge invariance dictates that in the decay $K^0 \to 3\gamma$, no pair of photons can have angular momentum zero, since that would correspond to a $0 \to 0$ radiative transition, which is forbidden for a real photon. Similarly, no pair of photons can have $J = 1$, since that conflicts with Bose statistics (Yang's theorem). It follows that the decays $K_{L,S} \to 3\gamma$ can only occur if each pair of photons in the final state has at least two units of angular momentum. The matrix element thus inevitably has a large number of angular momentum suppression factors. Using a simple model, we show below that the decays $K_{L,S} \to 3\gamma$ have rates that are **15 orders of magnitude lower** than the corresponding rates of $K_{L,S} \to 2\gamma$!

The model we employ is illustrated in Fig. 1. We assume that the $K_{L,S} \to 3\gamma$

transition is mediated by the decay $K_{L,S} \to \pi^0\pi^0\gamma$, with the two π^0's converting into two photons. The use of this particular channel is motivated by the fact that the decays $K_{L,S} \to \pi^0\pi^0\gamma$ are necessarily quadrupole transitions (E2 and M2 respectively [1]), so that the pion pair has $J = 2$, which is the minimum angular momentum required for the photon pairs in $K_{L,S} \to 3\gamma$. Other intermediate states are undoubtedly possible (including e.g. $K_{L,S} \to \pi^+\pi^-\gamma$, with pions in a D–wave). Our aim, however, is to expose the symmetry structure of the $K^0 \to 3\gamma$ amplitude and to obtain a rough estimate of its magnitude. The $\pi^0\pi^0\gamma$ intermediate state turns out to be adequate to this purpose.

To determine the matrix element of $K_L \to 3\gamma$ from the model shown in Fig. 1, we require the amplitudes for $K_L \to \pi^0\pi^0\gamma$ and $\pi^0\pi^0 \to \gamma\gamma$, which we parameterise as follows:

$$\begin{aligned} &\mathcal{M}\left(K_L(p_K) \to \pi^0(p_1)\,\pi^0(p_2)\,\gamma(k_3,\epsilon_3)\right) \\ =\; &\frac{h_L}{M_K^5}\left[\epsilon_3\cdot p_1 k_3\cdot p_2 - \epsilon_3\cdot p_2 k_3\cdot p_1\right]\, k_3\cdot(p_1-p_2) \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathcal{M}\left(\gamma(k_1,\epsilon_1)+\gamma(k_2,\epsilon_2) \to \pi^0(p_1)\,\pi^0(p_2)\right) \\ =\; &\frac{\tilde{G}s_{12}}{M_V^2}\cdot\left[\frac{p_1\cdot k_1\; p_1\cdot k_2}{k_1\cdot k_2}\,g_{\mu\nu} + p_{1\mu}p_{1\nu} - \frac{p_1\cdot k_1}{k_1\cdot k_2}\,k_{2\mu}p_{1\nu} - \frac{p_1\cdot k_2}{k_1\cdot k_2}\,k_{1\nu}p_{1\mu}\right]\epsilon_1^\mu\,\epsilon_2^\nu \end{aligned} \tag{2}$$

The structure in Eq. (1) is appropriate to an E2 transition, while that in Eq. (2) is obtained using a vector meson exchange model for $\pi^0\pi^0 \to \gamma\gamma$ (see, e.g., Ref. [2]), keeping only the leading term in s_{12}/M_V^2 ($s_{12} \equiv (k_1+k_2)^2$). (Numerically, the constant $\tilde{G}$ is given by $\tilde{G} = \tilde{G}_\rho + \tilde{G}_\omega$ with $\tilde{G}_\rho = \frac{1}{9}\,g^2_{\omega\pi\gamma}$, $\tilde{G}_\omega = g^2_{\omega\pi\gamma}$ and $g_{\omega\pi\gamma} = 7.7\cdot 10^{-4}\ \mathrm{MeV}^{-1}$.)

We now employ unitarity to obtain the absorptive part of $K_L \to 3\gamma$:

$$
\begin{aligned}
& Im\,\mathcal{M}\left[K_L(p_K) \to \gamma(\epsilon_1, k_1) + \gamma(\epsilon_2, k_2) + \gamma(\epsilon_3, k_3)\right] \\
= \; & \frac{1}{2} \int \frac{d^3 p_1}{2 p_{1\,0} (2\pi)^3} \, \frac{d^3 p_2}{2 p_{2\,0} (2\pi)^3} \, (2\pi)^4 \delta^{(4)}(p_K - p_1 - p_2 - k_3) \\
& \cdot \left\{ \frac{h_L}{M_K^2} (\epsilon_3 \cdot p_1 \; k_3 \cdot p_2 - \epsilon_3 \cdot p_2 \; k_3 \cdot p_1) \; k_3 \cdot (p_1 - p_2) \right\} \\
& \cdot \frac{\tilde{G} s_{12}}{M_V^2} \, \epsilon_1^\mu \epsilon_2^\nu \left\{ \frac{p_1 \cdot k_1 \; p_2 \cdot k_2}{k_1 \cdot k_2} \, g_{\mu\nu} + p_{1\mu} p_{2\nu} - \frac{p_1 \cdot k_1}{k_1 \cdot k_2} \, k_{2\mu} p_{1\nu} - \frac{p_1 \cdot k_2}{k_1 \cdot k_2} \, k_{1\nu} p_{1\mu} \right\} \\
& \cdot \Theta(s_{12} - 4 m_\pi^2) \\
& + \text{permutations of } (\epsilon_1, k_1), (\epsilon_2, k_2), (\epsilon_3, k_3) \qquad (3)
\end{aligned}
$$

To evaluate this, we need to calculate integrals of the form

$$
\begin{aligned}
K^{\mu\nu\rho\sigma} &= \int \frac{d^3 p_1}{2 p_{1\,0}} \, \frac{d^3 p_2}{2 p_{2\,0}} \, \delta^{(4)}(P - p_1 - p_2) f(p_1 \cdot p_2) \; p_1^\mu p_1^\nu p_1^\rho p_1^\sigma \; , \\
L^{\mu\nu\rho\sigma} &= \int \frac{d^3 p_1}{2 p_{1\,0}} \, \frac{d^3 p_2}{2 p_{2\,0}} \, \delta^{(4)}(P - p_1 - p_2) f(p_1 \cdot p_2) \; p_1^\mu p_1^\nu p_1^\rho p_2^\sigma \qquad (4)
\end{aligned}
$$

These are given in the appendix. The resulting expression for $\mathcal{M}_{abs} \equiv Im\,\mathcal{M}(K_L \to \gamma\gamma\gamma)$ is then squared, and the polarizations of the photons summed over, using the symbolic computation program FORM [3], with the result

$$
\begin{aligned}
\sum_{pol} |\mathcal{M}_{abs}|^2 &= V_{12}^2 F_{12}^2 \{ s_{12} s_{23}^3 s_{13} + s_{12} s_{23} s_{13}^3 \} \\
&+ V_{23}^2 F_{23}^2 \{ s_{12}^3 s_{23} s_{13} + s_{12} s_{23} s_{13}^3 \} \\
&+ V_{13}^2 F_{13}^2 \{ s_{12}^3 s_{23} s_{13} + s_{12} s_{23}^3 s_{13} \} \\
&- 2 V_{12} V_{23} F_{12} F_{23} s_{12} s_{23} s_{13}^3 \\
&- 2 V_{12} V_{13} F_{12} F_{13} s_{12} s_{23}^3 s_{13} \\
&- 2 V_{13} V_{23} F_{13} F_{23} s_{12}^3 s_{23} s_{13} \qquad (5)
\end{aligned}
$$

where the functions V_{ij} and F_{ij} are defined by

$$\begin{aligned} V_{ij} &= \frac{1}{16\pi}\frac{h_L}{M_K^5}\frac{\tilde{G}}{M_V^2}\frac{1}{s_{ij}^2}\sqrt{\lambda(s_{ij},m_\pi^2,m_\pi^2)} \\ F_{ij} &= \frac{1}{5}\,s_{ij}\left\{\frac{1}{8}\,s_{ij}^2 - \frac{29}{24}\,s_{ij}m_\pi^2 + 2m_\pi^4\right\} \\ \text{with} \qquad & \lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2xz \end{aligned} \tag{6}$$

The following features of the above result (5) should be noted:

(i) The Dalitz plot density, given by $\sum|\mathcal{M}_{abs}|^2$, is manifestly symmetric in s_{12}, s_{23} and s_{13}, as required by Bose statistics.

(ii) The density vanishes at the centre of the Dalitz plot ($s_{12}=s_{23}=s_{13}$), i.e. for the configuration in which the photons have equal energy.

(iii) Because of the overall factor $s_{12}s_{23}s_{13}$, the density vanishes along the boundaries of the Dalitz plot, defined by $s_{12}=0, s_{23}=0$ and $s_{13}=0$. These correspond to the configurations in which two of the three photons are collinear.

(iv) In the limit $m_\pi \to 0$, the factors $V_{ij}F_{ij}$ become proportional to s_{ij}^2, and the density simplifies to

$$\begin{aligned} \sum|\mathcal{M}_{abs}|^2 \overset{m_\pi=0}{\sim} s_{12}s_{23}s_{13} \quad & \left\{s_{12}^2\,(s_{12}^2s_{23}^2 + s_{12}^2s_{13}^2 - 2s_{23}^2s_{13}^2)\right. \\ & +s_{23}^2\,(s_{23}^2s_{12}^2 + s_{23}^2s_{13}^2 - 2s_{12}^2s_{13}^2) \\ & \left.+s_{13}^2\,(s_{13}^2s_{12}^2 + s_{13}^2s_{23}^2 - 2s_{12}^2s_{23}^2)\right\} \end{aligned} \tag{7}$$

(v) The result (7) has some resemblance to the matrix element for $\pi^0 \to 3\gamma$ obtained by Dicus [4]:

$$\sum |\mathcal{M}|^2 |_{\text{Dicus}} \sim s_{12}s_{13}s_{23} \{ s_{12}^2 s_{13}^2 + s_{12}^2 s_{23}^2 + s_{23}^2 s_{13}^2 - s_{13}s_{23}s_{12}^2 - s_{13}^2 s_{23}s_{12} - s_{13}s_{23}^2 s_{12} \} \tag{8}$$

In concordance with Ref. [4], we find that the 3γ matrix element contains a large number of momentum factors, which are ultimately responsible for an enormous suppression of the decay rate. (The result given by our model has two extra powers of s_{ij} compared to the expression in Eq. (8).)

Finally, we obtain the rate of $K_L \to 3\gamma$ using

$$\frac{d\Gamma}{ds_{12}\,ds_{23}} = \frac{1}{(2\pi)^3}\frac{1}{32M_K^3}\frac{1}{3!}\sum |\mathcal{M}|^2 \tag{9}$$

and assuming that $\sum |\mathcal{M}|^2$ is reasonably approximated by the absorptive part given in Eq. (5). The parameter h_L is determined by the decay rate of $K_L \to \pi^0\pi^0\gamma$, for which we use the theoretical estimate $B(K_L \to \pi^0\pi^0\gamma) \sim 1 \cdot 10^{-8}$ obtained in [1]. The resulting branching ratio for $K_L \to 3\gamma$ is

$$B(K_L \to 3\gamma) \sim 3 \cdot 10^{-19} \tag{10}$$

The above considerations can be repeated for the decay $K_S \to 3\gamma$, the only difference being that the E2 matrix element given in Eq. (1) has to be replaced by the M2 matrix element for $K_S \to \pi^0\pi^0\gamma$:

$$\mathcal{M}(K_S(p_K) \to \pi^0(p_1)\pi^0(p_2)\gamma(\epsilon, k) = \frac{h_S}{M_K^5}\,(p_1 - p_2)\cdot k\,\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu k^\nu p_1^\rho p_2^\sigma \tag{11}$$

The Dalitz plot density turns out to have exactly the same functional dependence on s_{12}, s_{23} and s_{13} as in Eq. (5). The branching ratio is estimated to be

$$B(K_S \to 3\gamma) \sim 5 \cdot 10^{-22} \tag{12}$$

The exceedingly low rates given by Eqs. (10) and (12) imply that the three photon decay of the neutral K meson cannot be a significant background to decays of the type $K^0 \to 2\pi^0 \to 4\gamma$, in which one photon is undetected.

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Appendix

We give here the integrals defined in Eq. (4), for the general case $p_1^2 = m_1^2,\ p_2^2 = m_2^2$.

$$
\begin{aligned}
K^{\mu\nu\rho\sigma} &= \int \frac{d^3p_1}{2p_{1\,0}} \frac{d^3p_2}{2p_{2\,0}} \delta^{(4)}\ (P - p_1 - p_2)\ f(p_1 \cdot p_2)\ p_1^\mu p_1^\nu p_1^\rho p_1^\sigma \\
&= \frac{\pi}{2} \frac{1}{s^3} \sqrt{\lambda(s, m_1^2, m_2^2)} f \left[\frac{1}{2}(s - m_1^2 - m_2^2)\right] \cdot \frac{1}{5} \\
&\cdot \left\{ \frac{1}{s^2} \left[(s + m_1^2 - m_2^2)^4 - 3sm_1^2(s + m_1^2 - m_2^2)^2 + s^2 m_1^4\right] P^\mu P^\nu P^\rho P^\sigma \right. \\
&- \left[\frac{1}{8}\frac{1}{s}(s + m_1^2 - m_2^2)^4 - \frac{7}{12} m_1^2 (s + m_1^2 - m_2^2)^2 + \frac{1}{3} s m_1^4\right] \\
&\cdot (P^\mu P^\nu g^{\rho\sigma} + P^\nu P^\rho g^{\mu\sigma} + P^\mu P^\rho g^{\nu\sigma} + P^\nu P^\sigma g^{\mu\rho} + P^\rho P^\sigma g^{\mu\nu} + P^\mu P^\sigma g^{\nu\rho}) \\
&+ \frac{1}{3} \left[\frac{1}{16}(s + m_1^2 - m_2^2)^4 - \frac{1}{2} s m_1^2 (s + m_1^2 - m_2^2)^2 + s^2 m_1^4\right] \\
&\left. \cdot (g^{\mu\nu} g^{\rho\sigma} + g^{\nu\rho} g^{\mu\sigma} + g^{\mu\rho} g^{\nu\sigma}) \right\}
\end{aligned}
$$

$$
\begin{aligned}
L^{\mu\nu\rho\sigma} &= \int \frac{d^3p_1}{2p_{1\,0}} \frac{d^3p_2}{2p_{2\,0}} \delta^{(4)}\ (P - p_1 - p_2)\ f(p_1 \cdot p_2)\ p_1^\mu p_1^\nu p_1^\rho p_2^\sigma \\
&= \frac{\pi}{2} \frac{1}{s^3} \sqrt{\lambda(s, m_1^2, m_2^2)} f \left[\frac{1}{2}(s - m_1^2 - m_2^2)\right] \cdot \frac{1}{5} \\
&\cdot \left\{ \frac{1}{s^2} \left[(s + m_1^2 - m_2^2)^3 (s - m_1^2 + m_2^2)^2 - \frac{3}{4} s \left\{(s + m_1^2 - m_2^2)^2 \right.\right.\right. \\
&\qquad \left. \cdot (s - m_1^2 - m_2^2) + m_1^2 (s + m_1^2 - m_2^2)(s - m_1^2 + m_2^2)\right\} \\
&\qquad \left. + \frac{1}{2} s^2 m_1^2 (s - m_1^2 - m_2^2)\right]\ P^\mu P^\nu P^\rho P^\sigma \\
&\quad - \left[\frac{1}{8}\frac{1}{s}(s + m_1^2 - m_2^2)^3 (s - m_1^2 + m_2^2) - \frac{7}{48} \left\{(s + m_1^2 - m_2^2)^2 \right.\right. \\
&\qquad \left. \cdot (s - m_1^2 - m_2^2) + m_1^2 (s + m_1^2 - m_2^2)(s - m_1^2 + m_2^2)\right\} \\
&\qquad \left. + \frac{1}{6} s m_1^2 (s - m_1^2 - m_2^2)\right] \\
&\cdot (P^\mu P^\nu g^{\rho\sigma} + P^\nu P^\rho g^{\mu\sigma} + P^\mu P^\rho g^{\nu\sigma} + P^\nu P^\sigma g^{\mu\rho} + P^\rho P^\nu g^{\mu\nu} + P^\mu P^\sigma g^{\nu\rho})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\left[\frac{1}{16}(s+m_1^2-m_2^2)^3(s-m_1^2+m_2^2)-\frac{1}{8}s\left\{(s+m_1^2-m_2^2)^2\right.\right.\\
&\quad\left.\cdot(s-m_1^2-m_2^2)+m_1^2(s+m_1^2-m_2^2)(s-m_1^2+m_2^2)\right\}\\
&\quad\left.\left.+\frac{1}{2}s^2m_1^2(s-m_1^2-m_2^2)\right]\ \cdot(g^{\mu\nu}g^{\rho\sigma}+g^{\nu\rho}g^{\mu\sigma}+g^{\mu\rho}g^{\nu\sigma})\right\}
\end{aligned}
$$

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