TTP93–9
March 1993

Mass Corrections to the Tau Decay Rate

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Abstract

In this note radiative corrections to the total hadronic decay rate of the τ-lepton are studied employing perturbative QCD and the operator product expansion. We calculate quadratic quark mass corrections to the decay rate ration R_τ to the order $\mathcal{O}(\alpha_s^2 m^2)$ and find that they contribute appreciably to the Cabbibo supressed decay modes of the τ-lepton. We also discuss corrections of mass dimension $D = 4$, where we emphasize the need of a suitable choice of the renormalization scale of the quark and gluon condensates.

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1 Introduction

Within the three lepton generations which to our present knowledge constitute the leptonic sector of the Standard Model, the τ-lepton is the only particle decaying into a semihadronic final state. As was pointed out [1, 2, 3, 4, 5, 6, 7, 8] some time ago, the methods of perturbative QCD can be applied to estimate the decay rate ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau \text{hadrons})}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)}. \tag{1}$$

Work has also concentrated on nonperturbative effects [1, 2, 3, 5, 8] as well as on electroweak corrections [9, 10] to this quantity.

In order to calculate QCD corrections to R_τ we consider the 2-point correlators for the vector ($j^V_{\mu,ij} = \bar{q}_i \gamma_\mu q_j$) and the axial vector ($j^A_{\mu,ij} = \bar{q}_i \gamma_\mu \gamma_5 q_j$) currents ($i, j = u, d, s$):

$$\Pi^{V/A}_{\mu\nu,ij}(q) = i \int dx e^{iqx} \langle T[j^{V/A}_{\mu,ij}(x) j^{V/A\dagger}_{\nu,ij}(0)] \rangle \tag{2}$$

As usual the correlation functions may be decomposed into a transversal and a longitudinal part

$$\Pi^{V/A}_{\mu\nu}(q) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi^{(1)}_{V/A}(q^2) + q_\mu q_\nu \Pi^{(0)}_{V/A}(q^2), \tag{3}$$

where the spectral functions $\Pi^{(0)}_{V/A}(q^2), \Pi^{(1)}_{V/A}(q^2)$ correspond to hadronic final states with respective angular momenta $J = 0, J = 1$ in the hadronic rest frame. The hadronic decay rate of the τ-lepton is obtained by integrating the absorptive parts of the spectral functions with respect to the invariant hadronic mass:

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \text{Im}\Pi^{(1)}(s + i\epsilon) + \text{Im}\Pi^{(0)}(s + i\epsilon) \right], \tag{4}$$

where

$$\Pi^{(J)} = |V_{ud}|^2 (\Pi^{(J)}_{ud,V} + \Pi^{(J)}_{ud,A}) + |V_{us}|^2 (\Pi^{(J)}_{us,V} + \Pi^{(J)}_{us,A}). \tag{5}$$

Due to the analyticity of the spectral functions in the complex s-plane, which is cut along the real positive s-axis, R_τ can be expressed as the contour integral along a circle C of radius $|s| = M_\tau^2$:

$$R_\tau = 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^{(0+1)}(s) - 2\frac{s}{M_\tau^2} \Pi^{(0)}(s) \right] \tag{6}$$

We have used in this equation the combination $\Pi^{(0+1)}(q^2) = \Pi^{(0)}(q^2) + \Pi^{(1)}(q^2)$.

R_τ may be expressed as the sum of different contributions corresponding to Cabbibo suppressed or allowed decay modes, vector or axial vector contributions and the mass dimension of the corrections:

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \tag{7}$$

with

$$
\begin{aligned}
R_V = & \frac{3}{2}|V_{ud}|^2\left(1+\delta^{(0)}+\sum_{D=2,4,\ldots}\delta^{(D)}_{V,ud}\right), \\
R_A = & \frac{3}{2}|V_{ud}|^2\left(1+\delta^{(0)}+\sum_{D=2,4,\ldots}\delta^{(D)}_{A,ud}\right), \\
R_S = & 3|V_{us}|^2\left(1+\delta^{(0)}+\sum_{D=2,4,\ldots}\delta^{(D)}_{us}\right).
\end{aligned}
\tag{8}
$$

Here D indicates the mass dimension of the fractional corrections $\delta^{(D)}_{ij,V/A}$ and $\delta^{(D)}_{ij}$ denotes the average of the vector and the axial vector contributions: $\delta^{(D)}_{ij,} = (\delta^{(D)}_{ij,V}+\delta^{(D)}_{ij,A})/2$.

In the literature results for perturbative QCD corrections to current-current correlators have been published by several groups. For the electromagnetic current correlator corrections were calculated in the massless limit up to $\mathcal{O}(\alpha_s^2)$ [11] and $\mathcal{O}(\alpha_s^3)$ [12], whereas quadratic mass corrections to this quantity were given to second $\mathcal{O}(\alpha_s^2)$ [13] and third $\mathcal{O}(\alpha_s^3)$ order [14]. For the axial current correlator mass corrections are known from two loop [15, 16, 17] and three loop calculations [18] for flavour non singlet type diagrams. Flavour singlet contributions to the axial correlator involving purely gluonic intermediate states were studied in the heavy top limit in [19, 20] and for the massive case in [21]. Finally second order massless corrections for the scalar current correlator can be found in [22].

A detailed analysis of QCD corrections for the τ decay into hadrons has been performed in [5], where much attention focussed on nonperturbative contributions to corrections of higher dimension.

The aim of this letter is twofold. First we extend the analysis of ref. [5] and calculate the order $\mathcal{O}(\alpha_s^2)$ term of the (dimension $D=2$) longitudinal spectral function $\Pi^{(0)}_{V/A}(q^2)$ in its power expansion with respect to α_s. This analysis is done in section 2.
The second part of this note is contained in section 3 where we consider corrections of mass dimension $D=4$ to R_τ. We estimate their size where we use a scale $\hat{\mu}$ for the quark condensates which corresponds to the energy scale of the process under consideration. Our choice $\hat{\mu}^2 = M_\tau^2$ seems to be more appropriate to us than $\hat{\mu}^2=\infty$ as used in [5].
Our numerical results are discussed in section 4. We finally list some formulae in the appendix.

2 Dimension $D=2$ Corrections

The way to derive the quadratic mass correction to the longitudinal spectral function $\Pi^{(0)}_{V/A;ij}$ is based on the knowledge of the anomalous dimensions $\gamma_m^{VV/AA}$ of the vector and the axial vector correlators as defined in [18]

$$\mu^2 \frac{d}{d\mu^2}\Pi^{V/A}_{\mu\nu} = \frac{q_\mu q_\nu - g_{\mu\nu}q^2}{16\pi^2}\gamma_q^{VV/AA}(\alpha_s) + (m_i \mp m_j)^2 \frac{g_{\mu\nu}}{16\pi^2}\gamma_m^{VV/AA}(\alpha_s). \tag{9}$$

Up to order $\mathcal{O}(\alpha_s^2)$ the anomalous dimension $\gamma_m^{VV/AA}$ is given by (with f denoting the number of quark flavours)

$$\gamma_m^{VV/AA} = 6\left(1 + \frac{5}{3}\frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left[\frac{455}{72} - \frac{1}{3}f - \frac{1}{2}\zeta(3)\right]\right) \tag{10}$$

and governs the renormalization group equation for the longitudinal spectral function

$$\mu^2 \frac{d}{d\mu^2}\Pi^{(0)}_{V/A;ij} = -\frac{(m_i(\mu^2) \mp m_j(\mu^2))^2}{Q^2}\gamma_m^{VV/AA}\frac{1}{16\pi^2}. \tag{11}$$

In order to get $\Pi^{(0)}_{V/A}$ the RGE eq.(9)must be integrated. The solution reads

$$\begin{aligned}\Pi^{(0)}_{V/A;ij} = \;& \frac{1}{16\pi^2}\frac{(m_i(\mu^2) \mp m_j(\mu^2))^2}{Q^2}\exp\left[2\int_{\frac{\alpha_s(\mu_0^2)}{\pi}}^{\frac{\alpha_s(\mu^2)}{\pi}} dA' \frac{\gamma_m}{\beta}\right] \\ & \cdot \int_{\frac{\alpha_s(\mu_0^2)}{\pi}}^{\frac{\alpha_s(\mu^2)}{\pi}} dA' \frac{-\gamma_m^{VV/AA}}{\beta}\exp\left[-2\int_{\frac{\alpha_s(\mu_0^2)}{\pi}}^{A'} dA'' \frac{\gamma_m}{\beta}\right] + C(\mu_0^2),\end{aligned} \tag{12}$$

where the β-function and the quark mass anomalous dimension γ_m are given in the appendix. The integration constant $C(\mu_0^2)$ may be fixed by a specific choice for the arbitrary scale μ_0^2 which we choose to be $\mu_0^2 = Q^2$. This leads to

$$C(Q^2) = \frac{3}{8\pi^2}\frac{(m_i(Q^2) \mp m_j(Q^2))^2}{Q^2}\left\{-2 + \frac{\alpha_s(Q^2)}{\pi}\left[4\zeta(3) - \frac{131}{12}\right]\right\}. \tag{13}$$

With the relation between the μ^2-dependent mass of the $\overline{\text{MS}}$-scheme and the Q^2-dependent running mass (see appendix)

$$m^2(\mu^2) = m^2(\mu_0^2)\exp\left[-2\int_{\frac{\alpha_s(\mu_0^2)}{\pi}}^{\frac{\alpha_s(\mu^2)}{\pi}} dA' \frac{\gamma_m}{\beta}\right] \tag{14}$$

we cast $\Pi^{(0)}_{V/A;ij}$ into the following form

$$
\begin{aligned}
\Pi^{(0)}_{V/A;ij} = & -\frac{1}{16\pi^2}\frac{(m_i(Q^2)\mp m_j(Q^2))^2}{Q^2}\Bigg\{\exp\left[2\int^{\frac{\alpha_s(Q^2)}{\pi}} dA'\frac{\gamma_m}{\beta}\right] \\
& \cdot\int^{\frac{\alpha_s(Q^2)}{\pi}} dA'\frac{-\gamma_m^{VV/AA}}{\beta}\exp\left[-2\int^{A'} dA''\frac{\gamma_m}{\beta}\right] - C(Q^2)\Bigg\} \\
& +\frac{1}{16\pi^2}\frac{(m_i(\mu^2)\mp m_j(\mu^2))^2}{Q^2}\exp\left[2\int^{\frac{\alpha_s(\mu^2)}{\pi}} dA'\frac{\gamma_m}{\beta}\right] \\
& \cdot\int^{\frac{\alpha_s(\mu^2)}{\pi}} dA'\frac{\gamma_m^{VV/AA}}{\beta}\exp\left[-2\int^{A'} dA''\frac{\gamma_m}{\beta}\right].
\end{aligned}
\tag{15}
$$

Inserting the series expansion for β and γ_m as given in the appendix we obtain after integration

$$
\begin{aligned}
\Pi^{(0)}_{V/A;ij} = & \frac{3}{2\pi^2}\frac{(m_i(Q^2)\mp m_j(Q^2))^2}{Q^2}\Bigg\{\left(\frac{\alpha_s(Q^2)}{\pi}\right)^{-1} - \frac{5}{2} \\
& + \left[-\frac{21373}{4896}+\frac{75}{34}\zeta(3)\right]\frac{\alpha_s(Q^2)}{\pi}\Bigg\} \\
& -\frac{3}{2\pi^2}\frac{(m_i(\mu^2)\mp m_j(\mu^2))^2}{Q^2}\Bigg\{\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^{-1} - 2 \\
& + \left[-\frac{8011}{4896}+\frac{41}{34}\zeta(3)\right]\frac{\alpha_s(\mu^2)}{\pi}\Bigg\}
\end{aligned}
\tag{16}
$$

The leading and next to leading terms of this expansion agree with [5] whereas the coefficient of α_s is new. The μ^2-dependent part of eq.(16) does not contribute to the hadronic τ-decay rate.

Mass corrections of the transversal spectral function $\Pi^{(0+1)}_{V/A;ij}$ were calculated to order $\mathcal{O}(\alpha_s^2 m^2)$ for the electromagnetic correlation function in [13]. Recently one of us [23] extended this calculation to the case of unequal masses. The obtained result reads

$$
\begin{aligned}
\Pi^{(0+1)}_{V/A;ij} = \quad & -\frac{3}{8\pi^2}\frac{(m_i(Q^2)\pm m_j(Q^2))^2}{Q^2}\left\{1+\frac{8}{3}\frac{\alpha_s(Q^2)}{\pi}\right.\\
&\left. +\left[\frac{17981}{432}+\frac{62}{27}\zeta(3)-\frac{1045}{54}\zeta(5)\right]\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2\right\}\\
& -\frac{3}{8\pi^2}\frac{(m_i(Q^2)\mp m_j(Q^2))^2}{Q^2}\left\{1+2\frac{\alpha_s(Q^2)}{\pi}\right.\\
&\left. +\left[\frac{4351}{144}+\frac{13}{3}\zeta(3)-\frac{115}{6}\zeta(5)\right]\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2\right\}\\
& +\frac{1}{12\pi^2}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2(32-24\zeta(3))\sum_k\frac{m_k(Q^2)}{Q^2}.
\end{aligned}
\tag{17}
$$

Even for the decay of the τ-lepton into non strange quarks is a dependence on m_s introduced by the second order contributions due to the s-quark circulating in a fermion loop. The term proportional to $\sum_k m_k(Q^2)/Q^2$ was first computed in [24] For the vector correlator with $m_i = m_j$ the result eq.(17) exactly reproduces the one of [13]. This means that a slightly different result (namely for the terms proportional to $\zeta(3)$) for the diagonal vector correlator given in [25] seems to be wrong. For a detailed discussion we refer the reader to [23].

3 Dimension $D = 4$ Corrections

Besides the perturbative radiative corrections to R_τ also nonperturbative QCD effects influence the hadronic τ decay rate. The short distance operator product expansion (OPE) for the spectral functions

$$
\Pi^{(J)}(-q^2) = \sum_{D=0,2,4,\cdots}\frac{1}{(-s)^{D/2}}\sum_{\dim\mathcal{O}=D} C^{(J)}(Q^2,\mu)\langle\mathcal{O}(\mu)\rangle \tag{18}
$$

may be used to take into account both perturbative and non perturbative contributions. For the fractional corrections $\delta^{(D)}_{V/A;ij}$ (see eq.(8)) we obtain

$$
\begin{aligned}
\delta^{(D)}_{V/A;ij} = \quad & \sum_{\dim\mathcal{O}=D}\frac{\langle\mathcal{O}(\mu)\rangle}{M_\tau^D}4\pi i\int_{|s|=M_\tau^2}\frac{ds}{M_\tau^2}\left(\frac{-s}{M_\tau^2}\right)^{-D/2}\left(1-\frac{s}{M_\tau^2}\right)^2\\
&\left[\left(1+2\frac{s}{M_\tau^2}\right)C^{(0+1)}_{ij,V/A}(s,\mu)-2\frac{s}{M_\tau^2}C^{(0)}_{ij,V/A}(s,\mu)\right].
\end{aligned}
\tag{19}
$$

The local operators in this expansion for the $D = 4$ perturbative corrections are the unit operator multiplied by a quartic product of quark masses and the vacuum expectation values of composite operators constructed with gluon and quark fields: $\langle GG(\mu)\rangle, \langle m_i(\mu)\bar{\Psi}_j\Psi_j(\mu)\rangle$. The latter contain nonperturbative contributions [26, 27] as well as mass logarithms of the form [16, 28] $m^4\alpha_s^n(\mu)\ln^k(m/\mu)$ and depend nontrivially on the renormalization scale μ via the corresponding renormalization group equations [28]. As usual, setting the renormalization scale $\mu = Q$ in eq.(18) allows to absorb all logarithms $\ln\mu^2/Q^2$ appearing in the coefficient functions $C^{(J)}$ into the running coupling constant. This procedure also leads to an implicit Q^2-dependence of VEV's, which is not convenient for a numerical analysis. The common remedy is to to solve the corresponding RG equations and express $\langle\mathcal{O}\rangle$ in terms of $\alpha_s(Q)$ and some RG invariant combination not depending on Q^2.

As has been shown in [28] it is possible to construct linear combinations of the operators which are invariant with respect to an arbitrary scale $\hat{\mu}$:

$$
\begin{aligned}
\langle\mathcal{I}_G\rangle \;&\equiv \left(1+\frac{16}{9}\frac{\alpha_s(\hat{\mu}^2)}{\pi}\right)\frac{\alpha_s(\hat{\mu}^2)}{\pi}\langle GG(\hat{\mu})\rangle \\
&-\frac{16}{9}\frac{\alpha_s(\hat{\mu}^2)}{\pi}\left(1+\frac{91}{24}\frac{\alpha_s(\hat{\mu}^2)}{\pi}\right)\sum_k\langle m_k(\hat{\mu})\bar{\Psi}_k\Psi_k(\hat{\mu})\rangle \\
&-\frac{1}{3\pi^2}\left(1+\frac{4}{3}\frac{\alpha_s(\mu^2)}{\pi}\right)\sum_k m_k^4(\hat{\mu}), \qquad (20)\\
\langle\mathcal{I}_{ij}\rangle \;&\equiv m_i(\hat{\mu})\langle\bar{\Psi}_j\Psi_j(\hat{\mu})\rangle \\
&+\frac{3}{7\pi\alpha_s(\hat{\mu})}\left(1-\frac{53}{24}\frac{\alpha_s(\hat{\mu}^2)}{\pi}\right)m_i(\hat{\mu})m_j^3(\hat{\mu}),
\end{aligned}
$$

These combinations were intensively used in the analysis of [5] under the names of the gluon condensate $\langle\frac{\alpha_s}{\pi}GG\rangle$ and the quark condensate $\langle m_i\bar{\Psi}_j\Psi_j\rangle$ respectively. Of course the choice is by no means unique. Moreover, it directly leads to an $1/\alpha_s$ enhancement factor in the dimension $D = 4$ correction to the ratio R_τ (see eq.(3.10) in [5]). This results in a partial cancellation between quartic mass corrections and those coming from the quark condensates, which in the final analysis causes some partial loss of accuracy of theoretical predictions [5]).

In addition one can easily check that

$$
\begin{aligned}
\langle\mathcal{I}_G\rangle &= \lim_{\hat{\mu}\to\infty}\frac{\alpha_s(\hat{\mu})}{\pi}\langle GG(\hat{\mu})\rangle, \\
\langle\mathcal{I}_{ij}\rangle &= \lim_{\hat{\mu}\to\infty}\langle m_i(\hat{\mu})\bar{\Psi}_j\Psi_j(\hat{\mu})\rangle.
\end{aligned} \qquad (21)
$$

Thus the choice of eq.(20) as RG invariant vacuum condensates features a quite high normalization scale for operators involved in a problem with a typical momentum transfer of about 1 GeV. No wonder that this causes an artificially large perturbative mass correction of order $\mathcal{O}(m^4)$ to various 2-point correlators as was found in [15].

Keeping all this in mind we have chosen just the very quark and gluon condensates normalized at the "natural" scale $\hat{\mu} = M_\tau$ as our RG invariant condensates:

$$\begin{aligned} \langle\frac{\alpha_s}{\pi}GG\rangle = & \quad \langle\frac{\alpha_s(M_\tau)}{\pi}GG(\hat{\mu} = M_\tau)\rangle, \\ \langle m_i\bar{\Psi}_j\Psi_j\rangle = & \quad \langle m_i(M_\tau)\bar{\Psi}_j\Psi_j(\hat{\mu} = M_\tau)\rangle. \end{aligned} \tag{22}$$

Using the RG invariance property of the combinations eq.(20) it is a straightforward exercise to express the Q^2-dependent VEV's in terms of our RG invariants and the running $\alpha_s(Q), m(Q)$.

The result for the fractional corrections after performing the contour integral reads

$$\begin{aligned} \delta^{(D=4)}_{V/A;ij} \cdot M_\tau^4 = & \ \frac{11}{4}\pi^2 \left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2 \langle\frac{\alpha_s}{\pi}GG\rangle \\ & +16\pi^2 \left[1 + \frac{9}{2}\left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2\right] \langle(m_i \mp m_j)(\bar{\Psi}_i\Psi_i \mp \bar{\Psi}_j\Psi_j)\rangle \\ & -18\pi^2 \left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2 \langle m_i\bar{\Psi}_i\Psi_i + m_j\bar{\Psi}_j\Psi_j\rangle \\ & -8\pi^2 \left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2 \sum_k \langle m_k\bar{\Psi}_k\Psi_k\rangle \\ & -\frac{84}{7}[m_i(M_\tau) \mp m_j(M_\tau)][m_i^3(M_\tau) \mp m_j^3(M_\tau)] \\ & \pm 6m_i(M_\tau)m_j(M_\tau)[m_i(M_\tau) \mp m_j(M_\tau)]^2 \\ & +36m_i^2(M_\tau)m_j^2(M_\tau) \end{aligned} \tag{23}$$

For some terms the leading coefficient is of order $\mathcal{O}(\alpha_s^2)$ due to the fact that a term of given order of the transversal spectral function is contributing only in higher orders to the fractional correction and order $\mathcal{O}(\alpha_s^0)$ terms may eventually integrate to zero. We have neglected terms of order $\alpha_s m^4$ in eq.(23). No enhanced term proportional to the inverse power of the coupling constant occurs as was the case for the choice of reference [5].

4 Numerical Discussion

The numerical discussion in this section is based on the parameters (quark masses, condensates etc.) as they are given in the appendix. Depending on the value of Λ_{QCD} the running coupling constant $\alpha_s(M_\tau^2)$ ranges between 0.16 and 0.44 and is thus sufficiently small for the perturbation expansion to be meaningful. The quadratic mass corrections $\delta_{V/A;ij}^{(D=2)}$ obtained from the contour integration of the spectral functions eqs.(16,17) are collected in Table 1. One observes that corrections from nonstrange decays are negligible, whereas the mass of the strange quark affects the ratio R_S for strange decays by -17% for an intermediate $\alpha_s(M_\tau^2) = 0.3$. When the total hadronic ratio R_τ is considered, this rather large strange mass contribution is reduced by the Cabbibo supression factor $|V_{us}|^2$ to -0.8% ($V_{ud} = 0.9753,\ V_{us} = 0.221$). The influence of the second order correction depends of course on the value of Λ_{QCD}. Even for nonstrange decays strange quark mass effects are present at order $\mathcal{O}(\alpha_s^2)$ due to a virtual strange quark loop. They are quite comparable in size to the leading mass corrections of order $m_u \pm m_d)^2$ and enter with opposite sign. Due to their increasing size for larger α_s the corrections $\delta_{V/A;ud}^{(D=2)}$ show little dependence on Λ_{QCD}. For strange decays of the τ-lepton we recall the numerical values of the coefficients entering the spectral functions (in the limit of vanishing masses of the light quarks)

$$\begin{aligned}\Pi_{V/A;us}^{(0)} = \; & \frac{3}{2\pi^2}\frac{m_s^2(Q^2)}{Q^2}\left\{\left(\frac{\alpha_s(Q^2)}{\pi}\right)^{-1} - \frac{5}{2} - 1.714\frac{\alpha_s(Q^2)}{\pi}\right\} \\ & -\frac{3}{2\pi^2}\frac{m_s^2(\mu^2)}{Q^2}\left\{\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^{-1} - 2 - 1.867\frac{\alpha_s(\mu^2)}{\pi}\right\},\end{aligned} \tag{24}$$

$$\Pi_{V/A;us}^{(0+1)} = -\frac{3}{4\pi^2}\frac{m_s^2(Q^2)}{Q^2}\left\{1 + \frac{7}{3}\frac{\alpha_s(Q^2)}{\pi} + 17.6\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2\right\} \tag{25}$$

and the fractional correction

$$\delta_{V/A;us}^{(2)} = -8\frac{m_s^2(M_\tau)}{M_\tau^2}\left\{1 + \frac{16}{3}\frac{\alpha_s(M_\tau)}{\pi} + 11.03\left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2\right\}. \tag{26}$$

Second order contributions to $\delta^{(D=2)}$ originate not only from order $\mathcal{O}(\alpha_s^2)$ terms of the spectral functions, but are also induced by lower order terms. The contributions from subsequent orders in α_s are showing a reasonable convergence behaviour. For $\Lambda_{QCD} = 52$ MeV the order $\mathcal{O}(\alpha_s^0)$ term in eq.(26) is -5.4%, whereas the higher order terms of $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ amount to -1.5% and -0.16% (and for $\Lambda_{QCD} = 308$ MeV: -2.8% and -0.55%; for $\Lambda_{QCD} = 547$ MeV: -4.1% and -1.2% respectively).

In Table 2 the corrections of mass dimension $D = 4$ are shown. The change of scale from $\hat{\mu} = \infty$ to $\hat{\mu} = M_\tau$ significantly affects only the Cabbibo supressed vector contribution. The small corrections $\delta_{V;us}^{(D=2)}$ in the analysis of [5] were due to a numerical cancellation between the leading quark condensate and an $1/\alpha_s$-enhanced mass term m_s^4. This resulted in a large relative uncertainty for this correction. When the condensates are defined at the scale $\hat{\mu} = M_\tau$ a similar compensation of numerically large terms does not occur due to the absence of the $1/\alpha_s$-enhanced mass term. Compared to the corresponding numbers of [5] the fractional corrections $\delta^{(D=2)}_{V;us}$ are increased by an order of magnitude which reduces the relative uncertainty considerably. In addition they do not change their sign at large values of Λ_{QCD}. The strange $D = 4$ corrections contribute only little (namely -3%) to the strange decay ratio R_S and are negligible when the total τ-decay ratio R_τ is considered.

We now include all corrections to present the values for the separate contributions as well as for the total hadronic τ-decay rate in Table 3. Here electroweak as well as corrections of higher mass dimension have been taken into account from [5], where it has been pointed out that the biggest nonperturbative corrections to R_τ in fact come from dimension $D = 6$ condensates. Our estimation of the corresponding uncertainty on R_τ is more conservative than in [5], because the large cancellation between the vector and the axial vector contribution is based on the validity of vacuum saturation approximation and may not necessarily be translated into a similar compensation for the uncertainties of the separate contributions. Numerically R_τ is dominated by purely perturbative contributions of zero mass dimensions which survive in the massless limit. According to [31] a proper summation of the effects of analytical continuation from space-like momenta to time-like ones is a necessity for these terms. At the moment it is not quite clear for us how essential these effects are for power suppressed contributions that we are dealing with. We hope to return to this and related problems in future publications.

To conclude, we have studied power suppressed perturbative and nonperturbative QCD corrections to the semileptonic decay rate of the τ-lepton. Strange quark mass effects contribute considerably to the decay modes with strangeness content and still reach the percent level for the total hadronic decay rate. Second order $\mathcal{O}(\alpha_s^2)$ corrections introduce a strange mass dependence even for non strange decays. We also have studied corrections of mass dimension $D = 4$, where we discussed an appropriate choice for the renormalization scale of the quark and gluon condensates. In view of the accuracy of QCD predictions for the decay rate of the τ-lepton into hadrons, semileptonic τ decays remain an important and interesting tool for testing QCD.

Table 1: $\delta^{(D=2)}_{ij,V/A}$ fractional corrections

Λ_{QCD}/MeV	$\alpha_s(M_\tau)$	$\delta^{(D=2)}_{ud,V} \cdot 10^3$	$\delta^{(D=2)}_{ud,A} \cdot 10^3$	$\delta^{(D=2)}_{us,V}$	$\delta^{(D=2)}_{us,A}$
52	0.16	-0.20 ± 0.04	-0.33 ± 0.06	-0.070 ± 0.016	-0.072± 0.016
308	0.30	-0.34 ± 0.12	-0.60 ± 0.16	-0.166 ± 0.037	-0.170 ± 0.038
547	0.44	-0.34 ± 0.27	-0.71 ± 0.32	-0.314 ± 0.070	-0.321 ± 0.071

Table 2: $\delta^{(D=4)}_{ij,V/A}$ fractional corrections

Λ_{QCD}/MeV	$\alpha_s(M_\tau)$	$\delta^{(D=4)}_{ud,V} \cdot 10^3$	$\delta^{(D=4)}_{ud,A} \cdot 10^3$	$\delta^{(D=4)}_{us,V}$	$\delta^{(D=4)}_{us,A}$
52	0.16	0.17 ± 0.13	-5.00 ± 0.61	0.011 ± 0.005	-0.048 ± 0.007
308	0.30	0.60 ± 0.27	-4.68 ± 0.67	0.010 ± 0.005	-0.050 ± 0.008
547	0.44	1.30 ± 0.55	- 4.20 ± 0.83	0.009 ± 0.005	-0.055 ± 0.009

Table 3: Contributions to the hadronic τ decay rate R_τ

Λ_{QCD}/MeV	$\alpha_s(M_\tau)$	R_V	R_A	R_S	R_τ
52	0.16	1.59 ± 0.02	1.49 ± 0.03	0.145 ± 0.003	3.23 ± 0.04
308	0.30	1.73 ± 0.02	1.63 ± 0.03	0.145 ± 0.005	3.51 ± 0.05
547	0.44	1.95 ± 0.08	1.85 ± 0.08	0.145 ± 0.011	3.95 ± 0.16

Acknowledgments
We would like to thank A.Pich and J.H.Kühn for helpful discussions. One of us (K.G.Ch.) is also grateful to M.Jamin, K.Schilcher and D.Pirjol for numerous fruitful discussions of mass corrections and their relation to vacuum condensates. The work was supported by Bundesministerium für Forschung und Technologie (K.G.Ch., Project No. 055KA94P1).

A Appendix

The β-function and the quark mass anomalous dimension γ_m are defined in the usual way

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_s(\mu)}{\pi}\right) = \beta(\alpha_s) \equiv -\sum_{i\geq 0} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{i+2}, \tag{27}$$

$$\mu^2 \frac{d}{d\mu^2}\bar{m}(\mu) = -\bar{m}(\mu)\gamma_m(\alpha_s) \equiv -\bar{m}\sum_{i\geq 0} \gamma_i \left(\frac{\alpha_s}{\pi}\right)^{i+1}. \tag{28}$$

Their expansion coefficients up to order $\mathcal{O}(\alpha_s^2)$ are well known [29, 30] and read

$$\begin{aligned} \beta_0 &= \left(11 - \frac{2}{3}f\right)/4, \quad \beta_1 = \left(102 - \frac{38}{3}f\right)/16, \\ \beta_2 &= \left(\frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2\right)/64, \end{aligned} \tag{29}$$

$$\begin{aligned} \gamma_0 &= 1, \quad \gamma_1 = \left(\frac{202}{3} - \frac{20}{9}f\right)/16, \\ \gamma_2 &= \left(1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right]f - \frac{140}{81}f^2\right)/64. \end{aligned} \tag{30}$$

The Riemann zeta function has the values $\zeta(3) = 1.2020569, \zeta(5) = 1.036927$. The solution of eq. (27) is given by ($L \equiv \ln\mu^2/\Lambda^2_{\overline{MS}}$)

$$\frac{\alpha_s(\mu^2)}{\pi} = \frac{1}{\beta_0 L}\left[1 - \frac{1}{\beta_0 L}\frac{\beta_1 \ln L}{\beta_0} + \frac{1}{\beta_0^2 L^2}\left(\frac{\beta_1^2}{\beta_0^2}(\ln^2 L - \ln L - 1) + \frac{\beta_2}{\beta_0}\right)\right] \tag{31}$$

while eq. (28) is solved by

$$\begin{aligned} m(\mu^2) = \;& \hat{m}\left(\frac{\beta_0}{2}\frac{\alpha_s(\mu^2)}{\pi}\right)^{\gamma_0/\beta_0}\left\{1 + \frac{\alpha_s(\mu^2)}{\pi}\left[\frac{\gamma_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2}\right]\right. \\ & \left. + \frac{1}{2}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left[\left[\frac{\gamma_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2}\right]^2 + \frac{\gamma_2}{\beta_0} - \frac{\beta_1\gamma_1}{\beta_0^2} - \frac{\beta_2\gamma_0}{\beta_0^2} + \frac{\beta_1^2\gamma_0}{\beta_0^3}\right]\right\} \end{aligned} \tag{32}$$

For the renormalization invariant quark mass parameters we have taken the values [32, 33]

$$\hat{m}_u = (8.7 \pm 1.5)MeV, \quad \hat{m}_d = (15.4 \pm 1.5)MeV, \quad \hat{m}_s = (270. \pm 30)MeV. \tag{33}$$

As input value for the gluon condensate we have used [32]

$$\langle \alpha_s GG \rangle = (0.02 \pm 0.01) GeV^4, \tag{34}$$

whereas the quark mass condensates are parametrized by [32] $\langle m_i \bar{\Psi}_j \Psi_j \rangle = -\hat{m}_i \hat{\mu}_j^3$ with

$$\hat{\mu}_u = \hat{\mu}_d = (189 \pm 7) MeV, \quad \hat{\mu}_s = (160 \pm 10) MeV. \tag{35}$$

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