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Higgs boson decay to two photons and the dispersion relations

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We discuss the computation of the Higgs boson decay amplitude to two photons through the W-loop using dispersion relations. The imaginary part of the form factor $F_W(s)$ that parametrizes this decay is unambiguous in four dimensions. When it is used to calculate the unsubtracted dispersion integral, the finite result for the form factor $F_W(s)$ is obtained. However, the $F_W(s)$ obtained in this way differs by a constant term from the result of a diagrammatic computation, based on dimensional regularization. It is easy to accommodate the missing constant by writing a once-subtracted dispersion relation for $F_W(s)$ but it is unclear *why* the subtraction needs to be done. The goal of this paper is to investigate this question in detail. We show that the correct constant can be recovered within a dispersive approach in a number of ways that, however, either require an introduction of an ultraviolet regulator or unphysical degrees of freedom; unregulated and unsubtracted computations in the unitary gauge are insufficient, in spite of the fact that such computations give a finite result.

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I. INTRODUCTION

The decay rate of the Higgs boson to two photons through the W-loop was computed in the literature at least thirteen times [1–13]. The recent flurry of activity around this process, important for understanding Higgs boson properties, was caused by the fact that the original computations of the $H \to \gamma\gamma$ decay rate [1–3], performed almost forty years ago, were challenged in Refs. [4, 5]. Among the follow up computations [6–13], only Ref. [13] agreed with the findings of Refs. [4, 5].

A good way to describe the controversial situation is as follows. Consider the $H \to \gamma(k_1)\gamma(k_2)$ decay amplitude, focusing on the W-boson loop, and write it as

$$\mathcal{M} = \frac{\alpha}{4\pi v} F_W(m_H^2)(k_1^\mu \epsilon_1^\nu - k_1^\nu \epsilon_1^\mu)(k_{2\mu}\epsilon_{2\nu} - k_{2\nu}\epsilon_{2\mu})\,. \quad (1)$$

Here $v = 2m_W/g = \left(G_F\sqrt{2}\right)^{-1/2}$ is the Higgs field vacuum expectation value and $\epsilon_{1,2}$ are the photon polarization vectors. The form factor $F_W(s)$ reads

$$\begin{aligned} F_W(s) &= F_W^\infty + F_W^c(s), \\ F_W^c(s) &= 3\beta + 3\beta(2-\beta)f(\beta), \end{aligned} \quad (2)$$

where $\beta = 4m_W^2/s$ and

$$f(\beta) = -\frac{1}{4}\left[\ln\frac{1+\sqrt{1-\beta}}{1-\sqrt{1-\beta}} - i\pi\right]^2. \quad (3)$$

The constant term F_W^∞ in Eq. (2) is the gist of the current discussion: according to Refs. [1–3, 6–12] $F_W^\infty = 2$ and according to Refs. [4, 5, 13], $F_W^\infty = 0$. The two groups [4, 5, 13] that claim $F_W^\infty = 0$ have used two different techniques in their computations that, however, have two important features in common. Indeed, both groups refuse to use the dimensional regularization, so that all the algebraic manipulations are performed in four dimensions *and* both groups insist on using only physical degrees of freedom in their calculations, i.e. the unitarity gauge for the W-bosons.

The authors of Refs. [4, 5] do this in the context of Feynman diagrams and loop integrations. This is a delicate matter since all the individual diagrams are divergent and need to be combined before the actual integration over the loop momentum to ensure the finite result. It is understandable, that this method of calculation drew criticism from Refs. [6, 8, 9, 14]. Interestingly, the authors of Refs. [4, 5] recognize this issue and try to ameliorate it by imposing an *additional* requirement on their result. This requirement is the heavy Higgs boson decoupling condition $F_W(s \to \infty) = F_W^\infty = 0$ whose validity was, however, criticized in Refs. [6, 8–10]. In-

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deed, the decoupling limit, $\beta = 4m_W^2/m_H^2 \to 0$, can also be viewed as the limit $m_W \to 0$. It is well-known that in the $m_W \to 0$ limit the Higgs boson interaction with vector bosons, $2Hm_W^2 W_\mu^\dagger W^\mu/v$, does not vanish for the longitudinal polarizations of the W bosons. This is in contrast to the Higgs interactions with fermions that do vanish in the zero fermion mass limit.

On the other hand, the computation of Ref. [13], based on the dispersive approach, is well-grounded at first sight. If one wants to use the four-dimensional set up and physical degrees of freedom, the best thing to do is to use dispersion relations for the form factor $F_W(s)$ whose imaginary part can be computed from tree-level Feynman diagrams. As it is seen from Eqs. (2, 3), $\mathrm{Im}\, F_W(s)$ does not depend on the ambiguity in F_W^∞ and equals to

$$\mathrm{Im}\, F_W(s) = \frac{3\pi}{2}\,\theta(1-\beta)\,\beta(2-\beta)\ln\frac{1+\sqrt{1-\beta}}{1-\sqrt{1-\beta}}\,. \quad (4)$$

Note, that the imaginary part does vanish in the $\beta \to 0$ limit.

The full function $F_W(s)$ is then reconstructed using the unsubtracted dispersion relation in s,

$$F_W(s) = \frac{1}{\pi}\int\limits_{4m_W^2}^{\infty} \frac{\mathrm{d}s_1 \mathrm{Im}[F_W(s_1)]}{s_1 - s - i0}\,. \quad (5)$$

The result of the integration in Eq. (5) is the form factor F_W^c shown in Eq. (2), which implies that $F_W^\infty = 0$. The authors of Ref. [13] interpret this result as the supporting evidence for the computation reported in Refs. [4, 5]. However, it should be recognized that the use of the unsubtracted dispersion relation assumes that the form factor $F_W(s)$ vanishes at $s \to \infty$, i.e. $F_W^\infty = 0$. In other words, decoupling is assumed, rather than proved in Ref. [13]. Without such an assumption, one can just add any real constant to the right hand side of Eq. (5).

The constant F_W^∞ then either needs to be computed with a method that is different from the dispersion relations *or* one should have a physical argument that determines the value of the form factor $F_W(s)$ for *one* value of s. The most well-known example of the latter is the requirement that the Dirac form factor of the electron equals to one at zero momentum transfer.

In case of the form factor $F_W(s)$, the low-energy theorem of Ref. [3] fixes its value at $s = 0$ to be the W-boson contribution to the coefficient of the one-loop QED β-function b_W

$$\lim_{s\to 0} F_W = b_W = 7\,. \quad (6)$$

It is straightforward to check, using Eq. (2), that this condition at $s = 0$ implies that $F_W^\infty = 2$.

Nevertheless, we can ask under which conditions the dispersion relations without the integral over the infinitely remote contour and the subtraction constant can be used in general. The answer to this question is well-known. Such a possibility should exist if a *finite* form factor is computed in a *renormalizable* theory since each independent subtraction term corresponds to an independent renormalization condition that usually are fixed by considering divergent, rather than finite, quantities. Also, the use of unsubtracted dispersion relations should be possible if one combines an ultraviolet (UV) regularization, such as dimensional or Pauli-Villars, with the dispersion relations. Indeed, taking the dimensional regularization as an example, any integral over the infinitely remote integration contour can be discarded since $F_W(s) \sim s^{-\epsilon}$ for dimensional reasons and ϵ can always chosen in such a way that such an integral vanishes. In case of the Pauli-Villars regularization, $F_W(s)$ is also decreasing for values of $\sqrt{s}$ that are larger than the ultraviolet cut-off, given by the regulator mass M_{PV}.

Combining these observations with the fact that the $\mathrm{Im}\, F_W(s)$ in Eq. (4) is finite and integrable in the dispersion integral, and that the Standard Model is, obviously, a renormalizable theory, we conclude that something unusual should occur in $\mathrm{Im}\, F_W(s)$ in the limit when the regulators are taken to their limiting values ($\epsilon \to 0$ or $M_{\mathrm{PV}} \to \infty$). Indeed, as we will see, this is exactly what happens and an additional contribution to the imaginary part of the form factor is generated at $\sqrt{s}$ of the order of the ultraviolet cut-off. This additional contribution to the dispersion integral changes $F_W(s)$ if s is in the range $m_W \ll \sqrt{s} \ll M_{\mathrm{PV}}$, effectively leading to a non-vanishing "constant" contribution to F_W.

Although this approach may look somewhat unphysical because it refers to the behavior of the theory for values of Higgs masses that are larger than the UV cut-off of the theory, we will see that it is consistent with an infrared condition, e.g. the fixed value of F_W at $s = 0$. We investigate how this happens in detail in this paper.

II. LONGITUDINAL POLARIZATIONS

The issue of non-decoupling at $m_W = 0$ refers to the longitudinally-polarized W bosons. To describe these polarizations at large energies, $E \gg m_W$, one can substitute $W_\mu = \partial_\mu \phi/m_W$ where ϕ is the charged scalar field; this statement is the essence of the equivalence theorem [15–17]. When written in terms of ϕ-fields, the interaction of the W-bosons with the Higgs field $2(H/v)m_W^2 W_\mu^\dagger W^\mu$ takes the form

$$S_{\mathrm{int}} = \int \mathrm{d}^4x\, \frac{H}{v}\,\partial_\mu\partial^\mu\left(\phi^\dagger\phi\right). \quad (7)$$

Technically, this interaction looks as a dimension-five, i.e. non-renormalizable, operator. This fact alone should act like a warning sign for the application of unsubtracted dispersion relations, even if the result of the computation turns out to be finite.

We will study the contribution of the ϕ particles to the form factor for the two-photon Higgs decay assuming that Higgs-ϕ interaction is given by Eq. (7) and denoting their masses as m_ϕ. We will see that this toy model captures all the essential features of the problem discussed in the Introduction. The counter-part of the full form factor $F_W(s)$ of Eq. (2) in our toy model is denoted by $F_\phi(s)$.

There are two ways to deal with the operator in Eq. (7). The first one is based on the observation that for the purpose of computing $H \to \gamma\gamma$ decay amplitude, it is possible to improve the ultraviolet properties of the action in Eq. (7). To this end, we integrate by parts in Eq. (7), use equations of motion for the Higgs particle $\partial_\mu \partial^\mu H = -m_H^2 H$ and obtain

$$S_{\rm int} = -\frac{m_H^2}{v} \int {\rm d}^4 x\, H\, \phi^\dagger \phi \,. \tag{8}$$

This transformation makes the interaction between the Higgs and the ϕ's explicitly renormalizable and guarantees that an unsubtracted dispersion relations for suitably defined form factor should be applicable.

To proceed further, we parametrize the matrix element $\langle \gamma\gamma | \phi^\dagger \phi | 0 \rangle$ as follows

$$\langle \gamma\gamma | \phi^\dagger \phi | 0 \rangle = -\Phi(s) \cdot \frac{\alpha}{4\pi}\, f_1^{\mu\nu} f_{2,\mu\nu} \,, \tag{9}$$

where $f_i^{\mu\nu} = k_i^\mu \epsilon_i^\nu - k_i^\nu \epsilon_i^\mu$. The physical form factor is then $F_\phi(s) = m_H^2 \Phi(s) = s\,\Phi(s)$.

The form factor $\Phi(s)$ at large $s = (k_1 + k_2)^2 \gg m_\phi^2$ equals to [3, 18, 19]

$$\Phi(s) = \frac{2}{s} \,. \tag{10}$$

After multiplying Eq. (9) by the "coupling constant" m_H^2, identifying m_H^2 with s and taking the $s \to \infty$ limit, we obtain $\lim_{s\to\infty} F_\phi(s) = 2$, which reproduces the non-decoupling constant in Eq. (2).

It is straightforward to reproduce this result in the dispersive approach. Indeed, by unitarity the imaginary part of the $\langle \gamma\gamma | \phi^\dagger \phi |\, 0 \rangle$ amplitude is

$$2\,{\rm Im}\, \langle \gamma\gamma | \phi^\dagger \phi | 0 \rangle = \int {\rm dLips}(p_1, p_2, K_{12})\, M_{\phi\bar\phi}^{\gamma\gamma} \,, \tag{11}$$

where dLips denotes the element of standard Lorentz invariant phase space of two ϕ particles with momenta p_1 and p_2 and $M_{\phi\bar\phi}^{\gamma\gamma}$ is the amplitude of $\phi(p_1) + \bar\phi(p_2) \to \gamma(k_1) + \gamma(k_2)$ annihilation,

$$M_{\phi\bar\phi}^{\gamma\gamma} = 2e^2 \Big\{ (\epsilon_1 \epsilon_2) - \frac{(p_1 \epsilon_1)(p_2 \epsilon_2)}{(p_1 k_1)} - \frac{(p_1 \epsilon_2)(p_2 \epsilon_1)}{(p_1 k_2)} \Big\}. \tag{12}$$

After integration over the phase space of two ϕ particles, we obtain

$${\rm Im}\, \Phi(s) = -\pi \theta(1 - \beta_\phi)\, \frac{\beta_\phi}{s} \ln \frac{1 + \sqrt{1 - \beta_\phi}}{1 - \sqrt{1 - \beta_\phi}} \,. \tag{13}$$

We use this result in the unsubtracted dispersion relation and find

$$\Phi(s) = \frac{1}{\pi} \int\limits_{4m_\phi^2}^{\infty} \frac{{\rm d}s_1 {\rm Im}\, \Phi(s_1)}{s_1 - s - i0} = \frac{2}{s} \left(1 - \beta_\phi f(\beta_\phi) \right) , \tag{14}$$

where $\beta_\phi = 4m_\phi^2/s$. This expression coincides with Eq. (9) at large s but it is valid for all s. To obtain the form factor $F_\phi(s)$, we multiply the real and imaginary parts of Φ by m_H^2 and identify m_H^2 with s. We find

$$\begin{aligned} & F_\phi(s) = s\Phi(s), \\ & {\rm Im}\, F_\phi(s) = -\pi\, \theta(1 - \beta_\phi) \beta_\phi \ln \frac{1 + \sqrt{1 - \beta_\phi}}{1 - \sqrt{1 - \beta_\phi}} \,, \\ & F_\phi(s) = 2\big(1 - \beta_\phi f(\beta_\phi)\big). \end{aligned} \tag{15}$$

Note that the physical form factor $F_\phi(s)$ contains the constant contribution in the limit $s \to \infty$ and, therefore, does not support the $s \to \infty$ decoupling condition.

We can now ask what is the dispersion relation that the form factor $F_\phi(s)$ satisfies, provided that $\Phi(s)$ satisfies an unsubtracted dispersion relation. It is straightforward to answer this question. We start from the unsubtracted relation for $\Phi(s)$ in Eq. (14), write $\Phi = F_\phi/s$, and obtain

$$F_\phi(s) = \frac{s}{\pi} \int_{4m_\phi^2}^{\infty} \frac{ds_1 {\rm Im}\, F_\phi(s_1)}{s_1(s_1 - s - i0)} \,, \tag{16}$$

which is a once-subtracted dispersion relation for the form factor $F_\phi(s)$. Therefore, the subtraction of the dispersion relation for $F_\phi(s)$ at $s = 0$, which enforces the condition $F_\phi(s = 0) = 0$, appears *automatically* provided that we use the unsubtracted dispersion relations only for quantities (e.g. $\Phi(s)$) that are computed in a theory where all interactions are renormalizable by naive power-counting. This is *not* the case for both, the toy model with the interaction term as in Eq. (7) and the Standard Model in the unitary gauge, so that the use of the unsubtracted dispersion relations in both of these cases leads to incorrect results.

We elaborate on the last statement. Suppose that we do not perform the integration by parts in the interaction term Eq. (7) and use it directly to compute $H \to \gamma\gamma$ amplitude. Roughly speaking, this is a situation that corresponds to calculations in the unitary gauge in the full Standard Model. The imaginary part of this amplitude is given by the imaginary part of the physical form factor $F_\phi(s)$. If we now use this imaginary part in the unsubtracted dispersion relation, we obtain a result that differs from $F_\phi(s)$ in Eq. (16) by a subtraction constant

$$-\int \frac{{\rm d}s_1}{s_1} {\rm Im}[F_\phi(s_1)] = 2. \tag{17}$$

We will now check that we can get the correct result for the form factor using the unsubtracted dispersion relations even if we work with the non-renormalizable interaction in Eq. (7) but regulate the theory in the ultraviolet, in spite of the fact that the final result turns out to be finite.

A simple form of the UV regularization is an introduction of Pauli-Villars fields. In our case it means that a contribution of the loop of charged scalar particles with the mass $m_{\rm PV}$ should be subtracted from the loop of ϕ-fields. The introduction of the Pauli-Villars regulator leads to a change in the imaginary part of the form factor $\mathrm{Im}\, F_\phi(s)$ at $s \geq 4m_{\rm PV}^2$,

$$\Delta_{\rm PV}[\mathrm{Im}\, F_\phi] = \pi\, \theta(1-\beta_R)\beta_R \ln \frac{1+\sqrt{1-\beta_R}}{1-\sqrt{1-\beta_R}}\,, \qquad (18)$$

where $\beta_R = 4m_{\rm PV}^2/s$. We find

$$\Delta_{\rm PV} F_\phi(s) = \frac{1}{\pi}\int\limits_{4m_R^2}^{\infty} \frac{\mathrm{d}s_1 \Delta_{\rm PV}[\mathrm{Im}[F_\phi(s_1)]}{s_1 - s - i0} = 2\beta_R f(\beta_R)\,. \qquad (19)$$

We are interested in the limit $\beta_R = 4m_{\rm PV}^2/s \to \infty$; in that limit

$$\Delta_{\rm PV} F_\phi(s) = 2\beta_R f(\beta_R) \to 2\,, \qquad (20)$$

which is the same constant that appears in Eq. (17).

We will now demonstrate that the same result is obtained if dimensional regularization is used for the UV cut-off. It is convenient to choose the photon polarization vectors as

$$\epsilon_{1,2} = (0,1,\pm i, 0)/\sqrt{2}\,, \quad \epsilon_1 \cdot \epsilon_2 = -1, \qquad (21)$$

in the reference frame where the photon momenta are along z axis. Then from the unitarity relation

$$2\,\mathrm{Im}\,\langle\gamma\gamma|H\rangle = \int \mathrm{dLips}(p_1,p_2,K_{12})\langle\phi\bar{\phi}\,|H\rangle\, M_{\phi\bar{\phi}}^{\gamma\gamma} \qquad (22)$$

we obtain

$$\mathrm{Im}\, F_\phi = (4\pi)^2\mu^{2\epsilon}\!\int\! \mathrm{dLips}(p_1,p_2,K_{12})\Big\{-1+\frac{p_x^2+p_y^2}{p_0^2-p_z^2}\Big\}, \qquad (23)$$

where μ is the normalization point and the factor $\mu^{2\epsilon}$ restores a correct dimension.

At $d = 4$ this expression shows that $\mathrm{Im}\, F_\phi \propto m_\phi^2$ and leads to $\mathrm{Im}\, F_\phi$ given in Eq. (15). At $d = 4-2\epsilon$ we should split the ϕ particle momentum p_μ into the four-dimensional part and the part $\boldsymbol{p}_\epsilon$ living in remaining -2ϵ dimension. To determine an additional part $\Delta_\epsilon \mathrm{Im}\, F_\phi$ we put $m_\phi = 0$. Then,

$$\begin{aligned}\Delta_\epsilon \mathrm{Im}\, F_\phi &= (4\pi)^2\mu^{2\epsilon}\!\int\! \mathrm{dLips}(p_1,p_2,K_{12})\,\frac{\boldsymbol{n}_\epsilon^2}{\sin^2\theta}\\ &= \epsilon\,(4\pi)^2\mu^{2\epsilon}\!\int\! \mathrm{dLips}(p_1,p_2,K_{12}) \qquad (24)\\ &= \frac{\Omega^{(d-1)}}{2^{d-3}(2\pi)^{d-4}}\,\epsilon\left(\frac{s}{\mu^2}\right)^{-\epsilon} \approx 2\pi\epsilon\left(\frac{s}{\mu^2}\right)^{-\epsilon}.\end{aligned}$$

In these equations $\boldsymbol{n}_\epsilon = \boldsymbol{p}_\epsilon/|\boldsymbol{p}|$, the angle θ is between $\boldsymbol{p}$ and $\boldsymbol{k}$, and $\Omega^{(d-1)}$ is the solid angle in $d-1$ spatial dimensions. The correction to the imaginary part induces the following change in F_ϕ

$$\Delta_\epsilon F_\phi(s) = \frac{1}{\pi}\int\limits_{4m_R^2}^{\infty} \frac{\mathrm{d}s_1 \Delta_\epsilon \mathrm{Im}[F_\phi(s_1)]}{s_1 - s - i0} = 2\left(-\frac{s}{\mu^2}\right)^{-\epsilon}. \qquad (25)$$

If we consider value of $\sqrt{s}$ that are much smaller than the scale $\mu\exp(1/(2\epsilon))$, which plays a role of the UV cut-off, $\Delta_\epsilon F_\phi(s)$ adds the required constant 2 to $F_\phi(s)$, that is reconstructed from the unsubtracted and unregulated dispersion relation.

By considering the toy model for the interaction of the longitudinally-polarized gauge bosons with the Higgs boson, we showed that the reason for the appearance of the constant contribution to the $H \to \gamma\gamma$ form factor is the fact that interactions between the Higgs boson and the electroweak bosons in the unitary gauge are not renormalizable by power counting. The correct result can be obtained by either introducing explicit ultraviolet regulator, in spite of the fact that the computation of the form factor leads to a finite result, or by switching to a formulation of the theory where interactions are renormalizable by power counting. We also note that the UV regularization leads automatically to a result that is consistent with the low-energy constraint which is $F_\phi(s=0) = 0$ in our toy model . As we saw, imposing this condition was sufficient for the dispersive reconstruction. All of these approaches can be used to compute the complete form factor $F_W(s)$ in the dispersive approach; in the next Section, we will do that by performing the dispersive computation in a renormalizable R_ξ gauge and studying if the unitary gauge result is recoverd in the $\xi \to \infty$ limit.

III. IMAGINARY PART AND THE RENORMALIZABLE GAUGE

Our goal is to compute the form factor $F_W(s)$ using unsubtracted and unregulated dispersion relations. As we have seen, this requires a formulation of the theory where renormalizability is apparent. Hence, we are forced to consider the R_ξ gauges.

Similar to what has been done before, we will calculate the form factor using dispersion relations; for this we will need to compute its imaginary part for $s \neq m_H^2$. It is important to recognize that the amplitude that describes the transition of the off-shell Higgs to two photons becomes gauge-dependent; this applies to the dependence of the imaginary part on the electroweak gauge parameter ξ as well as to the loss of the transversality of the electromagnetic current.

The second problem is easy to avoid by choosing the non-linear R_ξ gauge where the electromagnetic gauge invariance is explicitly maintained. To this end, we can use

$$\mathcal{L}_{\rm gauge} = -\frac{1}{\xi}\left|D_\mu W^\mu - i\xi m_W \phi\right|^2, \qquad (26)$$

as the gauge fixing term with $D_\mu = \partial_\mu - ieA_\mu$. If we choose this gauge, some Feynman rules of a linear R_ξ gauge get modified but this is not important for us. The important point is that the gauge-fixing term $\mathcal{L}_{\rm gauge}$ eliminates the $\phi W\gamma$ vertex. In addition, it is important for what follows that in the R_ξ gauge Lagrangian, linear or not, the only interaction vertex that explicitly contains m_H^2 is the interaction vertex involving the Higgs boson and the two Goldstone ϕ-fields

$$\mathcal{L}_{H\phi^\dagger\phi} = -\frac{m_H^2}{v} H\phi^\dagger\phi \,. \tag{27}$$

The interaction of the ϕ-fields with the photons are that of the scalar QED and follow from the Lagrangian

$$\mathcal{L}_\phi = |D_\mu \phi|^2. \tag{28}$$

The final remark that we need to make is that the mass squared of the Goldstone boson ϕ is $m_\phi^2 = \xi m_W^2$.

As we will now show this information is all that we need to perform the computation of $F_W(s)$, given the results that we already presented in Section II. To facilitate the computation of the imaginary part of the form factor $F_W(s)$, we use the already-mentioned fact that among many diagrams that contribute to the form factor, the only interaction vertex that is proportional to m_H^2 comes from the $H\phi^+\phi$ interaction term in Eq. (27). Motivated by this observation, we write the imaginary part as the sum of two terms

$$\mathrm{Im}[F_W^{R_\xi}(r_H, \beta, \xi)] = r_H G_1(\beta, \xi) + G_2(\beta, \xi) \,, \tag{29}$$

where $r_H = m_H^2/s$ and $\beta = 4m_W^2/s$. The functions $G_{1,2}$ can be computed directly from Feynman diagrams, however this is not necessary. Indeed, there is one constraint on the two functions that is available to us since if we compute the imaginary part for $s = m_H^2$, we should recover the ξ-independent result for the imaginary part in the unitary gauge. This implies

$$G_1(\beta, \xi) + G_2(\beta, \xi) = \mathrm{Im}[F_W^c(\beta)], \tag{30}$$

where $\mathrm{Im}[F_W^c(\beta)]$ is the imaginary part of the form factor in the unitary gauge defined in Eq. (4). Next, since the only m_H^2-dependent term in the calculation of $\mathrm{Im}[F_W^{R_\xi}(r_H, \beta, \xi)]$ comes from the diagrams with $\phi^\dagger\phi$-intermediate state, we can read off G_1 from the imaginary part of the form factor F_ϕ in Eq. (15). We obtain

$$G_1(\beta, \xi) = \mathrm{Im}[F_\phi(\beta_\phi = \xi\beta)]. \tag{31}$$

We can use the above constraints to rewrite the imaginary part of the form factor in a general R_ξ gauge in a useful way. By adding and subtracting G_1, we find

$$\mathrm{Im}[F_W^{R_\xi}(r_H, \beta, \xi)] = \mathrm{Im}[F_W^c(\beta)] + (r_H - 1)\mathrm{Im}[F_\phi(\xi\beta)]. \tag{32}$$

The second term here shows that it is the off-shell behavior that differentiates the singular unitary gauge from the renormalizable R_ξ gauge.

We can now restore the real part of the form factor from its imaginary part using the *unsubtracted* dispersion relation for $s = m_H^2$. The result of the calculation should be correct since the theory in R_ξ gauge is renormalizable by power-counting. To this end, we need to compute

$$F_W(m_H^2) = \frac{1}{\pi} \int \frac{\mathrm{d}s_1 \, \mathrm{Im}[F_W^{R_\xi}(r_H, \beta_1, \xi)]}{s_1 - m_H^2} \,. \tag{33}$$

To compute this integral, we use the expression for the imaginary part as in Eq. (32) and realize that the dispersion integral of $\mathrm{Im}[F_W^c(\beta)]$ reconstructs F_W^c, see Eq. (2). We also substitute $m_H^2 \to s$, to conform with the previous notations, and write the final result for the form factor as

$$\begin{aligned} F_W(s) &= F_W^c(s) - \frac{1}{\pi} \int\limits_{4\xi m_W^2}^{\infty} \frac{\mathrm{d}s_1}{s_1} \mathrm{Im}[F_\phi(\xi\beta_1)] \\ &= F_W^c(s) + 2 \,. \end{aligned} \tag{34}$$

We note that the integral over $\mathrm{Im}F_\phi$ in the above equation is ξ independent and coincides with a similar integral in the toy model, see Eq. (17). In general, the above computation shows that the form factors calculated in the R_ξ gauge and the unitary gauge *differ by a constant*, related to the contribution of Goldstone bosons to the imaginary part of the $H \to \gamma\gamma$ amplitude. The mass of the Goldstone boson $m_\phi^2 = \xi m_W^2$ remains arbitrary in the calculation, so that the limit $\xi \to \infty$ can be studied. It follows from Eq.(34) that the Goldstone boson contribution to $F_W(s)$ *does not decouple* in the limit $\xi \to \infty$; this feature leads to a difference between the results of the calculations in the unitary and the R_ξ gauges. Finally, the Goldstone boson contribution does not have a pole at $s = m_H^2$ and, therefore, does not contribute to the discontinuity of the form factor; for all practical purposes, it is a subtraction term.

IV. CONCLUSIONS

In this paper, we discussed how the dispersion relation computation of the $H \to \gamma\gamma$ decay amplitude through the W-boson loop can be reconciled with the results of the diagrammatic computations that employ dimensional regularization. As was pointed in Ref. [13], if one computes the imaginary part of the form factor $F_W(s)$ in the four-dimensional space-time and then uses it in an unsubtracted dispersion integral to calculate the full form factor, one obtains the result that differs from the correct one by a constant term. The appearance of this constant can be interpreted as the need to perform a subtraction in a finite dispersion integral which is quite unusual.

We have shown that the need to perform the subtraction in the dispersion integral for form factors computed in the unitary gauge is a consequence of the fact that the SM in the unitary gauge is not explicitly renormalizable. If one regularizes the (apparently finite) calculation by either introducing explicit UV regulator or starts from the formulation of the theory where the renormalizability is, in fact, apparent, one always obtains an additional contribution to the real part of the form factor. For values of s below the ultraviolet cut-off, this contribution is, essentially, a constant and can be interpreted as the subtraction term in the dispersion relation. Unfortunately, unregulated and unsubtracted dispersion relation calculations, that employ unitary gauge, do not seem to be sufficient even if they lead to finite results.

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