α_s from $\sigma(e^+e^- \to$ hadrons)

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1 Introduction

The cross section for electron-positron annihilation into hadrons is one of the cleanest ways for measuring the strong coupling constant. In principle a wide range of energies exists which can be exploited for this purpose, ranging from the region below the charm threshold, say around 3 GeV, up to the highest energies available at electron-positron colliders.

Apart from the regions very close to the charm-, bottom- and top-threshold the cross section can be evaluated by taking the limit of massless quarks. The leading term for QCD corrections is thus given by the extremely simple form $(1 + a_s)$ with $a_s \equiv \alpha_s/\pi$. Higher order terms, however, exhibit a non-trivial dependence on the number of active flavours. Also the difference between vector- and axial vector current induced reactions starts (in the massless limit) in order α_s^2. Closely related to this quantity are QCD-corrections to τ-lepton decays. Although in principle extemely clean and simple, the predictions are, in this case, at the limit of applicability of perturbative QCD. This fact is well visible from the difference between the predictions based on "Fixed Order" and "Contour Improved" (see [1] and references therein) and will be discussed in more detail in [2].

The following discussion will, therefore, be limited to decays of the virtual photon into hadrons, i.e. the famous R-ratio, or the corresponding object at higher energies, in particular the decay rate of the Z-boson into hadrons, $\Gamma(Z \to \text{hadrons})$, and the closely related quantities for the W or the Higgs-boson decay.

2 The R-ratio below the bottom threshold

The R-ratio, given by $\sigma(e^+e^- \to \text{hadrons})/\sigma_{\text{pt}}$, can be measured at vastly different energies. In the low energy region the annihilation proceeds through the virtual photon only. A combined fit [3] to CLEO results [4], based on four-loop accuracy, gives

$$\alpha_s^{(4)}(9^2\,\text{GeV}^2) = 0.160 \pm 0.024 \pm 0.024 \tag{1}$$

which, at the scale of M_Z, corresponds to

$$\alpha_s^{(5)}(M_Z^2) = 0.110 \begin{pmatrix} +0.014 \\ -0.017 \end{pmatrix}. \tag{2}$$

*Presented by Johann H. Kühn

This CLEO result, in turn, can be combined with earlier measurements by BESS [5], MD-1 [6] and CLEO [7], and leads to [3]

$$\alpha_s^{(4)}(9^2\,\mathrm{GeV}^2) = 0.182 \left(\begin{array}{c} +0.022 \\ -0.025 \end{array} \right) \tag{3}$$

corresponding to

$$\alpha_s^{(5)}(M_Z^2) = 0.119 \left(\begin{array}{c} +0.009 \\ -0.011 \end{array} \right). \tag{4}$$

This result is in nice agreement with a more recent measurement of the BESS collaboration [8]

$$R_{uds} = 2.224 \pm 0.019 \pm 0.089 \tag{5}$$

taken at energy values below the charm threshold.

Let us emphasize again that more precise measurements in this region, based on the enormous statistic of the BESS and BELLE experiment, would be highly welcome.

3 Status and perspectives for $e^+e^- \to$ hadrons at the Z resonance

a.) QCD corrections

At present the most precise determination of R is based on the measurement of Z decays, more precisely on the measurement of Γ_{had} and $\Gamma_{had}/\Gamma_{lept}$, where corrections have been calculated to $\mathcal{O}(\alpha_s^4)$ [1, 9, 10, 11]. Effects from non-vanishing charm and bottom quark masses will be mentioned below.

Even in the massless limit there are three different types of contributions

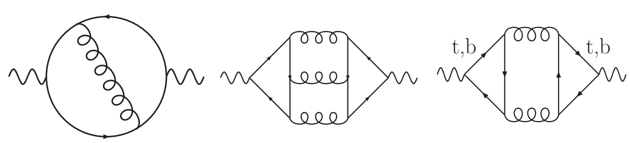

Figure 1: Non-singlet & singlet, vector & axial correlators.

i.) The non-singlet vector and axial vector correlators r_{NS} [1, 9].

ii.) The singlet correlator r_S^V, which starts in $\mathcal{O}(\alpha_s^3)$ and is present for the vector current only [10, 11].

iii.)A contribution to the axial correlator, r_S^A, resulting from the top-bottom doublet with $4m_t^2 \gg s \gg 4m_b^2$, which starts in order α_s^2 [12, 13, 11].

The corresponding predictions are available to order α_s^4. For an extremely conservative error estimate based on the variation of the scale parameter μ between $M_Z/3$ and $3M_Z$ this corresponds to the variation

$$\delta\Gamma_{NS} = 101\,\mathrm{KeV},\;\; \delta\Gamma_S^V = 2.7\,\mathrm{KeV},\;\; \delta\Gamma_S^A = 42\,\mathrm{KeV} \tag{6}$$

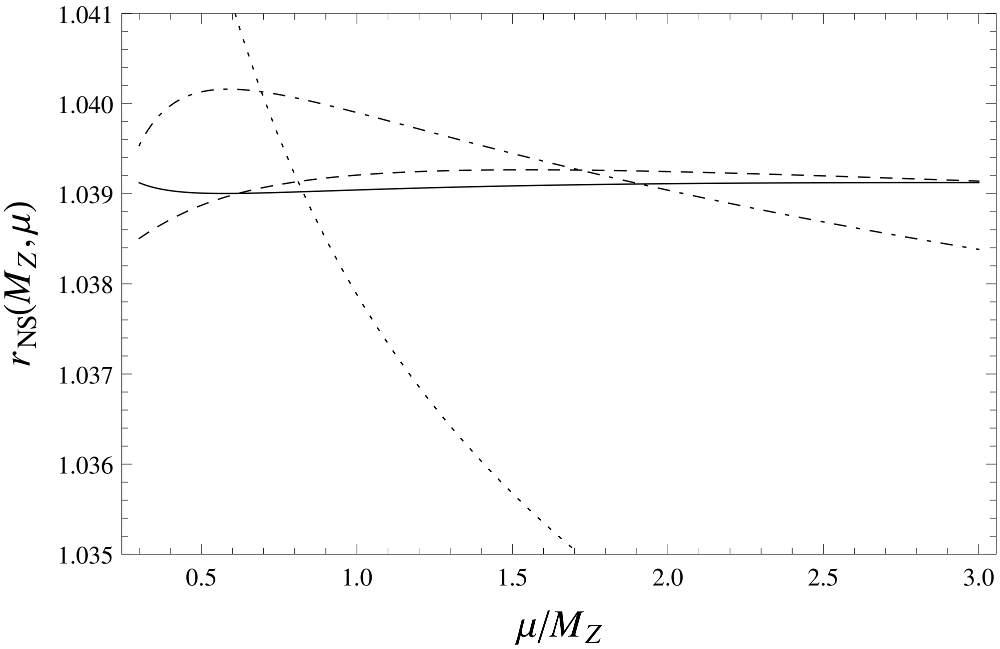

Figure 2: Predictions for r_{NS} in increasing orders of α_s [10].

and, correspondingly, to a variation of the strong coupling around $3 \cdot 10^{-4}$. This theory error is completely neglegeable compared to the present experimental uncertainty around 0.026. In fact, it is even comparable to the error of $\delta\Gamma_{had} \approx 100\,\mathrm{KeV}$ expected for an analysis at the FCC-ee. Various additional small corrections must be applied, which, however, have been calculated some time ago: quark mass corrections up to order $m_b^2\alpha_s^4$ and $m_b^4\alpha_s^3$ [14, 15] and mixed QCD and electroweak corrections that will be discussed in the following.

b.) Mixed electroweak and QCD corrections

For light quarks terms of order $\alpha\alpha_s$ have been evaluated some time ago [16]. The difference between the correct two-loop terms and the product of the corresponding two one-loop terms

$$\Delta\Gamma \equiv \Gamma(\text{two loop (EW} \star \text{QCD)}) - \Gamma_{\rm Born}\delta_{\rm EW}^{\rm NLO}\delta_{\rm QCD}^{\rm NLO} = -0.59(3) \tag{7}$$

is again of high importance, if a precision of $\delta\Gamma \approx 0.1$MeV should be reached at some point in the future. The next, presently unknow three-loop term is smaller by another factor of order α_s/π, times an unknown coefficient which should not exceed 5 in order to keep this term under control.

Similar comments apply to $\Gamma(Z \to b\bar{b})$. The present result, $\Gamma(Z \to b\bar{b})/\Gamma_Z \equiv R_b = 0.1512 \pm 0.0005$, obtained at LEP corresponds to an error of about 1.3 MeV. Pushing the relative error $\delta\Gamma(Z \to b\bar{b})/\Gamma_Z$ down to $2 \cdot 10^{-5}$, would imply a precision of 0.05 MeV for the partial width $\Gamma(Z \to b\bar{b})$. This should be compared to the present result of order $\alpha\alpha_s$, which is given by

$$\Gamma(Z \to b\bar{b}) - \Gamma(Z \to d\bar{d}) = (-5.69 - 0.79)_{\mathcal{O}(\alpha)} + (+0.50 + 0.06)_{\mathcal{O}(\alpha\alpha_s)})\,\mathrm{MeV} \tag{8}$$

separated into m_t^2 enhanced terms [17] and the rest [18].

The size of these terms has even motivated the evaluation of the m_t^2-enhanced corrections [19] of order $G_F m_t^2 \alpha_s^2$ with the result $\delta\Gamma_b \approx 0.1\,\mathrm{MeV}$ for the non-singlet contributions, a result drastically below the current precision and comparable in size to the planned precision of the FCC-ee.

Let us mention that many corrections seem to be significantly smaller if expressed in terms of the $\overline{\rm MS}$-mass [20, 21, 22]

$$\bar{m}_t(\bar{m}_t) = m_{pole}(1 - 1.33\, a_s - 6.46\, a_s^2 - 60.27\, a_s^3 - 704.28\, a_s^4). \tag{9}$$

Note that the $\overline{\rm MS}$-mass may well be directly accessible at an e^+e^- collider with high precision, e.g. through a measurement of the potential subtracted mass with a precision of 20 – 30 MeV.

4 M_W from G_F, M_Z, α and the rest

The precision of the present M_W-measurement amounts to 23 MeV For the FCC-ee option a precision of 0.5 to 1 MeV is forseen. In the Standard Model M_W can be considered a derived quantity that can be calculated from G_F, M_Z, M_t, $\Delta\alpha$ and m_H. Let us, for the moment, focus on the top mass dependence, which is obtained from the following chain of equations

$$\begin{aligned}
M_W^2 &= f(G_F, M_Z, m_t, \Delta\alpha, \ldots) && (10)\\
&= \frac{M_Z^2}{2(1-\delta\rho)}\left(1+\sqrt{1-\frac{4\pi\alpha(1-\delta\rho)}{\sqrt{2}G_F M_Z^2}\left(\frac{1}{1-\Delta\alpha}+\ldots\right)}\right), && (11)\\
\delta M_W &\approx M_W \frac{1}{2}\frac{\cos^2\theta_w}{\cos^2\theta_w - \sin^2\theta_w}\delta\rho \approx 5.7\times 10^4\,\delta\rho[MeV], && (12)\\
\delta\rho_t &= 3X_t\left(1 - 2.8599\left(\frac{\alpha_s}{\pi}\right) - 14.594\left(\frac{\alpha_s}{\pi}\right)^2 - 93.1\left(\frac{\alpha_s}{\pi}\right)^3\right). && (13)
\end{aligned}$$

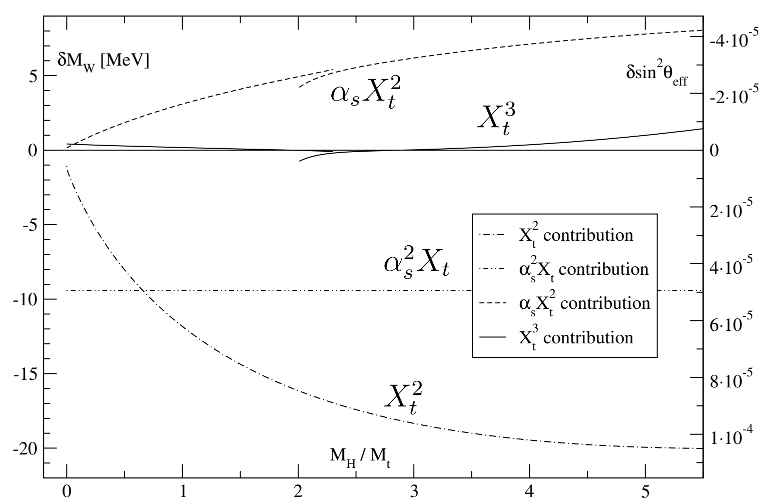

Fig. 3: Shifts in M_W and in $\sin^2\Theta_W$ from various corrections to $\Delta\rho$ [23].

In this case the three-loop term amounts to 9.5 MeV, the four-loop term to 2.1 MeV [24, 25]. In three loop approximation also purely weak ($\sim X_t^3$) and mixed (QCD*electroweak) of order $\alpha_s X_t^2$ three-loop terms are available, corresponding to $\delta M_W = 0.2\,\mathrm{MeV}$ and 2.5 MeV respectively. These are shown in Fig. 3 together with the terms of order $\alpha_s X_t^2$ and X_t^2.

Let us emphasize that there are still various uncalculated terms of order 0.1 MeV – 0.5 MeV, for example the four-loop term of order $\alpha_s^2 X_t^2$ or higher order terms relating pole and $\overline{\mathrm{MS}}$-mass.

5 Perspectives for $e{+}e^-\to$ hadrons above the Z-resonance

No detailled analysis of the cross section exists in the moment for the energy region above the Z-resonance. The situation is significantly more complicated than the one around the Z-boson or at low energies:

As a consequence of Furry's theorem the interference between Born and one-loop correction vanishes, as long as only vector currents are involved. This is no longer true for a mixture of vector and axial currents, such that the interference (relative to the Born term) is proportional to order α. In combination with the large radiative tail this enhances the radiative corrections drastically and no prediction for the R-ratio at a level required for a precise determination of the $Zf\bar{f}$ coupling is available at present.

6 Perspectives for $e^+e^-\to Z+H(\to$ hadrons$)$

a.) $\mathbf{H\to b\bar{b}}$

Up to this point only QCD corrections for vector and axial currents have been considered, where the first terms are equal for both cases and given by $(1+\alpha_s/\pi)$. In contrast, for the case of Higgs boson decay into a quark-antiquark pair, the leading QCD correction is of the form $(1+17/3\cdot\alpha_s/\pi)$. The full correction for the decay rate into b quarks is given by

$$\Gamma(H\to b\bar{b}) = \frac{G_F M_H}{4\sqrt{2}\pi} m_b^2(M_H) R^S(s=M_H^2,\mu^2=M_H^2) \tag{14}$$

$$R^S(M_H) = 1+1+5.667\,a_s+29.147\,a_s^2+41.758\,a_s^3-825.7\,a_s^4 = 1.2298 \tag{15}$$

for $\alpha_s(M_H)=0.108$, corresponding to $\alpha_s(M_Z)=0.118$.

The theory uncertainty of the correction factor has been reduced from 5 (four loop) to 1.5 (five loop) permille, where a variation of μ between $M_H/3$ and $3M_H$ has been assumed and $\alpha_s(M_H)$ has been fixed to 0.108.

The parametric uncertainties can be traced to the value of α_s, of m_b at low energies, say 10 GeV, and to the running of m_b from 10 GeV to M_H. For the values of α_s and m_b we adopt $\alpha_s(M_Z)=0.1189\pm0.002$ and $m_b(10\,\mathrm{GeV}) = 3610 - \frac{\alpha_s-0.1189}{0.002}$. Running m_b to the scale of M_H one obtains $m_b(M_H) = 2759\pm 8\,|_{m_b} \pm 27\,|_{\alpha_s}$ MeV.

While the quark mass anomallous dimension γ_4 has been obtained in five loop approximation aready some time ago [26] the corresponding result for the QCD beta-function is still missing at present. This corresponds to an uncertainty

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.4\times10^{-4}\,(\frac{\beta_4}{\beta_0}=0)\quad|\quad -4.3\times10^{-4}\,(\frac{\beta_4}{\beta_0}=100)\quad|\quad -7.3\times10^{-4}\,(\frac{\beta_4}{\beta_0}=200) \tag{16}$$

which should be compared to $\delta\Gamma(H \to b\bar{b})/\Gamma(H \to b\bar{b}) = 2.0 \times 10^{-4}$ expected for the FCC-ee. Assuming an improvement of our knowledge of the strong coupling by a factor 10, corresponding to $\delta\alpha_s = 2 \times 10^{-4}$ and for m_b by a factor 4 ($\delta m_b(M_H)/m(M_H) = 10^{-3}$) on finds

$$\frac{\delta\Gamma_{H\to b\bar{b}}}{\Gamma_{H\to b\bar{b}}} = \pm 2 \times 10^{-3}|_{m_b} \pm 1.3 \times 10^{-3}|_{\alpha_s,\mathrm{running}} \pm 1 \times 10^{-3}|_{\mathrm{theory}}. \tag{17}$$

b.) $\mathbf{H \to gg}$

Another hadronic decay mode of the Higgs boson with a large branching ratio is the one into two gluons, which starts in order α_s^2 and is presently known [27] to order α_s^5:

$$\begin{aligned} \Gamma(H \to gg) &= K \cdot \Gamma_{\mathrm{Born}}(H \to gg) \qquad (18) \\ K = 1 &+ 17.9167\, a_s' + (156.81 - 5.7083 \ln \frac{M_t^2}{M_H^2})(a_s')^2 \\ &+ (467.68 - 122.44 \ln \frac{M_t^2}{M_H^2} + 10.94 \ln^2 \frac{M_t^2}{M_H^2})\ (a_s')^3. \\ K &= 1 + 17.9167\, a_s' + 152.5(a_s')^2 + 381.5(a_s')^3 \\ &= 1 + 0.65038 + 0.20095 + 0.01825. \end{aligned}$$

where $M_t = 175 GeV$, $m_H = 120 GeV$ and $a_s' = \alpha_s^{(5)}(M_H)/\pi$ had been assumed. Note that an experimental precision for $\sigma(HZ) \times Br(H \to gg)$ of 1.4% has been claimed in [28]. Mass corrections to the Higgs-boson decay rate result can be found in [29].

7 Summary and Conclusions

Improved measurements of the total cross section for electron-positron annihilation just below the charm and the bottom threshold would lead to model independent tests of the running of α_s over a wide energy range. At high energies the precise control of QCD corrections is crucial for the detailled and precise comparison of theory predictions with experimental results at a future circular or linear electron-positron collider.

A significant improvement in the determination of the Z decay rate would immediately lead to a correspondingly improved value of the strong coupling constant. In combination with an improved determination of the fine structure constant at the scale of the Z-boson this would lead to a significantly improved prediction for M_W at the level of 1 MeV, corresponding to the anticipated precision of a future measurement.

A collider with an energy around 250 GeV would be ideally suited for the measurement of $e^+e^- \to ZH$, leading to the precise determination of the branching ratios of the Higgs boson into its various decay channels. Examples are presented which involve corrections up to order α_s^4 for the decay into $b\bar{b}$ and order α_s^5 for decays into gg. These demonstrate that predictions with a precision around one percent are well within reach for the dominant decay modes.

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