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Top-Yukawa effects on the β-function of the strong coupling in the SM at four-loop level

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Abstract: We present analytical results for the QCD β-function extended to the gaugeless limit of the unbroken phase of the Standard Model at four-loop level. Apart from the strong coupling itself we include the top-Yukawa contribution and the Higgs self-coupling. We observe a non-naive γ_5 contribution at order $y_t^4 g_s^4$, a feature not encountered in lower loop orders.

Keywords: Renormalization Group, Standard Model, QCD

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1 Introduction

An important feature of the perturbative treatment of any quantum field theory is the evolution of couplings, fields and masses with the renormalization scale μ, which is usually set to a characteristic energy scale of the physical process under consideration. This evolution is described by the Renormalization Group (RG) functions, i. e. β-functions for the couplings and anomalous dimensions for fields and masses.

The β-function for any coupling X is defined as

$$\beta_X(X, X_1, X_2, \ldots) = \mu^2 \frac{dX}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \beta_X^{(n)} \tag{1.1}$$

and the anomalous dimension of a field f as

$$\gamma_2^f = -\mu^2 \frac{d\ln Z_f^{-1}}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \gamma_2^{f\,(n)}, \tag{1.2}$$

where Z_f is the field strength renormalization constant[1] for the respective field. These functions are power series in all couplings $X, X_1, X_2, \ldots$ of the theory. The β-functions and the anomalous dimensions of masses are independent of all gauge parameters ξ whereas the anomalous dimensions of the fields are not.

Recently the RG functions of the Standard Model (SM) were computed at three-loop accuracy. In the $\overline{\text{MS}}$scheme β-functions do not depend on masses [1], hence they can be computed in the unbroken phase of the SM. For the gauge couplings g_s, g_2 and g_1 of

[1]Note that we take the inverse of Z_f in the definition of γ_2^f. For an n-point vertex V or a mass m the anomalous dimension is defined as $\gamma_n^V = -\mu^2 \frac{d\ln Z_V}{d\mu^2}$ or $\gamma_m = -\mu^2 \frac{d\ln Z_m}{d\mu^2}$.

the $SU_C(3)$, $SU_L(2)$ and $U_Y(1)$ subgroups of the SM the results were first published in [2, 3] and independently confirmed in [4]. For the top-Yukawa coupling y_t, which is the numerically largest Yukawa coupling by far, and the parameters of the Higgs potential λ and m^2 the β-functions were first computed in the gaugeless limit, i.e. $g_2, g_1 \to 0$, along with the anomalous dimensions of the fields involved [5]. Later β_λ and β_{m^2} were extended to the full SM [6], confirmed by [7, 8], as well as β_{y_t} [9], where the β-functions for the smaller Yukawa couplings were also added. The one- and two-loop β-functions for the gauge couplings [10–21], Yukawa couplings [18, 20, 22, 23] and Higgs potential parameters [18, 20, 21, 24] have been known for a long time as well as partial three-loop results [25–31]. At four-loop level only the QCD β-function, i.e. $\beta_{g_s}(g_s)$ or equivalently $\beta_{\alpha_s}(\alpha_s) = 2\frac{\alpha_s}{g_s}\beta_{g_s}$ with $\alpha_s = \frac{g_s^2}{4\pi}$ is known [32, 33].

Especially the evolution of the quartic Higgs self-coupling has received a lot of interest because of its close connection to the question of vacuum stability in the Standard Model. It has been shown that the stability of the SM vacuum up to some large energy scale $\Lambda \sim M_{Planck}$ is approximately equivalent to the requirement that the running coupling $\lambda(\mu) > 0$ for $\mu \leq \Lambda$ [34–36]. The function β_λ describing this evolution depends on all SM couplings an especially the large couplings y_t and g_s have a strong influence. As the evolution of all couplings is interdependent a precision calculation for the evolution of all - at least of the five largest (g_s, y_t, g_2, g_1 and λ) - is well motivated. Many analyses of this question have been performed [5, 37–49] during the last years.

In this paper we extend the QCD β-function to the gaugeless limit of the SM, i. e. we include the dependence on the top-Yukawa coupling y_t and the quartic Higgs self-coupling λ. This can be seen as a first step to all three gauge coupling β-functions in the full SM. To start with the gaugeless limit seems reasonable, first because at the energy-scales of our experiments y_t is the second largest coupling in the SM after g_s, followed by g_2, g_1 and λ. In order to renormalize fermion loops with four scalar legs we should also add counterterm $\propto \Phi^4$ to the Lagrangian of our simplified model. This is exactly a contribution to the renormalization of λ which makes it natural to include λ as well.

Secondly, the gaugeless limit of the SM provides an excellent opportunity to study the proper treatment of γ_5, which is introduced in the Yukawa-part of the Lagrangian. This matrix is not well-defined in $D = 4 - 2\varepsilon$ dimensions and hence constitutes a non-trivial challenge.

The paper is structured as follows: In the following section the technical details, especially the treatment of γ_5, as well as the automation of the calculation are discussed. Then the results are given and the relevance of the four-loop terms numerically determined at the scale of the top quark mass.

Note: During the finishing process of this paper a similar calculation was published by another group [50]. Their calculation was not performed with massive tadpole integrals but rather with massless propagator-like integrals and in the Background field gauge. Both results achieved with different methods agree but for one term which is numerically small.

2 Details of the calculation

2.1 Gaugeless limit of the SM

The Lagrangian of the SM in the unbroken phase can be decomposed into

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{SU(3)\times SU(2)\times U(1)}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\Phi}, \tag{2.1}$$

where $\mathcal{L}_{\mathrm{SU(3)\times SU(2)\times U(1)}}$ contains the kinetic terms of the fermions and gauge bosons, their interactions and the necessary gauge fixing and ghost terms. The Yukawa part $\mathcal{L}_{\mathrm{Yukawa}}$ describes the coupling of the fermions to a scalar SU(2) doublet $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ which results in fermion masses and the coupling of fermions to the Higgs boson after Spontaneous Symmetry Breaking as well as the mixing of the quark generations. The scalar part $\mathcal{L}_{\Phi}$ contains the kinetic term for the scalar field Φ, its potential and its coupling to the electroweak gauge bosons through the covariant derivative. In the gaugeless limit we neglect two smaller gauge couplings g_2 and g_1 (electroweak sector). We also approximate the small Yukawa couplings, i. e. all but the top-Yukawa coupling y_t, by zero and arrive at a simplified model which includes QCD and top-Yukawa effects as well as the scalar potential:

$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{y_t} + \mathcal{L}_{\Phi} \tag{2.2}$$

with

$$\begin{aligned}
\mathcal{L}_{QCD} &= -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2(1-\xi)} \left(\partial_\mu A^{a\,\mu}\right)^2 + \partial_\mu \bar{c}^a \partial^\mu c^a + g_s f^{abc}\, \partial_\mu \bar{c}^a A^{b\,\mu} c^c \\
&\quad + \sum_q \left\{ \frac{i}{2} \bar{q} \overleftrightarrow{\not\partial} q + g_s \bar{q} \not{A}^a T^a q \right\}, \qquad (2.3) \\
\mathcal{L}_{y_t} &= -y_t \left\{ \left(\bar{t} P_R t\right) \Phi_2^* + \left(\bar{t} P_L t\right) \Phi_2 - \left(\bar{b} P_R t\right) \Phi_1^* - \left(\bar{t} P_L b\right) \Phi_1 \right\}, \qquad (2.4) \\
\mathcal{L}_{\Phi} &= \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2. \qquad (2.5)
\end{aligned}$$

Here q runs over all quark flavours, the gluon field strength tensor is given by

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{2.6}$$

and f^{abc} are the structure constants of the colour gauge group with the generators T^a which satisfy

$$\left[T^a, T^b\right] = i f^{abc} T^c. \tag{2.7}$$

The Yukawa sector mixes left-handed (L) and right-handed (R) Weyl spinors which can be projected out from Dirac spinors used in our Feynman rules by the application of the projectors

$$P_L = \frac{1}{2}\left(1 - \gamma_5\right) \qquad P_R = \frac{1}{2}\left(1 + \gamma_5\right). \tag{2.8}$$

The left- and right-handed parts of the quark fields and vertices participating in the Yukawa interaction are renormalized differently.

The Lagrangian (2.2) is renormalized with the counterterms

$$\begin{aligned}
\delta\mathcal{L}_{QCD} &= -\frac{1}{4}\delta Z_3^{(2g)}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 - \frac{1}{2}\delta Z_1^{(3g)} g_s f^{abc}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A_\mu^b A_\nu^c \\
&- \frac{1}{4}\delta Z_1^{(4g)} g_s^2\left(f^{abc} A_\mu^b A_\nu^c\right)^2 + \delta Z_3^{(2c)}\partial_\mu \bar{c}^a \partial^\mu c^a + \delta Z_1^{(ccg)} g_s f^{abc}\,\partial_\mu \bar{c}^a A^{b\,\mu} c^c \qquad (2.9)\\
&+ \sum_q \left\{\frac{i}{2}\bar{q}\overleftrightarrow{\not{\partial}}\left[\delta Z_{2,L}^{(2q)} P_L + \delta Z_{2,R}^{(2q)} P_R\right] q + g_s \bar{q}\not{A}^a T^a\left[\delta Z_{1,L}^{(qqg)} P_L + \delta Z_{1,R}^{(qqg)} P_R\right] q\right\},
\end{aligned}$$

$$\delta\mathcal{L}_{Yukawa} = -\delta Z_1^{(tb\Phi)} y_t\left\{(\bar{t}P_R t)\,\Phi_2^* + (\bar{t}P_L t)\,\Phi_2 - \left(\bar{b}P_R t\right)\Phi_1^* - (\bar{t}P_L b)\,\Phi_1\right\}, \qquad (2.10)$$

$$\delta\mathcal{L}_\Phi = \delta Z_2^{(2\Phi)}\partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\,\delta Z_{\Phi^2}\Phi^\dagger\Phi + \delta Z_1^{(4\Phi)}\left(\Phi^\dagger\Phi\right)^2. \qquad (2.11)$$

All these renormalization constants were computed at three-loop level in the course of the calculations in [5]. The simplest way to derive the renormalization constant for the strong gauge coupling g_s is via

$$Z_{g_s} = \frac{Z_1^{(ccg)}}{Z_3^{(2c)}\sqrt{Z_3^{(2g)}}} \qquad (2.12)$$

where we use the renormalization constants $Z = 1 + \delta Z$ in the $\overline{\rm MS}$-scheme. All divergent integrals are regularized in $D = 4 - 2\varepsilon$ space time dimensions.

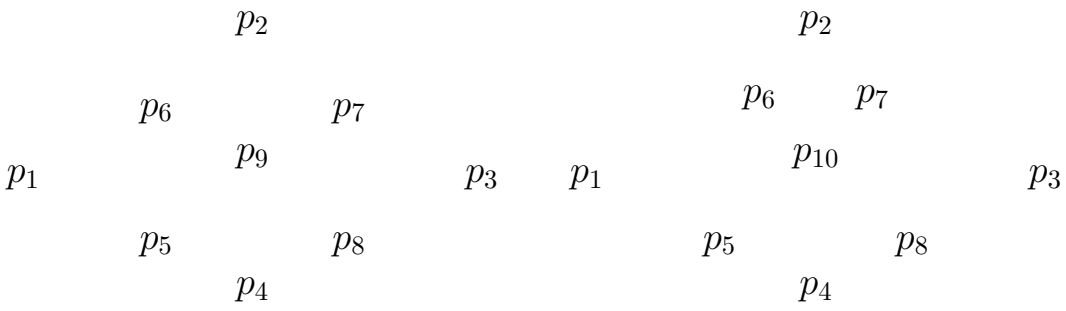
2.2 Automation and calculation with massive tadpoles

The calculation begins with the generation of all necessary 1PI Feynman diagrams with two external ghost or gluon legs for $Z_3^{(2c)}$ or $Z_3^{(2g)}$ and with two external ghost and one external gluon leg for $Z_1^{(ccg)}$. This was done with the program QGRAF [51].

The C++ programs Q2E and EXP [52, 53] are then used to identify the topology of the diagram. Later we will Taylor expand in the external momenta and use projectors on the integrals in order to make them scalar. For example the ghost-gluon vertex corrections are proportional to the outgoing ghost momentum q^μ, where μ is the Lorentz index of the gluon leg. Hence we expand to first order in q, use the projector $\frac{q^\mu}{q^2}$ on the integral and set $q \to 0$ after that. This is allowed as $\overline{\rm MS}$ renormalization constants do not depend on external momenta. After having set all external momenta to zero we are left with tadpole integrals. The fermion traces, the expansion in the external momenta and the insertion of counterterms in one-loop, two-loop and three-loop diagrams was performed using FORM [54, 55]. The colour factors were computed with the FORM package COLOR [56]. The tadpole integrals up to three-loop order were computed with the FORM-based package MATAD[57].

At four-loop level there are two independent tadpole topologies, see Fig. 1. [2] All scalar products $p_i{\cdot}_j$ $(i, j = 1, \ldots, 10)$ can be written as linear combinations of the p_i^2 which can be expressed in terms of the scalar propagators $D_i = \frac{1}{i}\frac{1}{M^2 - p_i^2}$ and the auxiliary Mass M^2

[2] All Feynman diagrams in this paper have been drawn with the Latex package Axodraw [58].



planar topology: tad4lp non-planar topology: tad4lnp

Figure 1: Four-loop tadpole topologies: p_1, p_2, p_3, p_4 are independent loop momenta, the others are linear combinations $p_5 = p_4 - p_1$, $p_6 = p_2 - p_1$, $p_7 = p_3 - p_2$, $p_8 = p_3 - p_4$, $p_9 = p_4 - p_2$ and $p_{10} = p_4 + p_2 - p_1 - p_3$.

(see below). Hence all four-loop integrals can be written in terms of functions

$$\text{TAD4l}(n_1, \ldots, n_{10}) := \int \mathrm{d}^D p_1 \int \mathrm{d}^D p_2 \int \mathrm{d}^D p_3 \int \mathrm{d}^D p_4 \prod_{i=1}^{10} D_i^{n_i}. \tag{2.13}$$

The integrals (2.13) can be reduced to Master Intgrals (MI) using FIRE [59]. For the huge number of integrals in such a calculation the C++ version of FIRE 5 [60] is necessary. All MI needed for this computation can be found in [33]. The program FIESTA 3 [61] was used to numerically cross check these MI and some unreduced integrals as a check for our setup.

In order to compute the divergent part of the needed self-energies and vertex corrections we use the same method as in our previous calculations [5, 6]. This method was suggested in [62] and further developed in [63]. A step-by-step explanation of this method can be found in [46]. An auxiliary mass parameter M^2 is introduced in every propagator denominator. A naive Taylor expansion in the external momenta is performed before applying the projector to scalar integrals. After that all external momenta are set to zero which leaves us with scalar tadpole integrals. Subdivergences $\propto M^2$ are canceled by counterterms

$$\frac{M^2}{2} \delta Z^{(2g)}_{M^2} A^a_\mu A^{a\,\mu} \text{ and } \frac{M^2}{2} \delta Z^{(2\Phi)}_{M^2} \Phi^\dagger \Phi. \tag{2.14}$$

which are computed order by order in perturbation theory and inserted in lower loop diagrams. Note that this method only yields the divergent part of the integrals correctly, not the finite part.

2.3 Treatment of γ_5

The most important issue of this calculation is the proper treatment of γ_5 in dimensional regularization. In $D = 4$ dimensions it can be defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \text{ with } \varepsilon_{0123} = 1 = -\varepsilon^{0123}. \tag{2.15}$$

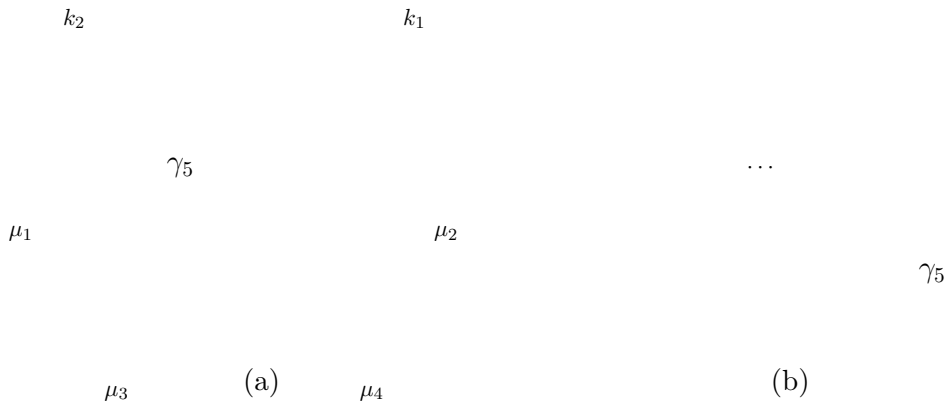


Figure 2: γ_5 on internal (a) and external (b) fermion lines

In most diagrams a naive treatment of γ_5 is sufficient, i. e. we use $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5^2 = 1$, valid in $D = 4$ dimensions, until only one or no γ_5 matrix remain on each fermion line, then discard diagrams with at least one γ_5. This is valid for fermion lines with less than four Lorentz indices and momenta flowing into the fermion line. Fig. 2 shows the schematic cases of γ_5 appearing on internal and external fermion lines. We start with the case of internal lines (see Fig. 2 (a)). In fact, for the calculation presented in this paper no external fermion lines appear.

As we set all momenta external to the whole Feynman diagram to zero for the computation of the UV divergent part of the diagram external momenta to a fermion line (k_1, k_2,...) are loop momenta from other loops. Taking the trace over the closed fermion loop in $D = 4$ dimensions yields a result with terms proportional to $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ and $\varepsilon_{\mu_1\mu_2\alpha\beta}\, k_1^\alpha\, k_2^\beta$ and so on. In order for the ε-tensors not to vanish at least 4 free Lorentz structures are needed. Else the diagram can be set to zero before we even employ dimensional regularization.

If we have only one internal fermion line with one γ_5 on it and the final result is known to be scalar (not pseudoscalar), as are the counterterms we want to compute here, we can discard these terms as well. The only possibility for a non-naive contribution to the final result can appear in the case of two (or more) fermion lines. Here the two ε-tensors can be contracted and expressed in terms of the metric tensor

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = -\sum_\pi \mathrm{sgn}(\pi) g^{\mu_{\pi(1)}}_{\nu_1} g^{\mu_{\pi(2)}}_{\nu_2} g^{\mu_{\pi(3)}}_{\nu_3} g^{\mu_{\pi(4)}}_{\nu_4}, \tag{2.16}$$

where the sum is taken over all permutations π of (1,2,3,4) and

$$\mathrm{sgn}(\pi) = \begin{cases} +1 & \text{for } \pi \text{ even} \\ -1 & \text{for } \pi \text{ odd} \end{cases}. \tag{2.17}$$

The lhs of (2.16) is composed of intrinsically four-dimensional objects whereas the rhs can be used in $D = 4 - 2\varepsilon$ dimensions, introducing an uncertainty of $\mathcal{O}(\varepsilon)$. However, if the integrals appearing in the calculation of the Feynman diagram in question have only $\frac{1}{\varepsilon}$ poles the divergent part, which we are interested in here, is unaffected.

For completeness we want to make a short remark about external fermion lines, such as the one shown in Fig. 2 (b), as well. Here we can anticommute the γ_5 to the end of the fermion

line and hence outside of all loops. But if we use a projector on the external fermion line in order to make the integral scalar and this involves taking a trace over the fermion line we have to treat it the same way as the internal ones. In the case of the three-loop β-function for the Yukawa couplings a non-naive γ_5 effect from the contraction of the ε-tensors from an internal and an external fermion line was observed [5].

In the the calculations needed for the renormalization constants in (2.12) only one type of diagram features two fermion lines with four external Lorentz indices or loop-momenta to them, namely in the gluon propagator, when each external leg is attached to a different fermion loop and the two fermion loops are connected by a gluon and two Φ-lines. A planar example is shown in Fig. 3

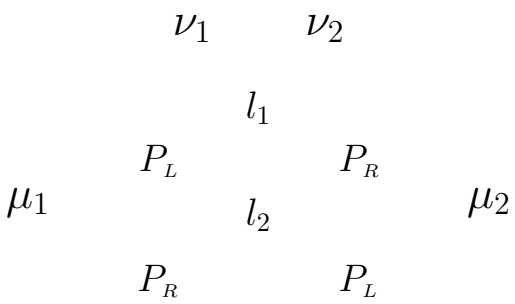


Figure 3: Diagram giving a non-naive γ_5-contribution to the gluon self-energy: each fermion line has two indices μ_i and ν_i (i=1,2) and two momenta l_1, l_2, which can support a $\varepsilon^{\mu_i\nu_i\alpha\beta}l_\alpha l_\beta$ term. The left- and right-handed projectors $P_{L,R}$ introduce γ_5 into the diagram.

It was checked explicitly that only $\frac{1}{\varepsilon}$ poles appear in the integrals needed for these diagrams. In fact, as an additional precaution we checked that at $\mathcal{O}(\varepsilon)$ completely antisymmetric and completely symmetric structures composed of the metric and the eight indices appearing in the ε tensors do not give contributions to the divergent part. This was implemented as

$$\begin{aligned}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = &-\sum_\pi \mathrm{sgn}(\pi) g^{\mu_{\pi(1)}}_{\nu_1} g^{\mu_{\pi(2)}}_{\nu_2} g^{\mu_{\pi(3)}}_{\nu_3} g^{\mu_{\pi(4)}}_{\nu_4} \left(1 + \varepsilon \cdot \mathrm{C}^{\mathrm{as}}\right) \\ &+ \varepsilon \cdot \mathrm{C}^{\mathrm{s}} \sum_\pi g^{\mu_{\pi(1)}}_{\nu_1} g^{\mu_{\pi(2)}}_{\nu_2} g^{\mu_{\pi(3)}}_{\nu_3} g^{\mu_{\pi(4)}}_{\nu_4}, \end{aligned} \tag{2.18}$$

where the labels $\mathrm{C}^{\mathrm{as,s}}$ parametrize the uncertainty introduced through (2.16) being applied in $D = 4 - 2\varepsilon$. As they drop out in the divergent term of our final result we are satisfied that γ_5 can be treated in this way. An alternative that was also checked is to use (2.16) and an $\tilde{\varepsilon}$ instead of ε in the fermion traces and check that $\frac{1}{\varepsilon^i}$ with $i > 1$ drop out in the sum of all diagrams. After that we set $\tilde{\varepsilon} = \varepsilon$.

3 Results

In this section we give the results for the four-loop β-function of the strong coupling g_s in the gaugeless limit of the SM. For a gerneric SU(N_c) gauge group the colour factors are expressed through the quadratic Casimir operators C_F and C_A of the fundamental and the

adjoint representation of the corresponding Lie algebra. The dimension of the fundamental representation is called N_c. The adjoint representation has dimension n_g and the trace T_F defined by $T_F\delta^{ab} = \mathbf{Tr}\left(T^aT^b\right)$ with the group generators T^a of the fundamental representation. In addition we need a few higher order invariants constructed from the symmetric tensors

$$\begin{aligned} d_F^{abcd} = \frac{1}{6}\mathrm{Tr}\Big(&T^aT^bT^cT^d + T^aT^bT^dT^c + T^aT^cT^bT^d \\ &+ T^aT^cT^dT^b + T^aT^dT^bT^c + T^aT^dT^cT^b\Big). \end{aligned} \tag{3.1}$$

from the generators of the fundamental representation and analogously d_A^{abcd} constructed from the generators of the adjoint representation. The combinations needed and their SU(N_c) values are

$$\begin{aligned} \frac{d_F^{abcd}d_F^{abcd}}{n_g} &= \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2}, \qquad \frac{d_F^{abcd}d_A^{abcd}}{n_g} = \frac{N_c(N_c^2+6)}{48}, \\ \frac{d_A^{abcd}d_A^{abcd}}{n_g} &= \frac{N_c^2(N_c^2+36)}{24}. \end{aligned} \tag{3.2}$$

Furthermore for SU(N_c) we have

$$T_F = \frac{1}{2}, \quad C_F = \frac{N_c^2-1}{2N_c}, \quad C_A = N_c, \quad n_g = N_c^2 - 1. \tag{3.3}$$

The number of active fermion flavours is denoted by n_f (=6 in the SM).

$$\begin{aligned} \frac{\beta_{g_s}^{(4)}}{g_s} =& g_s^8\left(\frac{40}{9}\frac{d_A^{abcd}d_A^{abcd}}{n_g} - \frac{150653}{972}C_A^4 - \frac{256}{9}n_f\frac{d_F^{abcd}d_A^{abcd}}{n_g} - 23n_fT_FC_F^3\right. \\ &+\frac{2102}{27}n_fC_AT_FC_F^2 - \frac{7073}{486}n_fC_A^2T_FC_F + \frac{39143}{162}n_fC_A^3T_F + \frac{352}{9}n_f^2\frac{d_F^{abcd}d_F^{abcd}}{n_g} \\ &-\frac{676}{27}n_f^2T_F^2C_F^2 - \frac{8576}{243}n_f^2C_AT_F^2C_F - \frac{3965}{81}n_f^2C_A^2T_F^2 - \frac{616}{243}n_f^3T_F^3C_F \\ &-\frac{212}{243}n_f^3C_AT_F^3 - \frac{352}{3}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{n_g} + \frac{22}{9}\zeta_3C_A^4 + \frac{832}{3}\zeta_3n_f\frac{d_F^{abcd}d_A^{abcd}}{n_g} \\ &-\frac{176}{9}\zeta_3n_fC_AT_FC_F^2 + \frac{328}{9}\zeta_3n_fC_A^2T_FC_F - \frac{68}{3}\zeta_3n_fC_A^3T_F - \frac{256}{3}\zeta_3n_f^2\frac{d_F^{abcd}d_F^{abcd}}{n_g} \\ &\left.+\frac{352}{9}\zeta_3n_f^2T_F^2C_F^2 - \frac{224}{9}\zeta_3n_f^2C_AT_F^2C_F - \frac{112}{9}\zeta_3n_f^2C_A^2T_F^2\right) \\ &+ g_s^6y_t^2\left(-3T_FC_F^2 - \frac{523}{18}C_AT_FC_F - \frac{985}{9}C_A^2T_F + \frac{322}{9}n_fT_F^2C_F\right. \\ &\left.+\frac{218}{9}n_fC_AT_F^2 + 72\zeta_3T_FC_F^2 + 36\zeta_3C_AT_FC_F\right) \\ &+ g_s^4y_t^4\left(-3T_FC_F + \frac{41}{2}T_FC_FN_c + 36C_AT_F + 25C_AT_FN_c\right. \\ &-24\zeta_3T_FC_FN_c + T_F^2\left(\frac{100}{3} - 16\zeta_3\right)\Big) \\ &+ g_s^2y_t^6\left(-\frac{21}{4}T_F - 29T_FN_c - \frac{3}{2}T_FN_c^2 - 6\zeta_3T_F\right) \\ &- 30g_s^2y_t^4\lambda T_F + 36T_Fg_s^2y_t^2\lambda^2. \end{aligned} \tag{3.4}$$

This is in agreement with [50] except for the term $\propto g_s^4 y_t^4 T_F^2$ (blue) which is the only one affected by non-naive γ_5 contributions as explained above. The naive and non-naive (i. e. stemming from the contraction of two ε-tensors) contributions are

$$g_s^4 y_t^4 T_F^2 \left(\frac{100}{3} - 16\zeta_3\right) = g_s^4 y_t^4 T_F^2 \left(\underbrace{24}_{\text{(naive)}} + \underbrace{\frac{28}{3}}_{\text{(non-naive)}} - \underbrace{48\zeta_3}_{\text{(naive)}} + \underbrace{32\zeta_3}_{\text{(non-naive)}} \right). \tag{3.5}$$

The lower loop results are

$$\begin{aligned} \frac{\beta_{g_s}^{(3)}}{g_s} &= g_s^6 \left(-\frac{2857}{108} C_A^3 - n_f T_F C_F^2 + \frac{205}{18} n_f C_A T_F C_F \right. \\ &\quad \left. + \frac{1415}{54} n_f C_A^2 T_F - \frac{22}{9} n_f^2 T_F^2 C_F - \frac{79}{27} n_f^2 C_A T_F^2 \right) \\ &\quad - g_s^4 y_t^2 \left(3 T_F C_F + 12 C_A T_F\right) + g_s^2 y_t^4 \left(+ \frac{9}{2} T_F + \frac{7}{2} T_F N_c \right), \end{aligned} \tag{3.6}$$

$$\frac{\beta_{g_s}^{(2)}}{g_s} = g_s^4 \left(-\frac{17}{3} C_A^2 + 2 n_f T_F C_F + \frac{10}{3} n_f C_A T_F \right) - 2 g_s^2 y_t^2, \tag{3.7}$$

$$\frac{\beta_{g_s}^{(1)}}{g_s} = g_s^2 \left(-\frac{11}{6} C_A + \frac{2}{3} n_f T_F \right). \tag{3.8}$$

in agreement with [5]. The pure QCD part of (3.4) agrees with [32, 33].

For convenience we also give the β-function for α_s. We absorb the loop factor $\frac{1}{16\pi^2}$ into

$$a_s = \frac{g_s^2}{(4\pi)^2} = \frac{\alpha_s}{4\pi}, \quad a_t = \frac{y_t^2}{(4\pi)^2}, \quad a_\lambda = \frac{\lambda}{(4\pi)^2} \tag{3.9}$$

and define

$$\beta_{\alpha_s}(a_s, a_t, a_\lambda) = \sum_{n=1}^{\infty} \beta_{\alpha_s}^{(n)}(a_s, a_t, a_\lambda). \tag{3.10}$$

We find

$$
\begin{aligned}
\frac{\beta^{(4)}_{\alpha_s}}{\alpha_s} =& a_s^4 \left(\frac{80}{9}\frac{d_A^{abcd}d_A^{abcd}}{n_g} - \frac{150653}{486}C_A^4 - \frac{512}{9}n_f\frac{d_F^{abcd}d_A^{abcd}}{n_g} - 46 n_f T_F C_F^3 \right. \\
&+ \frac{4204}{27}n_f C_A T_F C_F^2 - \frac{7073}{243}n_f C_A^2 T_F C_F + \frac{39143}{81}n_f C_A^3 T_F \\
&+ \frac{704}{9}n_f^2\frac{d_F^{abcd}d_F^{abcd}}{n_g} - \frac{1352}{27}n_f^2 T_F^2 C_F^2 - \frac{17152}{243}n_f^2 C_A T_F^2 C_F \\
&- \frac{7930}{81}n_f^2 C_A^2 T_F^2 - \frac{1232}{243}n_f^3 T_F^3 C_F - \frac{424}{243}n_f^3 C_A T_F^3 - \frac{704}{3}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{n_g} \\
&+ \frac{44}{9}\zeta_3 C_A^4 + \frac{1664}{3}\zeta_3 n_f\frac{d_F^{abcd}d_A^{abcd}}{n_g} - \frac{352}{9}\zeta_3 n_f C_A T_F C_F^2 C_F \\
&+ \frac{656}{9}\zeta_3 n_f C_A^2 T_F - \frac{136}{3}\zeta_3 n_f C_A^3 T_F - \frac{512}{3}\zeta_3 n_f^2\frac{d_F^{abcd}d_F^{abcd}}{n_g}C_F \\
&+ \frac{704}{9}\zeta_3 n_f^2 T_F^2 C_F^2 - \frac{448}{9}\zeta_3 n_f^2 C_A T_F^2 C_F - \frac{224}{9}\zeta_3 n_f^2 C_A^2 T_F^2 \Big) \\
&+ a_t a_s^3 \left(-6 T_F C_F^2 - \frac{523}{9}C_A T_F C_F - \frac{1970}{9}C_A^2 T_F + \frac{644}{9}n_f T_F^2 C_F \right. \\
&+ \frac{436}{9}n_f C_A T_F^2 + 144\zeta_3 T_F C_F^2 + 72\zeta_3 C_A T_F C_F \Big) \\
&+ a_t^2 a_s^2 \left(-6 T_F C_F + 41 T_F C_F N_c + 72 C_A T_F + 50 C_A T_F N_c \right. \\
&\left. -48\zeta_3 T_F C_F N_c + T_F^2\left(\frac{200}{3} - 32\zeta_3\right)\right) \\
&+ a_t^3 a_s \left(-\frac{21}{2}T_F - 58 T_F N_c - 3 T_F N_c^2 - 12\zeta_3 T_F \right) \\
&+ a_t^2 a_s a_\lambda \left(-60 T_F\right) + a_t a_s a_\lambda^2 \left(+72 T_F\right),
\end{aligned}
\tag{3.11}
$$

where

$$
a_t^2 a_s^2 T_F^2 \left(+\frac{200}{3}T_F^2 - 32\zeta_3 T_F^2 \right) = a_t^2 a_s^2 T_F^2 \left(\underbrace{48}_{\text{(naive)}} + \underbrace{\frac{56}{3}}_{\text{(non-naive)}} - \underbrace{96\zeta_3}_{\text{(naive)}} + \underbrace{64\zeta_3}_{\text{(non-naive)}} \right). \tag{3.12}
$$

and

$$
\begin{aligned}
\frac{\beta^{(3)}_{\alpha_s}}{\alpha_s} =& + a_s^3 \left(-\frac{2857}{54}C_A^3 - 2 n_f T_F C_F^2 + \frac{205}{9}n_f C_A T_F C_F \right. \\
&\left. + \frac{1415}{27}n_f C_A^2 T_F - \frac{44}{9}n_f^2 T_F^2 C_F - \frac{158}{27}n_f^2 C_A T_F^2 \right) \\
&+ a_t a_s^2 \left(-6 T_F C_F - 24 C_A T_F\right) + a_t^2 a_s \left(+9 T_F + 7 T_F N_c\right),
\end{aligned}
\tag{3.13}
$$

$$
\frac{\beta^{(2)}_{\alpha_s}}{\alpha_s} = + a_s^2 \left(-\frac{34}{3}C_A^2 + 4 n_f T_F C_F + \frac{20}{3}n_f C_A T_F \right) + a_t a_s \left(-4 T_F\right), \tag{3.14}
$$

$$
\frac{\beta^{(1)}_{\alpha_s}}{\alpha_s} = + a_s \left(-\frac{11}{3}C_A + \frac{4}{3}n_f T_F \right). \tag{3.15}
$$

Now we want to give a numerical evaluation of the β-functions at the scale of the top mass in order to get an idea of the size of the new terms. For $M_t \approx 173.34 \pm 0.76$ GeV [64], $M_H \approx 125.09 \pm 0.24$ GeV[65] and $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ [66] we get the couplings in the $\overline{\text{MS}}$-scheme at this scale using two-loop matching relations [48]

$$\begin{aligned} g_s(M_t) &= 1.1666 \pm 0.0035(\text{exp}), \\ y_t(M_t) &= 0.9369 \pm 0.0046(\text{exp}) \pm 0.0005(\text{theo}), \\ \lambda(M_t) &= 0.1259 \pm 0.0005(\text{exp}) \pm 0.0003(\text{theo}) \end{aligned} \tag{3.16}$$

where the experimental uncertainty (exp) stems from M_t, M_H and $\alpha_s(M_Z)$ and the theoretical one (theo) from the matching of on-shell to $\overline{\text{MS}}$ parameters (these are taken from [48]). We find[3]

$$\frac{\beta_{g_s}^{(2)}}{\beta_{g_s}^{(1)}} = \underbrace{3.20 \times 10^{-2}}_{g_s^4} \underbrace{+1.59 \times 10^{-3}}_{g_s^2 y_t^2}, \tag{3.17}$$

$$\frac{\beta_{g_s}^{(3)}}{\beta_{g_s}^{(1)}} = \underbrace{-3.45 \times 10^{-4}}_{g_s^6} \underbrace{+2.74 \times 10^{-4}}_{g_s^4 y_t^2} \underbrace{-6.62 \times 10^{-5}}_{g_s^2 y_t^4}, \tag{3.18}$$

$$\frac{\beta_{g_s}^{(4)}}{\beta_{g_s}^{(1)}} = \underbrace{2.26 \times 10^{-4}}_{g_s^8} \underbrace{+2.47 \times 10^{-5}}_{g_s^6 y_t^2} \underbrace{-1.06 \times 10^{-5}}_{g_s^4 y_t^4 (\text{naive})} \underbrace{-9.09 \times 10^{-7}}_{g_s^4 y_t^4 (\text{non-naive})} \tag{3.19}$$

$$\underbrace{+2.77 \times 10^{-6}}_{g_s^2 y_t^6} \underbrace{+1.06 \times 10^{-7}}_{g_s^2 y_t^4 \lambda} \underbrace{-1.82 \times 10^{-8}}_{g_s^2 y_t^2 \lambda^2} \tag{3.20}$$

We see that the top-Yukawa contributions have a sizable impact on the four-loop β-function for the strong coupling. The part $\propto g_s^6 y_t^2$ increases it by $\sim 10\%$ and the part $\propto g_s^4 y_t^4$ decreases it by $\sim 5\%$ at this scale compared to the pure QCD contribution $\propto g_s^8$. The non-naive term gives a $\sim 0.4\%$ contribution. Numerically the discrepancy with [50] is therefore negligible in any phenomenological analysis. At some point it should be resolved nevertheless.

4 Conclusions

We have presented an analytical result for the four-loop β-function of the strong coupling g_s in the gaugeless limit of the SM. This constitutes an important extension of the well-known QCD result as top-Yukawa coupling is numerically the next important coupling after g_s, at least at the electroweak scale. Furthermore, this is an important step towards a complete calculation of the four-loop β-functions of the gauge couplings in the full SM.

An important feature of this result is the non-naive γ_5 contribution $\propto g_s^4 y_t^4$. In the pure gauge boson and fermion sector of the SM, given by $\mathcal{L}_{\text{SU(3)}\times\text{SU(2)}\times\text{U(1)}}$, all non-naive contributions cancel in the sum of all diagrams, making this part of the SM anomaly free.

[3] The labels under the braces indicate from which part of the β-function the contributions come.

This has been explicitly checked during the calculation of the three-loop β-functions for the gauge couplings in the SM [2, 3]. Here we see that with the inclusion of a scalar field non-naive contributions may appear in higher orders and special care will have to be taken when a attempting a complete calculation of four-loop β-functions in the SM.

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