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Decoupling of heavy quarks at four loops and effective Higgs-fermion coupling

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Abstract

We compute the decoupling constant ζ_m relating light quark masses of effective n_l-flavour QCD to (n_l+1)-flavour QCD to four-loop order. Immediate applications are the evaluation of the $\overline{\rm MS}$ charm quark mass with five active flavours and the bottom quark mass at the scale of the top quark or even at GUT scales. With the help of a low-energy theorem ζ_m can be used to obtain the effective coupling of a Higgs boson to light quarks with five-loop accuracy. We briefly discuss the influence on $\Gamma(H \to b\bar{b})$.

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1 Introduction and notation

Perturbative calculations in QCD are quite advanced and have reached, at least for some observables, the four and even five-loop level (see Refs. [1, 2] for a recent review). This concerns in particular the renormalization group functions which have been computed at four loops in Refs. [3–7]. The first five-loop result has been obtained recently in Ref. [8] where the quark mass anomalous dimension has been computed to this order.

In order to consistently relate the quark masses and strong coupling constant evaluated at different energy scales, both the renormalization group functions and also the decoupling relations have to be available. The latter take care of integrating out heavy quark fields. In fact, N-loop running goes along with $N-1$-loop decoupling. Thus, besides the five-loop anomalous dimensions also the four-loop decoupling relations are needed. In Refs. [9,10] a first step has been undertaken in this direction and the four-loop decoupling constant for α_s has been computed (although the five-loop beta function is not yet available). In this paper we complement the result by computing the four-loop corrections to the decoupling constant for the light quark masses, which supplements the five-loop result for γ_m [8].

In Ref. [11] a formalism has been derived which allows for an effective calculation of the N-loop decoupling constants with the help of N-loop vacuum integrals. In the following we present the formulae which are relevant for the calculation of the quark mass decoupling constant.

The bare decoupling constant ζ_m^0 is defined via the relation

$$m_q^{0\prime} = \zeta_m^0 m_q^0 \,, \tag{1}$$

where m_q^0 and $m_q^{0\prime}$ are the bare quark mass parameters in the full n_f- and effective theory ($n_l \equiv n_f - 1$)-flavour theory. Introducing the renormalization constants in both theories leads to the equation

$$m_q^\prime(\mu) = \frac{Z_m}{Z_m^\prime}\zeta_m^0 m_q(\mu) = \zeta_m m_q(\mu) \,. \tag{2}$$

which relates finite quantities and defines ζ_m. Note that primed quantities depend on $\alpha_s^{(n_l)}$ and non-primed quantities on $\alpha_s^{(n_f)}$. Four-loop results for Z_m and $Z_m^\prime$ can be found in Refs. [3, 4, 7] and ζ_m^0 can be computed with the help

$$\zeta_m^0 = \frac{1 - \Sigma_S^{0h}(0)}{1 + \Sigma_V^{0h}(0)} \,, \tag{3}$$

where $\Sigma_S^{0h}(0)$ and $\Sigma_V^{0h}(0)$ are the scalar and vector parts of the light-quark self energy evaluated at zero external momentum. The superscript "h" reminds that one has to consider only the hard part which involves at least one propagator of the heavy quark.

In the next Section we discuss the calculation of ζ_m^0 and its renormalization to arrive at ζ_m. Section 3 applies a low-energy theorem to derive from the four-loop result of ζ_m the effective Higgs-fermion coupling constant to five-loop order. We summarize our findings in Section 4.

2 Decoupling for light quark masses

In this section, we compute the decoupling constant ζ_m^0 and combine it with the four-loop result for Z_m to obtain the finite quantity ζ_m. The computation of ζ_m^0 requires the knowledge of the hard contribution to the scalar and vector part of the light-quark propagator, see Fig. 1 for sample Feynman diagrams. The first non-vanishing contribution arises at two loops where one diagram contributes. At three-loop order there are 25 and at four loops we have 765 Feynman diagrams.

The perturbative expansion of Eq. (3) to four loops leads to

$$\zeta_m^0 = 1 - \Sigma_S^{0h}(0) - \Sigma_V^{0h}(0) + \Sigma_V^{0h}(0)\left[\Sigma_S^{0h}(0) + \Sigma_V^{0h}(0)\right] + \ldots \,, \tag{4}$$

where in the last term on the right-hand side only two-loop expressions for $\Sigma_S^{0h}(0)$ and $\Sigma_V^{0h}(0)$ have to be inserted.

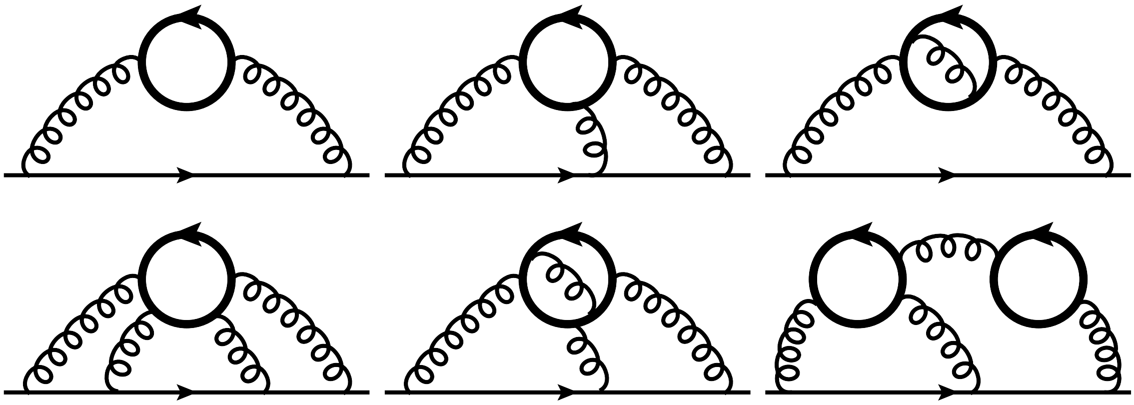

Figure 1: Sample Feynman diagrams contributing to the hard part of the light-quark propagator up to four loops. Solid and curly lines denote quarks and gluons, respectively. At least one of the closed fermion loops needs to be the heavy quark.

We generate the Feynman diagrams with the help of `QGRAF` [12] and pass the output via `q2e` [13,14], which transforms Feynman diagrams into Feynman amplitudes, to `exp` [13,14] that generates `FORM` [15, 16] code. After processing the latter one obtains the result as a linear combination of scalar functions which have a one-to-one relation to the underlying topology of the diagram. The functions contain the exponents of the involved propagators as arguments. At this point one has a large number of different functions. Thus, in a next step one passes them to a program which implements the Laporta algorithm [17] and performs a reduction to a small number of so-called master integrals. We use for the latter step the `C++` program `FIRE` [18]. Our four-loop result is expressed in terms of 13 master integrals which we take from Ref. [19] (see also [20–22] and references therein). All ϵ coefficients are known analytically except the ϵ^3 of integral $J_{6,2}$ (in the notation from Ref. [19]). We take the numerical value of this coefficient from Eq. (4.10) of Ref. [21], taking into account the different normalizations of the integrals.

Note that for our calculation we have used a general gauge parameter ξ of the gluon propagator. At four loops, in intermediate steps terms up to order ξ^6 are present, however, in the final result for ζ_m^0 all ξ terms drop out. The last term on the right-hand side of Eq. (4) is separately ξ-independent since at two loops $\Sigma_S^{0h}(0)$ and $\Sigma_V^{0h}(0)$ are individually ξ-independent. The results up to three-loop order have been checked with the help of `MATAD` [23].

To obtain ζ_m^0 we have to renormalize α_s and the heavy quark mass m_h to two-loop order. The corresponding $\overline{\text{MS}}$ counterterms are well-known (see, e.g.. Ref. [7]). ζ_m^0 still contains poles in ϵ which are removed by multiplying with the factor Z_m/Z_m' (see, Eq. (2)) which is needed to four-loop order [3, 4, 7]. Note that Z_m' depends on the strong coupling constant of the effective theory, $\alpha_s^{(n_l)}$, whereas Z_m and ζ_m^0 are expressed in terms of $\alpha_s^{(n_l+1)}$. In order to achieve the cancellation of the ϵ poles the same coupling constant has to be used in all three quantities. We have decided to replace $\alpha_s^{(n_l)}$ in favour of $\alpha_s^{(n_l+1)}$ which is done using the corresponding decoupling constant ζ_{α_s} up three-loop order [11]. Note, however, that also higher order terms in ϵ are needed since ζ_{α_s} gets multiplied by poles present in Z'_m. Up to two-loop order they can be found in Refs. [24, 25]; the three-loop terms of order ϵ can be extracted from Refs. [9, 10].

Our final result for the decoupling constant parametrized in terms of the $\overline{\mathrm{MS}}$ heavy quark mass reads

$$
\begin{aligned}
\zeta_m^{\overline{\mathrm{MS}}} = & \\
& 1+\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^2\left(\frac{89}{432}-\frac{5}{36}\ln\frac{\mu^2}{m_h^2}+\frac{1}{12}\ln^2\frac{\mu^2}{m_h^2}\right)+\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^3\left[\frac{2951}{2916}\right. \\
& +\frac{1}{9}\zeta_2\ln^2 2-\frac{1}{54}\ln^4 2-\frac{407}{864}\zeta_3+\frac{103}{72}\zeta_4-\frac{4}{9}a_4-\left(\frac{311}{2592}+\frac{5}{6}\zeta_3\right)\ln\frac{\mu^2}{m_h^2} \\
& \left.+\frac{175}{432}\ln^2\frac{\mu^2}{m_h^2}+\frac{29}{216}\ln^3\frac{\mu^2}{m_h^2}+n_l\left(\frac{1327}{11664}-\frac{2}{27}\zeta_3-\frac{53}{432}\ln\frac{\mu^2}{m_h^2}-\frac{1}{108}\ln^3\frac{\mu^2}{m_h^2}\right)\right] \\
& +\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^4\left[\frac{5158677029}{3292047360}+\frac{5635351}{4354560}\ln^4 2-\frac{601}{1620}\ln^5 2+\frac{4}{81}\ln^6 2-\frac{1445}{576}\zeta_2\right. \\
& -\frac{5635351}{725760}\zeta_2\ln^2 2+\frac{601}{162}\zeta_2\ln^3 2-\frac{25}{36}\zeta_2\ln^4 2+\frac{627120469}{40642560}\zeta_3-\frac{289}{81}\zeta_3^2 \\
& +\frac{985}{648}\zeta_3\zeta_2-\frac{125}{864}\zeta_3\zeta_4-\frac{12628999}{580608}\zeta_4+\frac{22015}{432}\zeta_4\ln 2-\frac{65}{3}\zeta_4\ln^2 2 \\
& -\frac{19499}{288}\zeta_5+\frac{179}{864}\zeta_5\zeta_2-\frac{283655}{6912}\zeta_6+\frac{17425}{12096}\zeta_7+\frac{5635351}{181440}a_4+\frac{10}{9}a_4\zeta_2 \\
& +\frac{1202}{27}a_5+\frac{320}{9}a_6+\frac{40}{3}s_6-\frac{5}{4608}J_{6,2}^{\epsilon^3}+\left(-\frac{2810855}{373248}-\frac{31}{216}\ln^4 2\right. \\
& \left.+\frac{31}{36}\zeta_2\ln^2 2-\frac{373261}{27648}\zeta_3+\frac{4123}{288}\zeta_4+\frac{575}{72}\zeta_5-\frac{31}{9}a_4\right)\ln\frac{\mu^2}{m_h^2}+\left(\frac{51163}{10368}\right. \\
& \left.-\frac{155}{48}\zeta_3\right)\ln^2\frac{\mu^2}{m_h^2}+\frac{301}{324}\ln^3\frac{\mu^2}{m_h^2}+\frac{305}{1152}\ln^4\frac{\mu^2}{m_h^2}+n_l\left(-\frac{2261435}{746496}+\frac{49}{2592}\ln^4 2\right. \\
& -\frac{1}{270}\ln^5 2-\frac{49}{432}\zeta_2\ln^2 2+\frac{1}{27}\zeta_2\ln^3 2-\frac{1075}{1728}\zeta_3-\frac{1225}{3456}\zeta_4+\frac{49}{72}\zeta_4\ln 2 \\
& +\frac{497}{288}\zeta_5+\frac{49}{108}a_4+\frac{4}{9}a_5+\left(\frac{16669}{31104}+\frac{1}{108}\ln^4 2-\frac{1}{18}\zeta_2\ln^2 2+\frac{221}{576}\zeta_3\right. \\
& \left.\left.-\frac{163}{144}\zeta_4+\frac{2}{9}a_4\right)\ln\frac{\mu^2}{m_h^2}-\frac{7825}{10368}\ln^2\frac{\mu^2}{m_h^2}-\frac{23}{288}\ln^3\frac{\mu^2}{m_h^2}-\frac{5}{144}\ln^4\frac{\mu^2}{m_h^2}\right) \\
& +n_l^2\left(\frac{17671}{124416}-\frac{5}{864}\zeta_3-\frac{7}{96}\zeta_4+\left(-\frac{3401}{46656}+\frac{7}{108}\zeta_3\right)\ln\frac{\mu^2}{m_h^2}\right. \\
& \left.\left.+\frac{31}{1296}\ln^2\frac{\mu^2}{m_h^2}+\frac{1}{864}\ln^4\frac{\mu^2}{m_h^2}\right)\right]
\end{aligned}
$$

$$\stackrel{\mu=m_h}{=} 1+\left(\frac{\alpha_s^{(n_f)}(m_h)}{\pi}\right)^2 0.2060+\left(\frac{\alpha_s^{(n_f)}(m_h)}{\pi}\right)^3 (1.848+0.02473 n_l)$$
$$+\left(\frac{\alpha_s^{(n_f)}(m_h)}{\pi}\right)^4 \left(6.850-1.466 n_l+0.05616 n_l^2\right)\,, \tag{5}$$

with $\alpha_s^{(n_f)} \equiv \alpha_s^{(n_f)}(\mu)$. In the analytic expression ζ_n denotes the Riemann zeta function and $a_n = \mathrm{Li}_n(1/2)$. Furthermore we have $s_6 \approx 0.987441426403299$ [19] and $J_{6,2}^{\epsilon^3} \approx -30697.269104102521023211937967728039575 7$ [21].

Often it is convenient to use ζ_m for on-shell heavy quark mass since in that case M_h is just a parameter and does not run. The corresponding result is obtained with the help of the two-loop relation between the $\overline{\mathrm{MS}}$ and on-shell quark mass [26–28]. We refrain from showing the corresponding analytical result and restrict the presentation to the numerical expression which is given by

$$\begin{aligned}
\zeta_m^{\mathrm{OS}} \;=\; & 1.+\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^2 \left(0.2060-0.1389\ln\frac{\mu^2}{M_h^2}+0.08333\ln^2\frac{\mu^2}{M_h^2}\right) \\
& +\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^3 \Bigg[1.477-0.9550\ln\frac{\mu^2}{M_h^2}+0.7384\ln^2\frac{\mu^2}{M_h^2}+0.1343\ln^3\frac{\mu^2}{M_h^2} \\
& +n_l\left(0.02473-0.1227\ln\frac{\mu^2}{M_h^2}-0.009259\ln^3\frac{\mu^2}{M_h^2}\right)\Bigg]+\left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^4 \Bigg[0.2233 \\
& +2.674\ln\frac{\mu^2}{M_h^2}+6.227\ln^2\frac{\mu^2}{M_h^2}+2.165\ln^3\frac{\mu^2}{M_h^2}+0.2648\ln^4\frac{\mu^2}{M_h^2} \\
& +n_l\left(-1.504-0.6470\ln\frac{\mu^2}{M_h^2}-0.9260\ln^2\frac{\mu^2}{M_h^2}-0.1632\ln^3\frac{\mu^2}{M_h^2}\right. \\
& \left.-0.03472\ln^4\frac{\mu^2}{M_h^2}\right)+n_l^2\left(0.05616+0.005016\ln\frac{\mu^2}{M_h^2}+0.02392\ln^2\frac{\mu^2}{M_h^2}\right. \\
& \left.+0.001157\ln^4\frac{\mu^2}{M_h^2}\right)\Bigg]\,.
\end{aligned} \tag{6}$$

On the webpage [29] we provide analytic results in computer-readable form for a general $SU(N_c)$ gauge group.

In the remaining part of this section we discuss two applications which involve the evaluation of light quark masses at high scales. In the first one we compute the running bottom quark mass at the scale $\mu = M_t$ which appears as an intermediate step in analyses concerned with Yukawa coupling unification. Here the role of the heavy quark is taken over by the top quark. In the second application we cross the bottom threshold and evaluate the charm quark mass for $\mu = M_Z$ using $m_c^{(4)}(3\text{ GeV})$ as input. As input parameters for the numerical analyses we use [30, 31]

$$\alpha_s^{(5)}(M_Z) \;=\; 0.1185\,,$$

$$
\begin{aligned}
m_b^{(5)}(m_b^{(5)}) &= 4.163 \text{ GeV}\,, \\
m_c^{(4)}(3 \text{ GeV}) &= 0.986 \text{ GeV}\,.
\end{aligned}
\tag{7}
$$

As a first phenomenological application we consider the evaluation of the bottom quark mass at the scale of the top quark with six active flavours using $m_b^{(5)}(m_b^{(5)})$ as input. We are interested in the dependence of $m_b^{(6)}(M_t)$ on the decoupling scale of the top quark. Since this scale is unphysical it should get weaker after including higher order corrections. Our results, which are shown in Fig. 2a, are obtained using the following scheme, where $N \in \{1,2,3,4,5\}$ refers to the number of loops:

- Use N-loop running: $m_b^{(5)}(m_b^{(5)}) \to m_b^{(5)}(\mu_t^{\rm dec})$
- Use $(N-1)$-loop decoupling: $m_b^{(5)}(\mu_t^{\rm dec}) \to m_b^{(6)}(\mu_t^{\rm dec})$
- Use N-loop running $m_b^{(6)}(\mu_t^{\rm dec}) \to m_b^{(6)}(M_t)$

The values for α_s involved in this procedure, $\alpha_s^{(5)}(m_b^{(5)}(m_b^{(5)}))$, $\alpha_s^{(5)}(\mu_t^{\rm dec})$, $\alpha_s^{(6)}(\mu_t^{\rm dec})$, and $\alpha_s^{(6)}(M_t)$, are obtained from $\alpha_s^{(5)}(M_Z)$ using the same loop-order for the running and decoupling as described above for the bottom quark mass.

In Fig. 2a $m_b^{(6)}(M_t)$ is shown as a function of the scale where the transition from five- to six-flavour QCD is performed, $\mu_t^{\rm dec}$, normalized to the on-shell top quark mass. For on-shell top quark mass we choose $M_t = 173.34$ GeV [32]. We vary $\mu_t^{\rm dec}/M_t$ by a factor of 10 around the central scale $\mu_t^{\rm dec}/M_t = 1$. The one-loop result leads to $m_b^{(6)}(M_t) \approx 2.9$ GeV and is not shown in the plot. One observes that already the result where two-loop running is used (short-dashed line) shows only a weak dependence on $\mu_t^{\rm dec}$. It becomes even weaker at three and four loops (results with higher perturbative order have longer dashes) and results in an almost flat curve at five loops (solid line) which can barely be distinguished from the four-loop curve. The five-loop results depends on the unknown five-loop coefficient β_4 of the beta function. Our default choice in Fig. 2a is $\beta_4 = 100\beta_0$ ($\beta_0 = 11/4 - n_f/6$). For $\beta_4 = 0$ and $\beta_4 = 200\beta_0$ one observes a shift of the five-loop result by about $+0.5$ MeV and -0.5 MeV, respectively.

It is interesting to look at the shift on $m_b^{(6)}(M_t)$ at the central scale $\mu_{\rm dec} = M_t$. The two-, three- and four-loop curve leads to shifts of about -201 MeV, -21 MeV and -2 MeV, respectively. For $\beta_4 = 100\beta_0$ the five-loop result leads to a shift of about -0.5 MeV.

In a second application we consider the evaluation of $m_c^{(5)}(M_Z)$ with $m_c^{(4)}(3 \text{ GeV})$ as input. The calculation proceeds in analogy to the bottom quark case discussed before, where for the on-shell bottom quark mass we use the value $M_b = 4.7$ GeV. Our results are shown in Fig. 2b. Again one observes a flattening of the curves after including higher order corrections. However, for $\mu_b^{\rm dec} \approx 1$ GeV, which corresponds to the left border of Fig. 2b, all curves show a strong variation which indicates the breakdown of perturbation theory for small scales. Around $\mu_b^{\rm dec}/M_b \gtrsim 0.3$ both the four- and five-loop curves are basically flat.

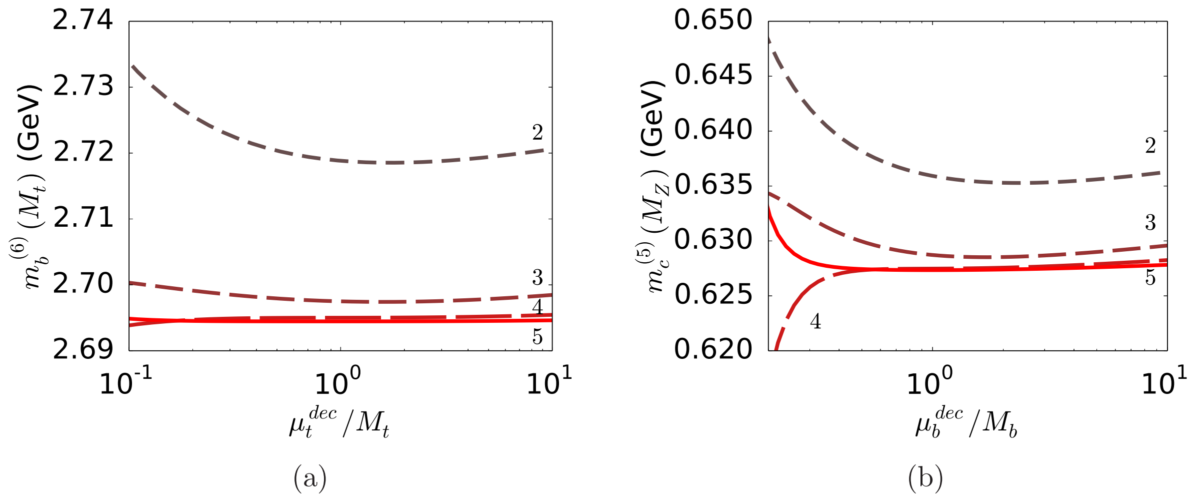

Figure 2: $m_b^{(6)}(M_t)$ as a function of $\mu_t^{\rm dec}$ (a) and $m_c^{(5)}(M_Z)$ as a function of $\mu_b^{\rm dec}$ (b). The numbers indicate the loop order used for the running.

At the central scale $\mu_b^{\rm dec} = M_b$ one observes shifts in $m_c^{(5)}(M_Z)$ of -55 MeV, -7 MeV and -1 MeV after including two-, three- and four-loop running accompanied by one-, two- and three-loop decoupling. The shift at five loops is below 1 MeV for $\beta_4 = 100\beta_0$ but also for $\beta_4 = 0$ and $\beta_4 = 200\beta_0$.

3 Low-energy theorem: Higgs-fermion coupling

The effective Lagrangian describing the coupling of a Higgs boson to gluons and light quarks can be written in the form

$$\mathcal{L}_{\rm eff} = -\frac{H^0}{v^0}\left(C_1\mathcal{O}_1' + C_2\mathcal{O}_2'\right) , \tag{8}$$

where the effective operators, which are constructed from light degrees of freedom [33], are given by

$$\begin{aligned} \mathcal{O}_1' &= \left(G^{a,\mu\nu}\right)^2 , \\ \mathcal{O}_2' &= \sum_{i=1}^{n_l} m_{q_i}^{0\prime}\bar{\psi}_{q_i}^{0\prime}\psi_{q_i}^{0\prime} . \end{aligned} \tag{9}$$

The residual dependence on the mass m_h of the heavy quark h is contained in the coefficient functions C_1^0 and C_2^0. In Eq. (8) H denotes the Higgs field and v the vacuum expectation value. The superscript "0" reminds that the corresponding quantities are bare. For the renormalization of $C_1^0, C_2^0, \mathcal{O}_1'$ and $\mathcal{O}_2'$ we refer to Ref. [11,33]; for the purpose of this paper it is of no further relevance. In Ref. [11] a low-energy theorem has been derived which relates the computation of the renormalized coefficient function C_2 to derivatives of ζ_m w.r.t the heavy mass m_h. It is given by

$$C_2 = 1 + \frac{\partial \ln \zeta_m}{\partial \ln m_h}. \tag{10}$$

It should be stressed that Eq. (10) is valid to all orders in α_s. Note that Eq. (10) contains the derivative w.r.t. $\ln m_h$. Since the m_h dependence of C_2 appears in the form $\ln(\mu/m_h)$ a derivative w.r.t. $\ln m_h$ is equivalent to a derivative w.r.t. $\ln \mu$. At this point we can exploit that the μ dependence of the next, not computed perturbative order can easily be reconstructed using renormalization group techniques. Thus, on the basis of our four-loop calculation for ζ_m we can compute C_2 to five-loop accuracy using the recently computed five-loop result for the quark mass anomalous dimension [8]. Note that the four-loop anomalous dimensions have been computed in Refs. [3, 4] (γ_m) and Refs. [5, 6] (β), respectively.

Inserting $\zeta_m^{\overline{\rm MS}}$ into Eq. (10) we obtain the following result

$$\begin{aligned} C_2^{\overline{\rm MS}} &= 1 + \left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^2 0.2778 + \left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^3 (2.243 + 0.2454\, n_l) \\ &\quad + \left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^4 \left(2.180 + 0.3096\, n_l - 0.01003\, n_l^2\right) \\ &\quad + \left(\frac{\alpha_s^{(n_f)}}{\pi}\right)^5 \left(66.71 + 13.44\, n_l - 3.642\, n_l^2 + 0.07556\, n_l^3\right), \end{aligned} \tag{11}$$

where we have chosen $\mu = m_h$ to obtain more compact expressions. Analytic result valid for general μ are provided from [29].

In practice, one often encounters the situation where C_2 has to be inserted in a formula expressed in terms of $\alpha_s^{(n_l)}$. If we furthermore transform the heavy quark mass to the on-shell scheme we obtain for $\mu = M_h$

$$\begin{aligned} C_2^{\rm OS} &= 1 + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 0.2778 + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3 (1.355 + 0.2454\, n_l) \\ &\quad + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^4 \left(-12.13 + 1.004\, n_l - 0.01003\, n_l^2\right) \\ &\quad + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^5 \left(-140.9 + 44.20\, n_l - 4.332\, n_l^2 + 0.07556\, n_l^3\right). \end{aligned} \tag{12}$$

Let us briefly discuss the influence of C_2 on the Higgs boson decay to bottom quarks where the role of the heavy quark is taken over by the top quark. We consider the contributions proportional to $(C_2)^2$ from Eq. (8) and use the result for the massless correlator from Ref. [34]. For convenience we identify the renormalization scale with the Higgs boson mass and set $\mu = M_H$. Then the decay rate of the Standard Model Higgs boson to bottom quarks can be written in the form

$$\Gamma(H \to b\bar{b}) = \frac{G_F M_H^2}{4\sqrt{2}\pi} m_b^2(M_H)\, R(M_H)\,, \tag{13}$$

$$\begin{aligned} R(M_H) &= 1 + 5.667 \left(\frac{\alpha_s}{\pi}\right) + (29.147 + 0.991) \left(\frac{\alpha_s}{\pi}\right)^2 + (41.758 + 13.105) \left(\frac{\alpha_s}{\pi}\right)^3 \\ &\quad + (-825.7 + 50.7) \left(\frac{\alpha_s}{\pi}\right)^4 + (??? + 224.8) \left(\frac{\alpha_s}{\pi}\right)^5 \qquad (14) \\ &= 1 + 0.20400 + (0.03777 + 0.00128) + (0.00195 + 0.00061) \\ &\quad + (-0.00139 + 0.00009) + (??? + 0.00001)\,, \end{aligned}$$

with $\alpha_s \equiv \alpha_s(M_H) \approx 0.1131$. The first number in the round brackets in Eq. (14) corresponds to the case $C_2 = 1$ [34] and the second one to the contribution from $(C_2 - 1)$. At three-loop order the top quark induced part amounts to about 30%, at order α_s^4 only 6%. Note that the massless correlator at order α_s^5 is currently unknown which is indicated by the three question marks. The α_s^5 term in Eq. (14) origins from the five-loop contribution in Eq. (12) and products of lower-order contributions.

Note that in this consideration the contribution of C_1 (cf. Eq. (8)) has been neglected. The corresponding corrections of order α_s^3 can be found in Ref. [35]. Corrections of order α_s^4 which are proportional to $C_1 C_2$ require the evaluation of massless four-loop two-point functions and are currently unknown. Corrections of order α_s^5 to the Higgs boson decay rate involving $(C_1)^2$ have been computed in Ref. [36].

In Refs. [9, 10] the five-loop result for C_1 is given in terms of $\alpha_s^{(n_f)}$ and the $\overline{\rm MS}$ quark mass. We complement this result by C_1 parametrized in terms of the effective coupling constant and the on-shell mass:

$$\begin{aligned} C_1^{\rm OS} &= -\frac{1}{12}\frac{\alpha_s^{(n_l)}}{\pi}\left\{1 + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right) 2.750 + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 (9.642 - 0.6979 n_l) \right. \\ &\quad + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3 \left(50.54 - 6.801 n_l - 0.2207 n_l^2\right) + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^4 \Big[- 625.2 + 149.8 n_l \\ &\quad \left. - 3.090 n_l^2 - 0.07752 n_l^3 + 6\left(\beta_4^{(n_l)} - \beta_4^{(n_l+1)}\right)\Big]\right\}. \end{aligned} \tag{15}$$

where $\mu = M_t$ has been chosen. The analytic version in computer-readable for can again be found in [29].

4 Summary and conclusions

In this paper we compute the four-loop corrections to the decoupling constant for light quark masses, ζ_m, which has to be applied every time heavy quark thresholds are crossed. It constitutes a fundamental constant of QCD and accompanies the five-loop quark anomalous dimension [8] in the "running and decoupling" procedure. Our results completes the calculation of the four-loop decoupling constants which has been started in Refs. [9, 10], however, the five-loop corrections to the QCD beta function is still lacking for establishing relations between $\alpha_s(\mu)$ and $m_q(\mu)$ at low and high energy scales.

As a by-product of our calculation we obtain the effective coupling of a scalar Higgs boson and light quarks to five-loop order. It is obtained from ζ_m with the help of an all-order low-energy theorem. We briefly investigate the influence on $\Gamma(H \to b\bar{b})$.

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