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Massless Propagators, $R(s)$ and Multiloop QCD

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Abstract

This is a short review of recent developments in calculation of higher order corrections to various two-point corelators and related quantities in (massless) QCD.

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1. Introduction

Precise determinations of parameters of the Standard Model (SM) and precise predictions for observables measured at present and future experiments are critical in testing the SM and may hint towards physics beyond the SM. Increasingly precise results from high energy experiments at LEP, LHC or a future electron-positron collider have been obtained during the past years or are expected for the coming decade, with production and decay rates or masses of gauge or Higgs bosons or of the top quark as characteristic examples. These are complemented by measurements at low energies, which lead to precise values of the strong coupling from τ lepton decay or the masses of strange, charm and bottom quarks.

To extract the fundamental parameters of the theory and relate the large number of experimental results, the knowledge of higher order perturbative corrections is crucial. Significant advances in this direction have been made during the past years, in particular in the framework of the collaborative research center "Computational Particle Physics" (SFB/TR-9).

Within perturbation theory quantum-theoretical amplitudes are described by Feynman Integrals (FI's). Improved precision, which is required both for strong and electroweak interactions, necessarily leads to a significant increase of the complexity of the calculations. This applies to the number of FI's, their increasing complexity and, consequently, to the effort required for their evaluation.

The complexity of a FI can be roughly measured by the sum of two quantities: (i) the number of (independent) external momenta and (ii) the number of so-called "loops", that is the number of integrations with respect to internal momenta which should be performed. In addition, the pattern of masses of (virtual) particles appearing inside of the FI also presents an important feature characterizing the complexity of the integral.

For instance, one-loop diagrams can be calculated in analytic form for arbitrary masses and external momenta, diagrams with three or more loops, however, only for configurations involving just one mass or energy scale (one-scale diagrams). Problems involving several scales, in particular those with pronounced scale hierarchy, can be treated approximately using asymptotic expansions (Hard Mass Expansion and/or Large Momentum Expansion, for example), where each of the diagrams is expressed through a nested sum of one-scale diagrams [1].

The present review will deal with a special class of one-scale FI's, namely massless *propagators*, that is integrals depending on only one external momentum, q, and with vanishing internal masses. In what follows we will customarily refer to massless propagator-type FI's as *p-integrals*.

2. Massless Propagators and Physics

2.1. RG functions and IR reduction

The method of the renormalization group (RG) [2–4] is of vital importance in modern quantum field theory. It is enough to recall that the famous idea of asymptotic freedom is based on the RG concept of the running coupling constant.

The RG functions — β-functions and various anomalous dimensions — serve as coefficients in the RG equations. They can be conveniently expressed in terms of p-integrals (see below) within the framework of Dimensional Regularization [5–7] and Minimal Subtraction (MS) schemes [8]. The naturalness and convenience of the MS-scheme for RG calculations comes from the following statement [9]:

Theorem 1. *Any UV counterterm for any FI integral and, consequently, any RG function in an arbitrary minimally renormalized model is a polynomial in momenta and masses.*

This observation was elaborated by A. Vladimirov [10] to simplify considerably the calculation of the RG functions. The method was further developed and named Infrared Rearrangement (IRR) in [11]. It essentially amounts to an appropriate transformation of the IR structure of FI's by setting zero some external momenta and masses (in some cases after the differentiation is performed with respect to the latter). As a result the calculation of UV counterterms is reduced to that of p-integrals. The method of IRR was ultimately refined and freed from unessential complications by inventing the so-called R^*-operation [12]. The main use of the R^* -operation is in the proof of the following statement [12]:

Theorem 2. *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Theorem 2 is a key tool for multiloop RG calculations as it reduces the general task of evaluation of (L+1)-loop UV counterterms to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals. In the following we shall refer to the latter as the L-loop Problem. A short account of the current status of the Problem can be found in Section 3.

2.2. Two-point correlators

Within perturbation theory, every two-point correlator

$$\Pi^{j_1 j_2} = \int \mathrm{d}x\, e^{iqx} \langle 0| T\left[j_2(x)\, j_1(0)\right] |0\rangle, \tag{1}$$

with j_1 and j_2 being in general elementary fields or (local) composite operators, is expressed within PT in terms of p-integrals provided the momentum transfer q is considered as large with respect to all relevant masses and, thus, the elementary field propagators contributing to $\Pi^{j_1 j_2}$ be effectively considered as massless.

An important class of two-point correlators is represented by the case of j_1 and j_2 being quark currents of the form:

$$j_1 = \overline{\psi}\Gamma\psi, \qquad j_2 = j_1^\dagger.$$

In particular, the total cross-section of e^+e^- annihilation into hadrons, the (inclusive) Higgs decay rate into hadrons, the semihadronic decay rate of the τ lepton coupling are all expressible in terms of absorptive parts of the quark current correlator (1) with Γ chosen as γ_μ, 1 and $(1-\gamma_5)\gamma_\mu$ respectively.

Clearly, one could compute a (L+1)-loop two-point correlator by computing the corresponding set of (L+1)-loop p-integrals. But one can do much better if the final aim is the absorptive part of the correlator.

Indeed, let Γ be a particular (L+1)-loop Feynman diagram contributing to the perturbative expansion of a massless correlator. The renormalized version of the corresponding Feynman integral can be generically written as[1]

$$R\,\langle\Gamma\rangle(Q^2) = \langle\Gamma\rangle(Q^2) + \boxed{\textstyle\sum_\gamma Z_\gamma \langle\Gamma/\gamma\rangle(Q^2) + \ldots} \tag{2}$$

Here Z_γ is the UV Z-factor corresponding to a 1PI subgraph γ of Γ and dots stand for contributions with two and more UV subtractions. The finiteness of the left part of eq. (2) means that the pole part in $\epsilon = (4-D)/2$ of $\langle\Gamma\rangle(Q^2)$ is completely fixed by poles in ϵ which appear in UV subtractions (the boxed terms in (2)). On the other hand, the UV subtractions could, obviously, contain L'-loop Z-factors with $L' \le L+1$ and the reduced p-integrals like $\langle\Gamma/\gamma\rangle(Q^2)$ with the loop number *not exceeding* L! Applying Theorem 2 we arrive at the conclusion that the pole part of $\langle\Gamma\rangle(Q^2)$ (and, consequently,

[1] Without essential loss of generality we assume that $\langle\Gamma\rangle(a_s, Q^2)$ is a scalar integral depending on the external momentum Q via its square, $Q^2 = Q_\nu Q^\nu$. In addition, we set the renormalization scale parameter $\mu = 1$.

its absorptive part) is completely expressed via L-loop p-integrals *only*.

We want also to stress that by high-energy limit we understand not only the case when all masses can be neglected but also the possibility to take into account mass effects by exploiting a small mass expansion. As a suitable example one could mention the calculation of the power suppressed (of order m_q^2/s, m_q^4/s^2 and so on) corrections for the correlators of (axial)vector quark currents in higher orders of pQCD [13–17].

2.3. OPE and DIS

The theoretically cleanest description of (inclusive) Deep Inelastic Scattering (DIS) can be achieved within Operator Product Expansion (OPE) of two composite operators (for a recent review see, e.g [18]). Here the main objects to compute are the so-called Coefficient Functions (CF) which can be always computed via p-integrals with the help of the so-called method of projectors [19, 20]. It is important to stress that within the method of projectors one needs no IRR: L-loop corrections to a CF can be expressed *directly* in terms of L-loop p-integrals.

A good example of an early multiloop OPE calculation is the one of the ${\alpha_s}^3$ corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule [21]. We will discuss later our calculations of the next, order ${\alpha_s}^4$, contributions to the Bjorken sum rule.

3. Calculational Methods

A significant number of higher order calculations are usually performed according to the following "standard" scenario. First, the Feynman amplitudes are reduced to a limited set of so-called *master integrals* (MI's). This step is based on recursion algorithms obtained by using Integration-by-Parts (IBP) identities (see, e.g. recent books and reviews [22–25] and references therein).

An important feature of the standard scenario is that the resulting set of master integrals should be computed only once and forever due to the well-established property of universality: for every given class of Feynman amplitudes characterized by the number of loops and the pattern of external momenta and masses the corresponding set of master integrals is universal in the following sense: every (even extremely complicated) amplitude from the class can be expressed in terms of one and the same (finite! [26, 27]) set of master integrals .

Thus, the task of evaluation of p-integrals at L-loops (L-loop Problem) is naturally decomposed in two: (A) reduction of a generic L-loop p-integral to masters and (B) evaluation of the latter.

Both A and B Problems were solved at two- and three-loop level long ago [11, 28]. The four-loop problem has been under active investigation in our group since the beginning of the current century. We will describe the current status in the next two Sections.

3.1. A: Reduction via $1/D$ expansion

The standard (Laporta) [29, 30] approach to solve IBP relations implies a step-by-step linear reduction of more complicated integrals to less complicated and finally to irreducible (master) integrals. Unfortunately this conceptually simple method could not be used in our case because of the extremely large amount (10^7 – 10^8) of 4-loop integrals appearing after IRR for a typical 5-loop problem. As the result we used a more sophisticated and laborious, but less demanding for computer resources method based on large D expansion [31] of the formal solutions of the IBP relations [32].

So assume that we need to perform the reduction, that is to calculate the coefficients in front of master integrals. The coefficients depend on indices of the original integral and fulfill the IBP identifies according to these indices. If we construct some "convenient" solutions of the identities, then the coefficients we need can be obtained as their linear combinations with proper boundary conditions.

As shown in [32] such "convenient" solutions can be constructed in the form of the integrals of polynomial raised to some degree (linear in dimension D). Unfortunately, although these integrals are simpler than the original FI, they are still too complicated for direct evaluation. From the other side, we are interested in linear combinations of these integrals which are rational over D (because in principle they can be calculated by the step-by-step reduction). So one can expand these integrals in the $1/D$ limit (resulting in integrals of Gaussian type [31]), calculate sufficiently many terms and finally reconstruct the exact D dependence.

Note, that we need not calculate each of the $10^7 - 10^8$ integrals involved in the specific problem individually (as it is necessary in the standard reduction). Instead we can calculate $1/D$ expansion coefficients of the total expression we are interested in, thus saving a lot of computer resources.

The construction of the large-D limit requires in general huge storage resources, which naturally constrains the structure of the input p-integrals: they should better not contain any extra parameters like color coefficients, n_f, the number of light quark flavours contributing to

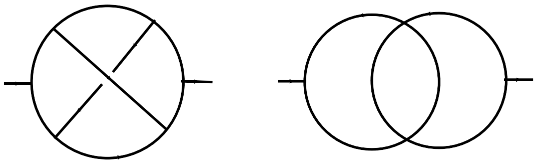

Figure 1: Non-trivial three-loop master p-integrals.

internal fermion loops, and so on. As a result we are forced to use a "slice" approach: that is to set all color coefficients to their numerical values and to fix n_f to some integer. Combining together different slices one can always reconstruct the full n_f-dependent structure (and, if necessary, even all colour coefficients)

3.2. B: Master p-integrals and their evaluation

At one- at two-loop levels the master p-integrals are trivial as they can be easily performed in terms of Γ-functions for a generic value of the running space-time dimension D (see, e.g. [33]. At three loops there exist only two non-trivial master p-integrals (see Fig. 1); their values are known since long [11].

The significantly more complicated problem of identifying and computing *all* 28 four-loop master p-integrals has been solved only recently. We refer the interested reader to the original publications [31, 34–36].

3.3. Computer Algebra & FORM

Higher order calculations dealing with thousands of diagrams already at 3-loop level require heavy use of computer algebra tools. We are using QGRAF [37] for automatic diagram generation as well as a collection of Mathematica and PERL scripts to automatically assign topologies and prepare input files for FORM.

The workhorse for all the complicated calculations discussed in the current paper is the computer algebra program `FORM` [38] and its parallel versions `ParFORM` [39] and `TFORM` [40]. The program offers excellent possibilities for dealing with gigantic data streams generated during the reduction procedure. The internal specifications allow FORM to deal with expression which are much larger than the available memory (RAM). The only restriction for the size of an expression is the disk space which nowadays is rather cheap. As a consequence, the complexity of a problem solvable by FORM is practically restricted only by time.

The FORM program, BAICER, intended for the reduction of complicated four-loop p-integrals implements the algorithms described in the previous Section. With increasing experience and using heuristic criteria about the pole structure of the coefficient functions in front of master integrals, BAICER has developed into an efficient tool which allows to calculate complicated four-loop massless propagator integrals, including their finite part. It runs routinely on ParFORM and TFORM using 8 to 16 cores with a speed-up between 6 and 12.

4. Scalar Correlator & $\Gamma(H \to \overline{q}q)$

The decay width of the Higgs boson into a pair of quarks can be written in the form

$$\Gamma(H \to \bar{f}f) = \frac{G_F\, M_H}{4\,\sqrt{2}\pi}\, m_f^2(\mu)\, R^S(s = M_H^2, \mu) \qquad (3)$$

where μ is the normalization scale and

$$R^S(s) = \mathrm{Im}\, \Pi^{SS}(-s - i\epsilon)/(2\pi\, s) \qquad (4)$$

is the spectral density of the scalar correlator

$$\Pi^{SS}(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\; T[\; J_f^{\mathrm{S}}(x) J_f^{\mathrm{S}}(0)\,]\; |0\rangle. \quad (5)$$

Here $Q^2 = -q^2$ and $J_f^{\mathrm{S}} = \overline{\Psi}_f \Psi_f$ is the scalar current for quarks with flavour f and mass m_f, coupled to the scalar Higgs boson. The $O(\alpha_s^4)$ result for R^S is known analytically since long [41, 42] (early results for orders α_s^2 and α_s^3 can be found in [43] and [44] respectively). For brevity we put below the final result in numerical form:

$$\begin{aligned} \widetilde{R}(s, \mu^2 = s) &= 1 + 5.6667 a_s + \left[35.94 - 1.359\, n_f\right] a_s^2 \\ &+ a_s^3 \left[164.14 - 25.77\, n_f + 0.259\, n_f^2\right] \qquad (6) \\ &+ a_s^4 \left[39.34 - 220.9\, n_f + 9.685\, n_f^2 - 0.0205\, n_f^3\right]. \end{aligned}$$

We will discuss the application of (6) for the dominant b-quark decay mode of the Higgs boson later in Section 7.

5. Vector Correlator & $R(s)$

The ratio

$$R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$$

is expressed through the absorptive part of the vector correlator

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i \int \mathrm{d}x e^{iqx} \langle 0|T[\; j_\mu^{\mathrm{em}}(x) j_\nu^{\mathrm{em}}(0)\;]|0\rangle = \\ &= (-g_{\mu\nu} q^2 + q_\mu q_\nu)\Pi(-q^2)\,, \qquad (7) \end{aligned}$$

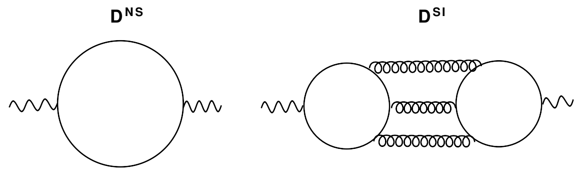

Figure 2: Lowest order non-singlet (a) and singlet (b) diagrams contributing to the polarization operator.

with the hadronic EM current $j^{\rm em}_\mu = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f$, and Q_f being the EM charge of the quark f. The optical theorem relates the inclusive cross-section and thus the function $R(s)$ to the discontinuity of Π in the complex plane

$$R(s) = 12\,\pi\,\mathrm{Im}\,\Pi(-s - i\delta)\,. \tag{8}$$

For the vector correlator the terms of order a_s^2 and a_s^3 are known since long [45, 46]. The next, a_s^4 order has been under investigation for more then 10 years [16, 17, 47–57] in A1 group. By now it is known in a complete form for a generic colour group G [55, 56]. We put below only physically relevant result for $G = SU(3)$:

$$\begin{aligned}
R(s) = {}& 3 \sum_f Q_f^2 \Big\{ 1 + a_s + \qquad\qquad (9)\\
&+ a_s^2 \Big(\tfrac{365}{24} - 11\,\zeta_3 - \tfrac{11}{12}\,n_f + \tfrac{2}{3}\,\zeta_3\,n_f \Big)\\
&+ a_s^3 \Big[n_f^2 \Big(\tfrac{151}{162} - \tfrac{1}{108}\pi^2 - \tfrac{19}{27}\,\zeta_3 \Big)\\
&\quad + n_f \Big(-\tfrac{7847}{216} + \tfrac{11}{36}\,\pi^2 + \tfrac{262}{9}\,\zeta_3 - \tfrac{25}{9}\,\zeta_5 \Big)\\
&\quad + \tfrac{87029}{288} - \tfrac{121}{48}\,\pi^2 - \tfrac{1103}{4}\,\zeta_3 + \tfrac{275}{6}\,\zeta_5 \Big]\\
&+ a_s^4 \Big[n_f^3 \Big(-\tfrac{6131}{5832} + \tfrac{11}{432}\,\pi^2 + \tfrac{203}{324}\,\zeta_3 - \tfrac{1}{54}\,\pi^2\,\zeta_3 + \tfrac{5}{18}\,\zeta_5 \Big)\\
&\quad + n_f^2 \Big(\tfrac{1045381}{15552} - \tfrac{593}{432}\,\pi^2 - \tfrac{40655}{864}\,\zeta_3\\
&\qquad + \tfrac{11}{12}\,\pi^2\,\zeta_3 + \tfrac{5}{6}\,\zeta_3^2 - \tfrac{260}{27}\,\zeta_5 \Big)\\
&\quad + n_f \Big(-\tfrac{13044007}{10368} + \tfrac{2263}{96}\,\pi^2 + \tfrac{12205}{12}\,\zeta_3 - \tfrac{121}{8}\,\pi^2\,\zeta_3\\
&\qquad - 55\,\zeta_3^2 + \tfrac{29675}{432}\,\zeta_5 + \tfrac{665}{72}\,\zeta_7 \Big)\\
&\quad + \tfrac{144939499}{20736} - \tfrac{49775}{384}\,\pi^2 - \tfrac{5693495}{864}\,\zeta_3 + \tfrac{1331}{16}\,\pi^2\,\zeta_3\\
&\qquad + \tfrac{5445}{8}\,\zeta_3^2 + \tfrac{65945}{288}\,\zeta_5 - \tfrac{7315}{48}\,\zeta_7 \Big] \Big\}\\
&+ \Big(\sum_f Q_f \Big)^2 \Big\{ a_s^3 \Big(\tfrac{55}{72} - \tfrac{5}{3}\,\zeta_3 \Big)\\
&\quad + a_s^4 \Big[n_f \Big(-\tfrac{745}{432} + \tfrac{65}{24}\,\zeta_3 + \tfrac{5}{6}\,\zeta_3^2 - \tfrac{25}{12}\,\zeta_5 \Big)\\
&\quad + \Big(\tfrac{5795}{192} - \tfrac{8245}{144}\,\zeta_3 - \tfrac{55}{4}\,\zeta_3^2 + \tfrac{2825}{72}\,\zeta_5 \Big) \Big] \Big\}\,,
\end{aligned}$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = s$; the results for generic values of μ can be easily recovered with standard RG techniques. Note that the two specific quark charge structures in (9) correspond to the so-called non-singlet (numerically dominant) and the singlet contributions (see Fig. 2) to the vector correlator (7). Numerically,

$$\begin{aligned}
R(s) = {}& 3 \sum_f Q_f^2 \Big\{ 1 + a_s + a_s^2 \big(1.986 - 0.1153\, n_f \big)\\
&+ a_s^3 \big(-6.637 - 1.200\, n_f - 0.00518\, n_f^2 \big) \qquad (10)\\
&+ a_s^4 \big(-156.608 + 18.7748\, n_f - 0.797434\, n_f^2\\
&\qquad + 0.0215161\, n_f^3 \big) \Big\}\\
&- \Big(\sum_f Q_f \Big)^2 \Big(1.2395\, a_s^3 + \big(17.8277 - 0.57489\, n_f \big)\, a_s^4 \Big)\,.
\end{aligned}$$

Specifically, for the particular values of $n_f = 3, 4$ and 5 one obtains (for the terms of order α_s^3 and α_s^4 we have explicitly decomposed the coefficient into non-singlet and singlet contributions):

$$R^{n_f=3}(s) = 2 \Big[1 + a_s + 1.6398 a_s^2 - 10.2839 a_s^3 - 106.8798 a_s^4 \Big] \tag{11}$$

$$\begin{aligned}
R^{n_f=4}(s) = {}& \frac{10}{3} \Big[1 + a_s + 1.5245 a_s^2\\
&+ a_s^3 \big(-11.686 = -11.52 - 0.16527^{\rm SI} \big)\\
&+ a_s^4 \big(-94.961 = -92.891 - 2.0703^{\rm SI} \big) \Big]\,,
\end{aligned} \tag{12}$$

$$\begin{aligned}
R^{n_f=5}(s) = {}& \frac{11}{3} \Big[1 + a_s + 1.40902 a_s^2\\
&+ a_s^3 \big(-12.80 = -12.767 - 0.037562^{\rm SI} \big)\\
&+ a_s^4 \big(-80.434 = -79.981 - 0.4531^{\rm SI} \big) \Big].
\end{aligned} \tag{13}$$

Note that for $n_f = 3$ the singlet contributions vanish in every order in α_s as the corresponding global coefficient $(\sum_f Q_f)^2$ happens to be zero. Implications of this result for the determination of α_s in electron-positron annihilation and in Z-boson decays are discussed in [53, 56].

As a by-product of the calculation of $R(s)$ the authors of [54] have obtained the five-loop β-function in pure QED, that is a theory with n_f single-charged fermions minimally coupled to the photon field. The result reads (the four-loop result is known since long from [58])

$$\beta^{QED}(A) = n_f\left[\frac{4\,A^2}{3}\right] + 4\,n_f A^3 - A^4\left[2\,n_f + \frac{44}{9}\,n_f^2\right]$$
$$+\ A^5\left[-46\,n_f + \frac{760}{27}\,n_f^2 - \frac{832}{9}\,\zeta_3\,n_f^2 - \frac{1232}{243}\,n_f^3\right]$$
$$+\ A^6\Bigg(n_f^3\left[-\frac{21758}{81} + \frac{16000}{27}\zeta_3 - \frac{416}{3}\zeta_4 - \frac{1280}{3}\zeta_5\right]$$
$$+n_f^2\left[-\frac{7462}{9} - 992\zeta_3 + 2720\zeta_5\right]$$
$$+n_f\left[\frac{4157}{6} + 128\zeta_3\right] + n_f^4\left[\frac{856}{243} + \frac{128}{27}\zeta_3\right]\Bigg). \quad (14)$$

Here the QED coupling constant

$$A(\mu) = \alpha(\mu)/(4\,\pi) = e(\mu)^2/(16\,\pi^2).$$

6. QCD RG-functions

Our starting point is the QCD Lagrangian with n_f quark flavours written in terms of renormalized fields, coupling constant g and quark masses m_f:

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{4}Z_3\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\\ &-\frac{1}{2}g\,Z_1^{3g}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\,(A_\mu \times A_\nu)^a\\ &-\frac{1}{4}g^2\,Z_1^{4g}\,(A_\mu \times A_\nu)^2 - \frac{1}{2\xi_L}(\partial_\nu A_\mu)^2\\ &+Z_3^c\,\partial_\nu \bar{c}\,(\partial_\nu c) + g\,Z_1^{ccg}\,\partial^\mu \bar{c}\,(A \times c)\\ &+\sum_{f=1}^{n_f}\bar{\psi}^f(\mathrm{i}Z_2\,\partial\!\!\!/ + g Z_1^{\psi\psi g} A\!\!\!/ - Z_{\psi\psi}\,m_f)\,\psi^f,\end{aligned} \quad (15)$$

with

$$(A \times B) = f^{abc}A^b B^c,\ \partial\!\!\!/ = \gamma^\mu\frac{\partial}{\partial x_\mu}$$

and with bare gluon, quark and ghost fields related to the renormalized ones as follows:

$$A_0^{a\mu} = \sqrt{Z_3}\,A^{a\mu},\ \ \psi_0^f = \sqrt{Z_2}\,\psi_0^f,\ \ c_0^a = \sqrt{Z_3^c}\,c^a. \quad (16)$$

The vertex Renormalization Constants (RCs)

$$Z_1^V,\quad V \in \{3\mathrm{g},\ 4\mathrm{g},\ \mathrm{ccg},\ \psi\psi\mathrm{g}\} \quad (17)$$

renormalize 3-gluon, 4-gluon, ghost-ghost-gluon, quark-quark-gluon vertex functions respectively. The Slavnov-Taylor identities allow one to express all vertex RCs in terms of wave function RCs and an independent charge RC, $Z_g = \frac{g_0}{g}$:

$$\begin{aligned} Z_\xi &= Z_3, & (18)\\ Z_g &= \sqrt{Z_1^{4g}}\,(Z_3)^{-1}, & (19)\\ Z_g &= Z_1^{3g}(Z_3)^{-3/2}, & (20)\\ Z_g &= Z_1^{ccg}(Z_3)^{-1/2}(Z_3^c)^{-1}, & (21)\\ Z_g &= Z_1^{\psi\psi g}(Z_3)^{-1/2}(Z_2)^{-1}. & (22)\end{aligned}$$

Within the commonly accepted $\overline{\mathrm{MS}}$ scheme RCs are independent of dimensional parameters (masses and momenta) and can be represented as follows

$$Z(h) = 1 + \sum_{i,j}^{1\le j\le i} Z_{ij}\frac{h^i}{\epsilon^j}, \quad (23)$$

where $h = g^2/(16\pi^2) = \alpha_s/(4\pi)$ and the parameter ϵ is related to the continuous space time dimension D via $D = 4-2\epsilon$. Given a RC $Z(h)$, the corresponding anomalous dimension is defined as

$$\gamma(h) = -\mu^2\frac{\mathrm{d}\log Z(h)}{\mathrm{d}\mu^2} = \sum_{n=1}^{\infty} Z_{n,1}\,n\,h^n = -\sum_{n=0}^{\infty}(\gamma)_n\,h^{n+1}. \quad (24)$$

The anomalous dimension of the quark-gluon coupling constant h is conventionally referred to as "QCD β-function"; equation (21) leads to the calculationally simplest presentation of the function:

$$\beta(h) = 2\gamma_1^{ccg} - 2\,\gamma_3^c - \gamma_3. \quad (25)$$

The quark mass anomalous dimension, γ_m, governs the evolution of the quark mass, viz.

$$\mu^2\frac{d}{d\mu^2}m|_{h^0,h^0} = m\gamma_m(h) \equiv -m\sum_{i\ge 0}\gamma_i\,h^{i+1}. \quad (26)$$

To calculate γ_m one needs to find the so-called quark mass renormalization constant, Z_m, which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m_f^0}{m_f} = \frac{Z_{\psi\psi}}{Z_2}. \quad (27)$$

The final formula for γ_m follows from the QCD Lagrangian (15) and reads

$$\gamma_m = \gamma_{\psi\psi} - \gamma_2. \quad (28)$$

Thus, to compute the QCD β-function and the quark mass anomalous dimension at five loops [2] one should evaluate five separate anomalous dimensions, viz.

$$\gamma_1^{ccg},\ \gamma_3^c,\ \gamma_3,\ g_{\psi\psi}, \gamma_2.$$

By now we have computed all of them except (most difficult) γ_3. The results are presented in the four next subsections[3].

6.1. Five-loop running of the ghost field

$$\gamma_3^c = -\sum_{i=0}^{\infty}\left(\gamma_3^c\right)_i h^{i+1}, \tag{29}$$

$$(\gamma_3^c)_0 = -\frac{3}{2}, \tag{30}$$

$$(\gamma_3^c)_1 = -\frac{147}{8} + \frac{5}{4}\, n_f, \tag{31}$$

$$(\gamma_3^c)_2 = -229 - \frac{81}{4}\zeta_3 + n_f\left(\frac{1085}{48} + \frac{33}{2}\zeta_3\right) + \frac{35}{36}\, n_f^2, \tag{32}$$

$$\begin{aligned}(\gamma_3^c)_3 = &-\frac{1016843}{192} - \frac{129825}{32}\zeta_3 + \frac{9963}{32}\zeta_4 + \frac{78705}{16}\zeta_5 \\ &+ n_f\left(\frac{198229}{192} + \frac{48461}{48}\zeta_3 - \frac{4797}{16}\zeta_4 - \frac{3355}{4}\zeta_5\right) \\ &+ n_f^2\left(-\frac{3385}{144} - \frac{49}{2}\zeta_3 + \frac{33}{2}\zeta_4\right) + n_f^3\left(\frac{83}{108} - \frac{4}{3}\zeta_3\right),\end{aligned} \tag{33}$$

$$\begin{aligned}(\gamma_3^c)_4 = &-\frac{193301287}{2048} - \frac{19562145}{128}\zeta_3 - \frac{2060829}{128}\zeta_3^2 + \frac{1101573}{16}\zeta_4 + \frac{66632427}{128}\zeta_5 \\ &- \frac{36327825}{256}\zeta_6 - \frac{140900823}{512}\zeta_7 \\ &+ n_f\left(\frac{633704171}{27648} + \frac{5166473}{144}\zeta_3 + \frac{233519}{64}\zeta_3^2 - \frac{764949}{32}\zeta_4 - \frac{32902291}{384}\zeta_5 + \frac{4123825}{128}\zeta_6 + \frac{14425075}{384}\zeta_7\right) \\ &+ n_f^2\left(-\frac{1326547}{3456} - \frac{1739167}{864}\zeta_3 - \frac{2659}{6}\zeta_3^2 + \frac{13485}{8}\zeta_4 + \frac{8074}{9}\zeta_5 - \frac{16775}{12}\zeta_6\right) \\ &+ n_f^3\left(-\frac{342895}{7776} - \frac{1211}{18}\zeta_3 - \frac{5}{2}\zeta_4 + \frac{284}{3}\zeta_5\right) \\ &+ n_f^4\left(\frac{65}{108} + \frac{20}{27}\zeta_3 - \frac{4}{3}\zeta_4\right),\end{aligned} \tag{34}$$

6.2. Five-loop running of the ghost-ghost-gluon vertex

$$\gamma_1^{ccg} = -\sum_{i=0}^{\infty}\left(\gamma_1^{ccg}\right)_i h^{i+1}, \tag{35}$$

$$(\gamma_1^{ccg})_0 = \frac{3}{2}, \tag{36}$$

$$(\gamma_1^{ccg})_1 = \frac{27}{4}, \tag{37}$$

$$(\gamma_1^{ccg})_2 = \frac{3375}{32} + -\frac{135}{16}\, n_f, \tag{38}$$

$$\begin{aligned}(\gamma_1^{ccg})_3 = &\frac{46945}{24} + \frac{6561}{8}\zeta_3 + \frac{243}{8}\zeta_4 - \frac{13095}{16}\zeta_5 \\ &+ n_f\left(-\frac{14675}{72} - \frac{177}{2}\zeta_3 - \frac{99}{4}\zeta_4\right) \\ &+ n_f^2\left(-\frac{251}{54} + 6\zeta_3\right),\end{aligned} \tag{39}$$

$$\begin{aligned}(\gamma_1^{ccg})_4 = &\frac{112928171}{2048} + \frac{11577699}{256}\zeta_3 + \frac{815103}{128}\zeta_3^2 - \frac{1539243}{256}\zeta_4 - \frac{23404221}{256}\zeta_5 \\ &+ \frac{2241675}{256}\zeta_6 + \frac{22895649}{1024}\zeta_7 \\ &+ n_f\left(-\frac{10723195}{1024} - \frac{1042157}{128}\zeta_3 - \frac{14361}{64}\zeta_3^2 + \frac{62571}{128}\zeta_4 + \frac{1137861}{128}\zeta_5 + \frac{77775}{128}\zeta_6 - \frac{59535}{64}\zeta_7\right) \\ &+ n_f^2\left(\frac{572723}{2304} + \frac{8105}{16}\zeta_3 - \frac{3789}{32}\zeta_4 - \frac{2109}{8}\zeta_5\right) \\ &+ n_f^3\left(-\frac{2989}{864} - \frac{5}{3}\zeta_3 + 6\zeta_4\right).\end{aligned} \tag{40}$$

[2] Up to and including four loop level they are known since long [59–72].

[3] Note that in all calculations we have used the simplest — Feynman — gauge fixing condition. The physically relevant γ_m and β functions do not depend on gauge.

Note that the leading renormalon contribution $\approx n_f^i\, a_s^{i+1}$ vanishes *(in any gauge!)* due to the Taylor theorem which states, in particular, that $\gamma_1^{ccg} \equiv 0$ in the Landau gauge.

6.3. Five-loop running of the quark field

$$\gamma_2 = -\sum_{i=0}^{\infty} \left(\gamma_2\right)_i h^{i+1}, \tag{41}$$

$$(\gamma_2)_0 = \frac{4}{3}, \tag{42}$$

$$(\gamma_2)_1 = \frac{94}{3} - \frac{4}{3} n_f, \tag{43}$$

$$(\gamma_2)_2 = \frac{24941}{36} - 26\zeta_3 - \frac{1253}{18} n_f + \frac{20}{27} n_f^2, \tag{44}$$

$$\begin{aligned}(\gamma_2)_3 = {} & \frac{19684159}{1296} - \frac{67469}{162}\zeta_3 + 501\zeta_4 - \frac{129380}{81}\zeta_5 \\ & + n_f\left(- \frac{53713}{24} - \frac{5306}{27}\zeta_3 - 54\zeta_4 - \frac{160}{3}\zeta_5\right) \\ & + n_f^2\left(\frac{10483}{243} + \frac{208}{9}\zeta_3\right) + \frac{140}{243} n_f^3,\end{aligned} \tag{45}$$

$$\begin{aligned}(\gamma_2)_4 = {} & \frac{2798900231}{7776} + \frac{17969627}{864}\zeta_3 + \frac{13214911}{648}\zeta_3^2 + \frac{16730765}{864}\zeta_4 - \frac{832567417}{3888}\zeta_5 \\ & + \frac{40109575}{1296}\zeta_6 + \frac{124597529}{1728}\zeta_7 \\ & + n_f\left(- \frac{861347053}{11664} - \frac{274621439}{11664}\zeta_3 + \frac{1960337}{972}\zeta_3^2 + \frac{465395}{1296}\zeta_4 + \frac{22169149}{5832}\zeta_5 \right. \\ & \left. + \frac{1278475}{1944}\zeta_6 + \frac{3443909}{216}\zeta_7\right) \\ & + n_f^2\left(\frac{37300355}{11664} + \frac{1349831}{486}\zeta_3 - \frac{128}{9}\zeta_3^2 - \frac{27415}{54}\zeta_4 - \frac{12079}{27}\zeta_5 - \frac{800}{9}\zeta_6 - \frac{1323}{2}\zeta_7\right) \\ & + n_f^3\left(- \frac{114049}{8748} - \frac{1396}{81}\zeta_3 + \frac{208}{9}\zeta_4\right) \\ & + n_f^4\left(\frac{332}{729} - \frac{64}{81}\zeta_3\right).\end{aligned} \tag{46}$$

6.4. Five-loop running of the quark mass

$$\gamma_m = -\sum_{i=0}^{\infty} \left(\gamma_m\right)_i h^{i+1}, \tag{47}$$

$$(\gamma_m)_0 = 4, \tag{48}$$

$$(\gamma_m)_1 = \frac{202}{3} - \frac{20}{9} n_f \tag{49}$$

$$(\gamma_m)_3 = 1249 - n_f\left(\frac{2216}{27} + \frac{160}{3}\zeta_3\right) - \frac{140}{81} n_f^2, \tag{50}$$

$$\begin{aligned}(\gamma_m)_3 = {} & \frac{4603055}{162} + \frac{135680}{27}\zeta_3 - 8800\zeta_5 \\ & + n_f\left(- \frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\zeta_4 + \frac{18400}{9}\zeta_5\right) \\ & + n_f^2\left(\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right) \\ & + n_f^3\left(- \frac{332}{243} + \frac{64}{27}\zeta_3\right),\end{aligned} \tag{51, 52}$$

$$\begin{aligned}(\gamma_m)_4 = {} & \frac{99512327}{162} + \frac{46402466}{243}\zeta_3 + 96800\zeta_3^2 - \frac{698126}{9}\zeta_4 \\ & - \frac{231757160}{243}\zeta_5 + 242000\zeta_6 + 412720\zeta_7 \\ & + n_f\left(- \frac{150736283}{1458} - \frac{12538016}{81}\zeta_3 - \frac{75680}{9}\zeta_3^2 + \frac{2038742}{27}\zeta_4 + \frac{49876180}{243}\zeta_5 \right. \\ & \left. - \frac{638000}{9}\zeta_6 - \frac{1820000}{27}\zeta_7\right) \\ & + n_f^2\left(\frac{1320742}{729} + \frac{2010824}{243}\zeta_3 + \frac{46400}{27}\zeta_3^2 - \frac{166300}{27}\zeta_4 - \frac{264040}{81}\zeta_5 + \frac{92000}{27}\zeta_6\right) \\ & + \boxed{n_f^3\left(\frac{91865}{1458} + \frac{12848}{81}\zeta_3 + \frac{448}{9}\zeta_4 - \frac{5120}{27}\zeta_5\right)} \\ & + \boxed{n_f^4\left(- \frac{260}{243} - \frac{320}{243}\zeta_3 + \frac{64}{27}\zeta_4\right)}.\end{aligned} \tag{53}$$

Note that in four-loop order we exactly[4] reproduce well-known results obtained in [71, 72]. The boxed

[4]This agreement can be also considered as an important check of

n_f	3	4	5	6
$(\gamma_m)_4^{\rm exact}$	198.9	111.6	41.8	-9.8
$\frac{1}{4^5}(\gamma_m)_4^{\rm APAP}$ [76]	162.0	67.1	-13.7	-80.0
$\frac{1}{4^5}(\gamma_m)_4^{\rm APAP}$ [77]	163.0	75.2	12.6	12.2
$\frac{1}{4^5}(\gamma_m)_4^{\rm APAP}$ [78]	164.0	71.6	-4.8	-64.6

Table 1: The exact results for $(\gamma_m)_4$ together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

terms in (6.4) are in full agreement with the results derived previously on the basis of the $1/n_f$ method in [73–75].

In numerical form γ_m reads

$$\gamma_m = -a_s - a_s^2\left(4.20833 - 0.138889 n_f\right) - a_s^3\left(19.5156 - 2.28412 n_f - 0.0270062 n_f^2\right) - a_s^4\left(98.9434 - 19.1075 n_f + 0.276163 n_f^2 + 0.00579322 n_f^3\right) - a_s^5\left(559.7069 - 143.6864\, n_f + 7.4824\, n_f^2 + 0.1083 n_f^3 - 0.000085359\, n_f^4\right). \tag{54}$$

Let us compare our numerical result for $(\gamma_m)_4$

$$\frac{1}{4^5}(\gamma_m)_4 = 559.71 - 143.6\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4, \tag{55}$$

with an old prediction based on the "Asymptotic Páde Approximants" (APAP) method [76] (the boxed term below was used as the input)

$$\frac{1}{4^5}(\gamma_m)_4^{\rm APAP} = 530 - 143\, n_f + 6.67\, n_f^2 + 0.037\, n_f^3 - \boxed{0.00008535\, n_f^4} \tag{56}$$

However, this good agreement is broken for fixed values of n_f due to severe cancellations between different powers of n_f as one can see from the Table 1.

all our setup which is completely different from the ones utilized at the four-loop calculations.

7. Phenomenological applications of γ_m

7.1. RG invariant quark mass

The solution of eq. (26) reads:

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp\left\{\int dx' \frac{\gamma_m(x')}{\beta(x')}\right\}, \tag{57}$$

$$c(x) = (x)^{\bar\gamma_0}\Big\{1 + d_1 x + (d_1^2/2 + d_2)\, x^2 \tag{58}$$

$$+(d_1^3/6 + d_1 d_2 + d_3)\, x^3 + (d_1^4/24 + d_1^2 d_2/2 + d_2^2/2 + d_1 d_3 + d_4)\, x^4 + O(x^5)\Big\}, \tag{59}$$

$$d_1 = -\bar\beta_1\,\bar\gamma_0 + \bar\gamma_1, \tag{60}$$
$$d_2 = \bar\beta_1^2\,\bar\gamma_0/2 - \bar\beta_2\,\bar\gamma_0/2 - \bar\beta_1\,\bar\gamma_1/2 + \bar\gamma_2/2, \tag{61}$$
$$d_3 = -\bar\beta_1^3\,\bar\gamma_0/3 + 2\bar\beta_1\bar\beta_2\,\bar\gamma_0/3 - \bar\beta_3\,\bar\gamma_0/3 + \bar\beta_1^2\,\bar\gamma_1/3 - \bar\beta_2\,\bar\gamma_1/3 - \bar\beta_1\,\bar\gamma_2/3 + \bar\gamma_3/3, \tag{62}$$
$$d_4 = \bar\beta_1^4\,\bar\gamma_0/4 - 3\bar\beta_1^2\bar\beta_2\,\bar\gamma_0/4 + \bar\beta_2^2\,\bar\gamma_0/4 + \bar\beta_1\bar\beta_3\,\bar\gamma_0/2 - \bar\beta_4\,\bar\gamma_0/4 - \bar\beta_1^3\,\bar\gamma_1/4 + \bar\beta_1\bar\beta_2\,\bar\gamma_1/2 - \bar\beta_3\,\bar\gamma_1/4 + \bar\beta_1^2\,,\bar\gamma_2/4 - \bar\beta_2\,\bar\gamma_2/4 - \bar\beta_1\,\bar\gamma_3/4 + \bar\gamma_4/4. \tag{63}$$

Here $\bar\gamma_i = (\gamma_m)_i/\beta_0, \bar\beta_i = \beta_i/\beta_0$ and

$$\beta(a_s) = -\sum_{i\geq 0}\beta_i\, a_s^{i+2} = -\beta_0\left\{\sum_{i\geq 0}\bar\beta_i\, a_s^{i+2}\right\}$$

is the QCD β-function. Unfortunately, the coefficient d_4 in eq. (63) does depend on the yet unknown *five-loop* coefficient β_4 (up to four loops the β-function is known from [59–67]).

Numerically, the c-function reads:

$$c(x) \underset{n_f=3}{=\!=} x^{4/9}\, c_s(x), \quad c(x) \underset{n_f=4}{=\!=} x^{12/25}\, c_c(x),$$
$$c(x) \underset{n_f=5}{=\!=} x^{12/23}\, c_b(x), \quad c(x) \underset{n_f=6}{=\!=} x^{4/7}\, c_t(x),$$

with

$$c_s(x) = 1 + 0.8950\, x + 1.3714\, x^2 + 1.9517\, x^3 + (15.6982 - 0.11111\bar\beta_4)\, x^4, \tag{64}$$

$$c_c(x) = 1 + 1.0141\, x + 1.3892\, x^2 + 1.0905\, x^3 + (9.1104 - 0.12000\bar\beta_4)\, x^4, \tag{65}$$

$$c_b(x) = 1 + 1.1755\, x + 1.5007\, x^2 \tag{66}$$

$$+0.17248\,x^3 + (2.69277 - 0.13046\bar{\beta}_4)\,x^4,$$

$$c_t(x) = 1 + 1.3980\,x + 1.7935\,x^2 \qquad (67)$$
$$-0.68343\,x^3 + (-3.5130 - 0.14286\bar{\beta}_4)\,x^4.$$

Eq. (57) could be used to define the important concept of the RGI mass

$$m^{\mathrm{RGI}} \equiv m(\mu_0)/c(a_s(\mu_0)), \qquad (68)$$

A remarkable property of the RGI mass is μ and *scheme* independency: in *any* (mass-independent) scheme

$$\lim_{\mu\to\infty} a_s(\mu)^{-\bar{\gamma}_0}\; m(\mu) = m^{\mathrm{RGI}}.$$

Due to this property the RGI mass and function $c(x)$ are often used in the context of lattice simulations. For example, the lattice **ALPHA**collaboration uses (68) to find the $\overline{\mathrm{MS}}$ mass of the strange quark at a lower scale, say $m_s(2\,\mathrm{GeV})$, from the m_s^{RGI} mass determined from lattice simulations (see, e.g. [79]). For example, setting $a_s(\mu = 2\,\mathrm{GeV}) = \frac{\alpha_s(\mu)}{\pi} = 0.1$, we arrive at (here the formal parameter $h = 1$ counts loops):

$$m_s(2\,\mathrm{GeV}) = m_s^{\mathrm{RGI}}\,(a_s(2\,\mathrm{GeV}))^{\frac{4}{9}} \times$$
$$\Big(1 + 0.0895\,h^2 + 0.0137\,h^3 + 0.00195\,h^4$$
$$+ (0.00157 - 0.000011\,\bar{\beta}_4)\,h^5\Big). \qquad (69)$$

In order to have an idea of effects due to the five-loop term in (69) one should make a guess about $\bar{\beta}_4$. By inspecting the available four-loop result

$$\beta(n_f = 3) = -\left(\frac{4}{9}\right)\times \qquad (70)$$
$$\Big(a_s + 1.777\,a_s^2 + 4.4711\,a_s^3 + 20.990\,a_s^4 + \bar{\beta}_4\,a_s^5\Big),$$

we conclude that $\bar{\beta}_4 = 50 - 100$ could serve as a natural estimate of $\bar{\beta}_4$. With this choice we conclude that the (apparent) convergence of the above series is quite good even at a rather small energy scale of 2 GeV. On the other hand, the authors of [77] cite an estimation $\bar{\beta}_4 = -850(!)$ for the $n_f = 3$ QCD. With such a huge value of $\bar{\beta}_4$ the five loop term in (69) would amount to 0.01092 and, thus, would significantly exceed the four-loop contribution (0.00195).

7.2. *Higgs decay into quarks*

The (inclusive) decay width of the Higgs boson into a pair of quarks is related to the spectral density of the scalar correlator according to (3-5). For $n_f = 5$ which corresponds to the newly discovered Higgs boson we get

$$R^S(s = M_H^2, \mu = M_H) = 1 + 5.667\,a_s \qquad (71)$$
$$+29.147\,a_s^2 + 41.758\,a_s^3 - 825.7\,a_s^4$$
$$= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140,$$

where we set $a_s = \alpha_s/\pi = 0.0360$ (for the Higgs mass value $M_H = 125$ GeV and $\alpha_s(M_Z) = 0.118$). The decay rate (3) depends on two phenomenological parameters $\alpha_s(M_H)$ and the quark running mass m_q. Let us consider, for definiteness, the dominant decay mode $H \to \bar{b}b$. To avoid the appearance of large logarithms of the type $\ln\mu^2/M_H^2$ the parameter μ should be chosen around M_H. However, the starting value of m_b is usually determined at a much smaller scale (typically around 5-10 GeV [80]). The evolution of $m_b(\mu)$ from a lower scale to $\mu = M_h$ is governed by eqs. (57-63) which depend on the quark mass anomalous dimension $\gamma(\alpha_s)$ and the QCD beta function $\beta(\alpha_s)$ (for QCD with $n_f = 5$). In order to match the $O({\alpha_s}^4)$ accuracy of (7.2) one should know *both* RG functions β and γ_m in the five-loop approximation.

Let us assume, conservatively, that $0 \leq \bar{\beta}_4^{n_f=5} \leq 200$. The value of $m_b(\mu = M_H)$ is obtained with RG running from $m_b(\mu = 10\,\mathrm{GeV})$ and, thus, depends on β and γ_m. Using the Mathematica package RunDec[5] [81] and eq. (7.1) we find for the shift from the five-loop term

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.3 \cdot 10^{-4}(\bar{\beta}_4 = 0) \qquad (72)$$

$$| - 4.3 \cdot 10^{-4}(\bar{\beta}_4 = 100)| - 7.3 \cdot 10^{-4}(\bar{\beta}_4 = 200) \qquad (73)$$

If we set $\mu = M_H$, then the total effect of $O(\alpha_s^4)$ terms as coming from the five-loop running and four-loop contribution to R^S on $\Gamma(H \to \bar{b}b)$ would be around -2‰ (for $\bar{\beta}_4 = 100$). This is to be compared to the parametric uncertainties coming from the input parameters $\alpha_s(M_Z) = 0.1185(6)$ [82] and $m_b(m_b) = 4.169(8)$ GeV [83] which correspond to ± 1‰ and ± 4‰ respectively.

8. Deep Inenalsic Scattering (DIS)

8.1. *Bjorken sum rule*

The Bjorken sum rule (for polarized DIS) expresses the integral over the spin distributions of quarks inside

[5] We have extended the package by including the five-loop effects to the running of α_s and quark masses.

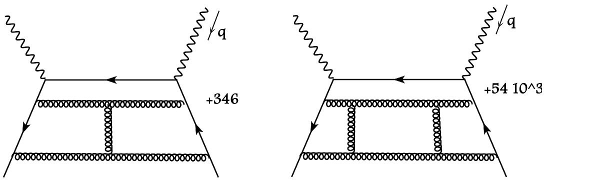

Figure 3: Examples of diagrams contributing to the coefficient function C_{NS}^{Bjp} at three and four loops.

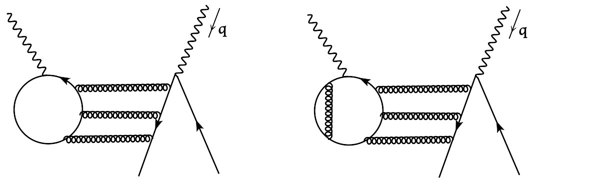

Figure 4: Examples of diagrams contributing to the coefficient function C_{SI}^{Bjp} at three and four loops.

of the nucleon in terms of its axial charge times a coefficient function C^{Bjp}:

$$\begin{aligned}\Gamma_1^{p-n}(Q^2) &= \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx \\ &= \frac{g_A}{6} C^{Bjp}(a_s) + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}}, \end{aligned} \tag{74}$$

where g_1^{ep} and g_1^{en} are the spin-dependent proton and neutron structure functions, g_A is the nucleon axial charge as measured in neutron β-decay. The coefficient function $C^{Bjp}(a_s) = 1 + O(a_s)$ is fixed by the OPE of two EM currents (for a more detailed discussion, see [84]):

$$\begin{aligned}\int T[J_\alpha^E(x) J_\beta^E(0)] e^{iqx} dx|_{q^2\to-\infty} \approx \\ \frac{q^\sigma}{q^2} \epsilon_{\alpha\beta\rho\sigma} C_a^{Bjp}(a_s)\, A_\rho^c(0) + \ldots,\end{aligned} \tag{75}$$

$$\begin{aligned}&C_a^{Bjp}(a_s) = \\ &\mathrm{Tr}[E^2 t_a]\, C_{NS}^{Bjp}(a_s) + \mathrm{Tr}(E)\,\mathrm{Tr}[E t_a]\, C_{SI}^{Bjp}(a_s) = \\ &\left(C_{NS}^{Bjp}(a_s) + 3\,\mathrm{Tr}[E]\, C_{SI}^{Bjp}(a_s)\right) \mathrm{Tr}[E^2 t_a] \\ &\qquad = C^{Bjp}(a_s)\,\mathrm{Tr}[E^2 t_a].\end{aligned} \tag{76}$$

Here $E = diag(Q_i)$ is the quark charge matrix, $J_\alpha^E = \overline{\psi} E \gamma_\rho \psi$ is the quark EM current, $A_\rho^c = \overline{\psi}\,\gamma_\rho\, t^c \gamma_5\, \psi$ is the (flavor non-singlet!) axial current and $Q^2 = -q^2$.

As one can see from eqs. (76), the coefficient function $C^{Bjp}(a_s)$ receives contributions from two types of diagrams, viz. the singlet and non-singlet ones (see Figs. 4 and 3 respectively). Let us discuss them in turn.

The coeffcient function $C_{NS}^{Bjp}(a_s)$ starts from 1 which corresponds to the parton approximation. The four-loop result was published in [85] for the case of a general gauge group. The QCD result reads:

$$\begin{aligned}C_{NS}^{Bjp}(a_s) = {}& 1 - a_s + a_s^2\Big(-\frac{55}{12} + \frac{n_f}{3}\Big) \\ &+ a_s^3\Big[-\frac{13841}{216} - \frac{44}{9}\zeta_3 + \frac{55}{2}\zeta_5 \\ &\quad + n_f\Big(\frac{10339}{1296} + \frac{61}{54}\zeta_3 - \frac{5}{3}\zeta_5\Big) - n_f^2\,\frac{115}{648}\Big] \\ &+ a_s^4\Big[-\frac{17865665}{20736} + \frac{8213}{48}\zeta_3 - \frac{363}{8}\,\zeta_3^2 \\ &\quad + \frac{343175}{864}\zeta_5 - \frac{2695}{16}\,\zeta_7 \\ &\quad + n_f\Big(\frac{10134475}{62208} - \frac{32743}{2592}\zeta_3 + \frac{11}{2}\,\zeta_3^2 \\ &\qquad - \frac{53215}{1296}\zeta_5 + \frac{245}{24}\,\zeta_7\Big) \\ &\quad + n_f^2\Big(-\frac{169523}{20736} + \frac{103}{432}\zeta_3 - \frac{1}{6}\,\zeta_3^2 + \frac{5}{12}\zeta_5\Big) \\ &\quad + n_f^3\,\frac{605}{5832}\Big],\end{aligned} \tag{77}$$

$$\begin{aligned}C_{NS}^{Bjp} = {}& 1 - a_s + a_s^2\big(-4.583 + 0.3333\,n_f\big) \\ &+ a_s^3\big(-41.44 + 7.607\,n_f - 0.1775\,n_f^2\big) \\ &+ a_s^4\big(-479.4 + 123.4\,n_f - 7.697\,n_f^2 + 0.1037\,n_f^3\big).\end{aligned} \tag{78}$$

Note that for phenomenologically relevant values of $Q^2 \leq 3\,\mathrm{GeV}^2$ one should work in an effective QCD with three active flavours. In this case the singlet contrubution vanishes identically as $\mathrm{Tr}[E] \equiv 0$ and

$$\begin{aligned}C^{Bjp}(n_f = 3) \equiv C_{NS}^{Bjp}(n_f = 3) = \\ 1 - a_s - 3.583 a_s^2 - 20.22 a_s^3 - 175.7\, a_s^4.\end{aligned} \tag{79}$$

Phenomelogical implications of (79) have been studied in works [86, 87]. Their results can be summarized as follows.

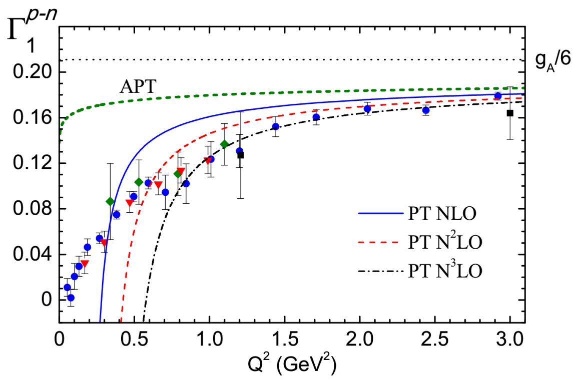

Figure 5: Perturbative part of the Bjorken sum rule (74) as a function of the momentum transfer squared Q^2 in different orders against the combined set of the Jefferson Lab (taken from V.L. Khandramai, R.S. Pasechnik, D.V. Shirkov, O.P. Solovtsova, O.V. Teryaev, *Four-loop QCD analysis of the Bjorken sum rule vs data*, Phys.Lett.B706:340-344,2012).

First, by comparing experimental data (see Fig. 5) with the theoretical prediction (79) the authors have arrived at the following conclusion [86]: "One can see that at $Q^2 \geq 0.7$ GeV^2 the four-loop approximation describes the data quite well. Moreover, the corresponding curve passes close to the central values of several data points, although the experimental accuracy (which is of the same order as both the three- and four-loop contributions) does not allow one to make a definite choice between four- and three-loop approximations."

Second, a certain *duality* between higher orders and higher-twist contributions to the Bjorken sum rule (74) has been detected[6] in [87]. Indeed, the fitted value of first non-zero higher twist contribution, μ_4, has proved to be strongly dependent on the order of PT terms kept in (79). For example, at leading order (that is with all terms in (79) except first two set to zero) μ_4 was found to be -0.037 ± 0.003 GeV^2. At next-leading-order μ_4 is decreased to -0.025 ± 0.04 GeV^2 and, finally, at $O(\alpha_s^4)$ it becomes compatible to zero: $\mu_4 = 0.005 \pm 0.008$ GeV^2.

The singlet contributions to C^{Bjp} formally starts at two loops but the corresponding diagrams sums to identical zero due to Furry's Theorem. The next, three-loop term also happen to be zero [21]. This fact has been explained in [84] with the help of generalized Crewther relation [89–91]. Here it has been also predicted that at four loops the singlet contribution should have the form:

$$X \beta_0 \, d^{abc} d^{abc} \left(\frac{\alpha_s}{\pi}\right)^4, \tag{80}$$

[6]The phenomenon in a more general context was earlier discussed in [88].

with $\beta_0 = \frac{11}{12} C_A - \frac{T_f n_f}{3}$, $d^{abc} = 2\,\mathrm{Tr}(\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\}\frac{\lambda^c}{2})$, and X being a constant.

We have performed a direct calculation of C^{Bjp}_{SI} at order α_s^4. Our result reads:

$$C^{Bjp}_{SI} = \frac{1}{9} \beta_0 \, d^{abc} d^{abc} \left(\frac{\alpha_s}{\pi}\right)^4$$

in full agreement[7] with (80).

9. Anomalous Dimensions of twist 2 operators

Recently there has been a lot of progress in three-loop QCD calculations of the moments (and the corresponding anomalous dimensions) of deep inelastic structure functions [92–96]). In particular, the anomalous dimension of γ^{NS}_N of the twist-two non-singlet operator γ^N_{NS} (ψ and ψ' refers to the two *different* quark species)

$$O^{\{\mu_1,\ldots,\mu_N\}} = \bar{\psi}' \, \gamma^{\{\mu_1,} D^{\mu_2,\ldots,\mu_N\}} \, \psi \tag{81}$$

has been analytically found for a generic value of spin N at the three loop level.

In fact, a general consistency argument requires the use of *four*-loop splitting functions in applications of the results of [94, 96] to the phenomenological analysis of deep inelastic experimental data. Unfortunately, the formidable problem of the *four-loop* calculation for generic N seems to be out of reach for available technologies. On the other hand, fixed N calculations are now possible (at least for not too large values of N).

The first result in this direction was reported in [97] where the four loop anomalous dimension of the operator $O^{\{\mu_1,\mu_2\}}$ was computed for a particular number of quark species $n_f = 3$. Later this result was confirmed and generalized to a generic gauge group in [98]. Note that at four-loop level the calculation of γ^{NS}_N with the use of BAICER *does not require* application of any IRR: one just computes a diagonal matrix element

$$\langle p | O^{\{\mu_1,\ldots,\mu_N\}}(0) | p \rangle,$$

with $|p\rangle$ being an off-shell quark state. *In principle* one could even compute the five-loop anomalous dimension γ^{NS}_N for low N. However, this would require significantly more computer as well as human power (the latter due to quite complicated IRR).

[7]In fact, paper [84] has also guessed a particular value of $X = -\frac{1}{3}\left(-\frac{179}{384} + \frac{25}{48}\zeta_3 - \frac{5}{24}\zeta_5\right)$ which happens to be very different from our result.

Two years ago the present authors computed γ_N^{NS} for $N = 2, 3$ and 4 in QCD with full n_f dependence[8]. Our results read:

$$\gamma_N^{NS} = \sum_{i\geq 0} \left(\gamma_N^{NS}\right)_i h^i,$$

$$\left(\gamma_2^{NS}\right)_0 = \frac{32}{9}, \tag{82}$$

$$\left(\gamma_2^{NS}\right)_1 = \frac{11744}{243} - n_f\,\frac{256}{81}, \tag{83}$$

$$\left(\gamma_2^{NS}\right)_2 = -n_f^2\,\frac{896}{729} + n_f\left(-\frac{1280}{27}\zeta_3 - \frac{167200}{2187}\right) + \frac{1280}{81}\zeta_3 + \frac{5514208}{6561}, \tag{84}$$

$$\begin{aligned}\left(\gamma_2^{NS}\right)_3 = {}& \frac{26060864}{6561}\zeta_3 - \frac{7040}{27}\zeta_4 - \frac{1249280}{243}\zeta_5 + \frac{3100369144}{177147} \\ & + n_f\left(-\frac{6322976}{2187}\zeta_3 + \frac{64640}{81}\zeta_4 + \frac{14720}{9}\zeta_5 - \frac{167219672}{59049}\right) \\ & + n_f^2\left(\frac{2560}{27}\zeta_3 - \frac{1280}{27}\zeta_4 + \frac{1084904}{19683}\right) \\ & + n_f^3\left(\frac{512}{243}\zeta_3 - \frac{4096}{6561}\right),\end{aligned} \tag{85}$$

$$\left(\gamma_3^{NS}\right)_0 = \frac{50}{9}, \tag{86}$$

$$\left(\gamma_3^{NS}\right)_1 = \frac{17225}{243} - n_f\,\frac{415}{81}, \tag{87}$$

$$\left(\gamma_3^{NS}\right)_2 = \frac{64486199}{52488} + \frac{1100}{81} - n_f\left(\frac{967495}{8748} + \frac{2000}{27}\zeta_3\right) - n_f^2\,\frac{2569}{1458}, \tag{88}$$

$$\begin{aligned}\left(\gamma_3^{NS}\right)_3 = {}& \frac{69231923065}{2834352} + \frac{73641835}{13122}\zeta_3 - \frac{6050}{27}\zeta_4 - \frac{1834550}{243}\zeta_5 \\ & + n_f\left(-\frac{1978909951}{472392} - \frac{9638360}{2187}\zeta_3 + \frac{100100}{81}\zeta_4 + \frac{23000}{9}\zeta_5\right) \\ & + n_f^2\left(\frac{1733306}{19683} + \frac{12200}{81}\zeta_3 - \frac{2000}{27}\zeta_4\right) \\ & + n_f^3\left(-\frac{23587}{26244} + \frac{800}{243}\zeta_3\right),\end{aligned} \tag{89}$$

[8] The results given below were first presented on the 19 Meeting of SFB/TR9 "Computational Particle Physics" 19.03.2013 (Aachen). Very recently γ_3^{NS} and γ_4^{NS} have been computed for a case of a generic gauge group [99]. For the QCD case gauge we have found full agreement between eqs. (82-93) and results of [99].

$$\left(\gamma_4^{NS}\right)_0 = \frac{314}{45}, \tag{90}$$

$$\left(\gamma_4^{NS}\right)_1 = \frac{2620957}{30375} - n_f\,\frac{13271}{2025}, \tag{91}$$

$$\left(\gamma_4^{NS}\right)_2 = \frac{245787905651}{164025000} + \frac{5756}{405}\zeta_3 - n_f\left(\frac{726591271}{5467500} + \frac{2512}{27}\zeta_3\right) - n_f^2\,\frac{384277}{182250}, \tag{92}$$

$$\begin{aligned}\left(\gamma_4^{NS}\right)_3 = {}& \frac{1267599127484293}{44286750000} + \frac{58681291019}{8201250}\zeta_3 - \frac{31658}{135}\zeta_4 - \frac{32178794}{3645}\zeta_5 \\ & + n_f\left(-\frac{7539856966909}{1476225000} - \frac{1495404568}{273375}\zeta_3 + \frac{627476}{405}\zeta_4 + \frac{1289656}{405}\zeta_5\right) \\ & + n_f^2\left(\frac{6771192712}{61509375} + \frac{8584}{45}\zeta_3 - \frac{2512}{27}\zeta_4\right) \\ & + n_f^3\left(-\frac{17813699}{16402500} + \frac{5024}{1215}\zeta_3\right).\end{aligned} \tag{93}$$

Numerically all 3 anomalous dimensions display a remarkable similarity (modulo global normalization):

$$\gamma_2^{NS}(n_f = 3) = \frac{32}{9\cdot 4}\left(a_s + 2.7319a_s^2 + 7.8763a_s^3 + 28.706a_s^4\right),$$

$$\gamma_3^{NS}(n_f = 3) = \frac{50}{9\cdot 4}\left(a_s + 2.4982a_s^2 + 7.0891a_s^3 + 23.587a_s^4\right),$$

$$\gamma_4^{NS}(n_f = 3) = \frac{314}{45\cdot 4}\left(a_s + 2.3871a_s^2 + 6.8288a_s^3 + 22.294a_s^4\right).$$

10. Conclusions

The problem of analytical evaluation of massless propagators at four loops has been under investigation since long [47]. It has been solved using reduction via $1/D$ expansion. As a result a number of important four and five loop calculations have been done. In this short review we have briefly discussed some of them related to the R-ratio, Higgs decays into quarks, deep inelastic scattering and QCD renormalization group functions.

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References

[1] V. A. Smirnov, Applied asymptotic expansions in momenta and masses, Springer, Berlin, 2002.
[2] E. Stueckelberg, A. Petermann, La normalisation des constantes dans la theorie des quanta, Helv. Phys. Acta. 26 (1953) 499–520.
[3] M. Gell-Mann, F. Low, Quantum electrodynamics at small distances, Phys.Rev. 95 (1954) 1300–1312. doi:10.1103/PhysRev.95.1300.
[4] N. Bogolyubov, D. Shirkov, Charge renormalization group in quantum field theory, Nuovo Cim. 3 (1956) 845–863. doi:10.1007/BF02823486.
[5] J. F. Ashmore, A method of gauge invariant regularization, Lett. Nuovo Cim. 4 (1972) 289–290.
[6] G. M. Cicuta, E. Montaldi, Analytic renormalization via continuous space dimension, Nuovo Cim. Lett. 4 (1972) 329–332.
[7] G. 't Hooft, M. J. G. Veltman, Regularization and Renormalization of Gauge Fields, Nucl. Phys. B44 (1972) 189–213. doi:10.1016/0550-3213(72)90279-9.
[8] G. 't Hooft, Dimensional regularization and the renormalization group, Nucl. Phys. B61 (1973) 455–468. doi:10.1016/0550-3213(73)90376-3.
[9] J. C. Collins, Normal Products in Dimensional Regularization, Nucl. Phys. B92 (1975) 477. doi:10.1016/S0550-3213(75)80010-1.
[10] A. A. Vladimirov, Method For Computing Renormalization Group Functions In Dimensional Renormalization Scheme, Theor. Math. Phys. 43 (1980) 417. doi:10.1007/BF01018394.
[11] K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique, Nucl. Phys. B174 (1980) 345–377. doi:10.1016/0550-3213(80)90289-8.
[12] K. G. Chetyrkin, V. A. Smirnov, R^* Operation Corrected, Phys. Lett. B144 (1984) 419–424. doi:10.1016/0370-2693(84)91291-7.
[13] K. G. Chetyrkin, J. H. Kühn, Mass corrections to the z decay rate, Phys. Lett. B248 (1990) 359–364.
[14] K. G. Chetyrkin, R. Harlander, J. H. Kühn, M. Steinhauser, Mass corrections to the vector current correlator, Nucl. Phys. B503 (1997) 339–353. arXiv:hep-ph/9704222, doi:10.1016/S0550-3213(97)00383-0.
[15] K. G. Chetyrkin, R. V. Harlander, J. H. Kühn, Quartic mass corrections to R_{had} at $O(\alpha_s^3)$, Nucl. Phys. B586 (2000) 56–72. arXiv:hep-ph/0005139, doi:10.1016/S0550-3213(00)00393-X.
[16] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Vacuum polarization in pQCD: First complete $O(\alpha_s^4)$ result, Nucl. Phys. Proc. Suppl. 135 (2004) 243–246. doi:10.1016/j.nuclphysbps.2004.09.013.
[17] P. Baikov, K. Chetyrkin, J. Kühn, R(s) and hadronic tau-Decays in Order α_s^4: Technical aspects, Nucl.Phys.Proc.Suppl. 189 (2009) 49–53. arXiv:0906.2987, doi:10.1016/j.nuclphysbps.2009.03.010.
[18] J. Blumlein, The Theory of Deeply Inelastic Scattering, Prog.Part.Nucl.Phys. 69 (2013) 28–84. arXiv:1208.6087, doi:10.1016/j.ppnp.2012.09.006.
[19] S. G. Gorishny, S. A. Larin, F. V. Tkachov, The Algorithm For OPE Coefficient Functions In The MS Scheme, Phys. Lett. B124 (1983) 217–220. doi:10.1016/0370-2693(83)91439-9.
[20] S. G. Gorishny, S. A. Larin, Coefficient Functions Of Asymptotic Operator Expansions In Minimal Subtraction Scheme, Nucl. Phys. B283 (1987) 452. doi:10.1016/0550-3213(87)90283-5.
[21] S. A. Larin, J. A. M. Vermaseren, The α_s^3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule, Phys. Lett. B259 (1991) 345–352.
[22] V. A. Smirnov, Feynman Integral Calculus, Springer, Berlin, 2006.
[23] V. A. Smirnov, Analytic tools for Feynman integrals, Springer, Berlin, 2012.
[24] A. Grozin, Lectures on QED and QCD: Practical calculation and renormalization of one- and multi-loop Feynman diagrams, Hackensack, USA: World Scientific, 2007.
[25] A. Grozin, Integration by parts: An Introduction, Int.J.Mod.Phys. A26 (2011) 2807–2854. arXiv:1104.3993, doi:10.1142/S0217751X11053687.
[26] A. Smirnov, A. Petukhov, The Number of Master Integrals is Finite, Lett.Math.Phys. 97 (2011) 37–44. arXiv:1004.4199, doi:10.1007/s11005-010-0450-0.
[27] R. N. Lee, A. A. Pomeransky, Critical points and number of master integrals, JHEP 1311 (2013) 165. arXiv:1308.6676, doi:10.1007/JHEP11(2013)165.
[28] K. G. Chetyrkin, F. V. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, Nucl. Phys. B192 (1981) 159–204. doi:10.1016/0550-3213(81)90199-1.
[29] S. Laporta, E. Remiddi, The analytic value of g(e)-2 at three loops in qed, Nucl. Phys. Proc. Suppl. 51C (1996) 142–147.
[30] S. Laporta, Calculation of master integrals by difference equations, Phys. Lett. B504 (2001) 188–194. arXiv:hep-ph/0102032.
[31] P. A. Baikov, A practical criterion of irreducibility of multi-loop feynman integrals, Phys. Lett. B634 (2006) 325–329. arXiv:hep-ph/0507053.
[32] P. A. Baikov, Explicit solutions of the 3–loop vacuum integral recurrence relations, Phys. Lett. B385 (1996) 404–410. arXiv:hep-ph/9603267, doi:10.1016/0370-2693(96)00835-0.
[33] A. G. Grozin, Lectures on multiloop calculations, Int.J.Mod.Phys. A19 (2004) 473–520. arXiv:hep-ph/0307297, doi:10.1142/S0217751X04016775.

[34] P. A. Baikov, K. G. Chetyrkin, Four-Loop Massless Propagators: an Algebraic Evaluation of All Master Integrals, Nucl. Phys. B837 (2010) 186–220. arXiv:1004.1153, doi:10.1016/j.nuclphysb.2010.05.004.

[35] A. V. Smirnov, M. Tentyukov, Four-Loop Massless Propagators: a Numerical Evaluation of All Master Integrals, Nucl. Phys. B837 (2010) 40–49. arXiv:1004.1149, doi:10.1016/j.nuclphysb.2010.04.020.

[36] R. N. Lee, A. V. Smirnov, V. A. Smirnov, Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve, Nucl. Phys. B856 (2012) 95–110. arXiv:1108.0732, doi:10.1016/j.nuclphysb.2011.11.005.

[37] P. Nogueira, Automatic Feynman graph generation, J. Comput. Phys. 105 (1993) 279–289. doi:10.1006/jcph.1993.1074.

[38] J. A. M. Vermaseren, New features of formarXiv:math-ph/0010025.

[39] M. Tentyukov, et al., ParFORM: Parallel Version of the Symbolic Manipulation Program FORMarXiv:cs/0407066.

[40] M. Tentyukov, J. A. M. Vermaseren, The multithreaded version of FORMarXiv:hep-ph/0702279.

[41] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Scalar correlator at $O(\alpha_s^4)$, Higgs decay into b- quarks and bounds on the light quark masses, Phys. Rev. Lett. 96 (2006) 012003. arXiv:hep-ph/0511063, doi:10.1103/PhysRevLett.96.012003.

[42] K. G. Chetyrkin, A. Khodjamirian, Strange Quark Mass from Pseudoscalar Sum Rule with $O(\alpha_s^4)$ Accuracy, Eur. Phys. J. C46 (2006) 721–728. arXiv:hep-ph/0512295, doi:10.1140/epjc/s2006-02508-8.

[43] S. G. Gorishny, A. L. Kataev, S. A. Larin, L. R. Surguladze, Corrected three loop qcd correction to the correlator of the quark scalar currents and gamma (tot) (h0 $\to$ hadrons), Mod. Phys. Lett. A5 (1990) 2703–2712.

[44] K. G. Chetyrkin, Correlator of the quark scalar currents and $\Gamma_{\rm tot}(H \to \text{hadrons})$ at $O(\alpha_s^3)$ in pQCD, Phys. Lett. B390 (1997) 309–317. arXiv:hep-ph/9608318, doi:10.1016/S0370-2693(96)01368-8.

[45] K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Higher Order Corrections to $\sigma_{tot}(e^+e^- \to \text{Hadrons})$ in Quantum Chromodynamics, Phys. Lett. B85 (1979) 277. doi:10.1016/0370-2693(79)90596-3.

[46] S. G. Gorishny, A. L. Kataev, S. A. Larin, The $O(\alpha_s^3)$ corrections to $\sigma_{\rm tot}(e^+e^- \to \text{hadrons})$ and $\sigma(\tau \to \nu_\tau + \text{hadrons})$ in QCD, Phys. Lett. B259 (1991) 144–150.

[47] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, The cross section of e+ e- annihilation into hadrons of order $\alpha_s^4 n_f^2$ in perturbative QCD, Phys. Rev. Lett. 88 (2002) 012001. arXiv:hep-ph/0108197, doi:10.1103/PhysRevLett.88.012001.

[48] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Towards order α_s^4 accuracy in tau decays, Phys. Rev. D67 (2003) 074026. arXiv:hep-ph/0212299, doi:10.1103/PhysRevD.67.074026.

[49] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Five-loop vacuum polarization in pQCD: $O(m_q^2\alpha_s^4 n_f^2)$ contribution, Phys. Lett. B559 (2003) 245–251. arXiv:hep-ph/0212303, doi:10.1016/S0370-2693(03)00186-2.

[50] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, QCD corrections to hadronic Z and tau decays, Eur. Phys. J. C33 (2004) s650–s652. arXiv:hep-ph/0311137, doi:10.1140/epjcd/s2004-03-1839-8.

[51] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Perturbative QCD and tau-decays, Nucl. Phys. Proc. Suppl. 144 (2005) 81–87. doi:10.1016/j.nuclphysbps.2005.02.011.

[52] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Multiloop calculations: Towards R at order α_s^4, Nucl. Phys. Proc. Suppl. 157 (2006) 27–31. arXiv:hep-ph/0602126, doi:10.1016/j.nuclphysbps.2006.03.005.

[53] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Order α_s^4 QCD Corrections to Z and τ Decays, Phys. Rev. Lett. 101 (2008) 012002. arXiv:0801.1821, doi:10.1103/PhysRevLett.101.012002.

[54] P. Baikov, K. Chetyrkin, J. Kühn, J. Rittinger, Vector Correlator in Massless QCD at Order $O(\alpha_s^4)$ and the QED β-function at Five Loop, JHEP 1207 (2012) 017. arXiv:1206.1284, doi:10.1007/JHEP07(2012)017.

[55] P. Baikov, K. Chetyrkin, J. Kühn, Adler Function, DIS sum rules and Crewther Relations, Nucl.Phys.Proc.Suppl. 205-206 (2010) 237–241. arXiv:1007.0478, doi:10.1016/j.nuclphysbps.2010.08.049.

[56] P. Baikov, K. Chetyrkin, J. Kühn, J. Rittinger, Complete $O(\alpha_s^4)$ QCD Corrections to Hadronic Z-Decays, Phys.Rev.Lett. 108 (2012) 222003. arXiv:1201.5804, doi:10.1103/PhysRevLett.108.222003.

[57] P. Baikov, K. Chetyrkin, J. Kühn, J. Rittinger, Adler Function, Sum Rules and Crewther Relation of Order $O(\alpha_s^4)$: the Singlet Case, Phys.Lett. B714 (2012) 62–65. arXiv:1206.1288, doi:10.1016/j.physletb.2012.06.052.

[58] S. G. Gorishny, A. L. Kataev, S. A. Larin, L. R. Surguladze, The analytical four loop corrections to the qed beta function in the ms scheme and to the qed psi function: Total reevaluation, Phys. Lett. B256 (1991) 81–86.

[59] D. J. Gross, F. Wilczek, Ultraviolet behavior of non-abelian gauge theories, Phys. Rev. Lett. 30 (1973) 1343–1346.

[60] H. D. Politzer, Reliable perturbative results for strong interactions?, Phys. Rev. Lett. 30 (1973) 1346–1349.

[61] W. E. Caswell, Asymptotic behavior of nonabelian gauge theories to two loop order, Phys. Rev. Lett. 33 (1974) 244.

[62] D. R. T. Jones, Two loop diagrams in yang-mills theory, Nucl. Phys. B75 (1974) 531.

[63] E. Egorian, O. V. Tarasov, Two loop renormalization of the qcd in an arbitrary gauge, Theor. Math. Phys. 41 (1979) 863–867.

[64] O. V. Tarasov, A. A. Vladimirov, A. Y. Zharkov, The gell-mann-low function of qcd in the three loop approximation, Phys. Lett. B93 (1980) 429–432.

[65] S. A. Larin, J. A. M. Vermaseren, The three loop qcd beta function and anomalous dimensions, Phys. Lett. B303 (1993) 334–336. arXiv:hep-ph/9302208.

[66] T. van Ritbergen, J. A. M. Vermaseren, S. A. Larin, The four-loop beta function in quantum chromodynamics, Phys. Lett. B400 (1997) 379–384. arXiv:hep-ph/9701390.

[67] M. Czakon, The four-loop QCD beta-function and anomalous dimensions, Nucl. Phys. B710 (2005) 485–498. arXiv:hep-ph/0411261.

[68] R. Tarrach, The pole mass in perturbative qcd, Nucl. Phys. B183 (1981) 384.

[69] O. V. Tarasov, Anomalous dimensions of quark masses in three loop approximationJINR-P2-82-900.

[70] S. A. Larin, The renormalization of the axial anomaly in dimensional regularization, Phys. Lett. B303 (1993) 113–118. arXiv:hep-ph/9302240.

[71] K. G. Chetyrkin, Quark mass anomalous dimension to $O(\alpha_s^4)$, Phys. Lett. B404 (1997) 161–165. arXiv:hep-ph/9703278.

[72] J. A. M. Vermaseren, S. A. Larin, T. van Ritbergen, The 4-loop quark mass anomalous dimension and the invariant quark mass, Phys. Lett. B405 (1997) 327–333. arXiv:hep-ph/9703284.

[73] A. Palanques-Mestre, P. Pascual, The 1/n-f expansion of the gamma and beta functions in qed, Commun. Math. Phys. 95 (1984) 277.

[74] M. Ciuchini, S. E. Derkachov, J. Gracey, A. Manashov, Computation of quark mass anomalous dimension at O(1 / N**2(f)) in quantum chromodynamics, Nucl.Phys. B579 (2000) 56–100. arXiv:hep-ph/9912221, doi:10.1016/S0550-3213(00)00209-1.

[75] M. Ciuchini, S. E. Derkachov, J. Gracey, A. Manashov, Quark mass anomalous dimension at O(1/N(f)**2) in QCD,

Phys.Lett. B458 (1999) 117–126. arXiv:hep-ph/9903410, doi:10.1016/S0370-2693(99)00573-0.

[76] J. R. Ellis, I. Jack, D. Jones, M. Karliner, M. Samuel, Asymptotic Pade approximant predictions: Up to five loops in QCD and SQCD, Phys.Rev. D57 (1998) 2665–2675. arXiv:hep-ph/9710302, doi:10.1103/PhysRevD.57.2665.

[77] V. Elias, T. G. Steele, F. Chishtie, R. Migneron, K. B. Sprague, Pade improvement of QCD running coupling constants, running masses, Higgs decay rates, and scalar channel sum rules, Phys.Rev. D58 (1998) 116007. arXiv:hep-ph/9806324, doi:10.1103/PhysRevD.58.116007.

[78] A. Kataev, V. Kim, Higgs boson decay into bottom quarks and uncertainties of perturbative QCD predictionsarXiv:0804.3992.

[79] M. Della Morte, et al., Non-perturbative quark mass renormalization in two-flavor qcd, Nucl. Phys. B729 (2005) 117–134. arXiv:hep-lat/0507035.

[80] K. Chetyrkin, J. Kühn, A. Maier, P. Maierhofer, P. Marquard, et al., Charm and Bottom Quark Masses: An Update, Phys.Rev. D80 (2009) 074010. arXiv:0907.2110, doi:10.1103/PhysRevD.80.074010.

[81] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser, Rundec: A mathematica package for running and decoupling of the strong coupling and quark masses, Comput. Phys. Commun. 133 (2000) 43–65. arXiv:hep-ph/0004189.

[82] J. Beringer, et al., Review of Particle Physics (RPP), Phys.Rev. D86 (2012) 010001. doi:10.1103/PhysRevD.86.010001.

[83] A. A. Penin, N. Zerf, Bottom Quark Mass from Υ Sum Rules to $O(\alpha_s^3)$arXiv:1401.7035.

[84] S. Larin, The singlet contribution to the Bjorken sum rule for polarized deep inelastic scattering, Phys.Lett. B723 (2013) 348–350. arXiv:1303.4021, doi:10.1016/j.physletb.2013.05.026.

[85] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory, Phys. Rev. Lett. 104 (2010) 132004. arXiv:1001.3606, doi:10.1103/PhysRevLett.104.132004.

[86] V. Khandramai, R. Pasechnik, D. Shirkov, O. Solovtsova, O. Teryaev, Four-loop QCD analysis of the Bjorken sum rule vs data, Phys.Lett. B706 (2012) 340–344. arXiv:1106.6352, doi:10.1016/j.physletb.2011.11.023.

[87] V. Khandramai, O. Solovtsova, O. Teryaev, Polarized Bjorken Sum Rule Analysis: Revised, Nonlin.Phenom.Complex Syst. 16 (2013) 93–98. arXiv:1302.3952.

[88] S. Narison, V. Zakharov, Duality between QCD Perturbative Series and Power Corrections, Phys.Lett. B679 (2009) 355–361. arXiv:0906.4312, doi:10.1016/j.physletb.2009.07.060.

[89] R. J. Crewther, Nonperturbative evaluation of the anomalies in low-energy theorems, Phys. Rev. Lett. 28 (1972) 1421.

[90] D. J. Broadhurst, A. L. Kataev, Connections between deep inelastic and annihilation processes at next to next-to-leading order and beyond, Phys. Lett. B315 (1993) 179–187. arXiv:hep-ph/9308274.

[91] R. J. Crewther, Relating inclusive e+ e- annihilation to electroproduction sum rules in quantum chromodynamics, Phys. Lett. B397 (1997) 137–142. arXiv:hep-ph/9701321, doi:10.1016/S0370-2693(97)00157-3.

[92] S. Moch, J. A. M. Vermaseren, A. Vogt, The three-loop splitting functions in QCD: The non-singlet case, Nucl. Phys. B688 (2004) 101–134. arXiv:hep-ph/0403192.

[93] A. Vogt, S. Moch, J. A. M. Vermaseren, The three-loop splitting functions in QCD: The singlet case, Nucl. Phys. B691 (2004) 129–181. arXiv:hep-ph/0404111.

[94] S. Moch, J. A. M. Vermaseren, A. Vogt, The longitudinal structure function at the third order, Phys. Lett. B606 (2005) 123–129. arXiv:hep-ph/0411112.

[95] J. Blumlein, J. A. M. Vermaseren, The 16th moment of the non-singlet structure functions f2(x,q**2) and f(l)(x,q**2) to o(alpha(s)**3), Phys. Lett. B606 (2005) 130–138. arXiv:hep-ph/0411111.

[96] J. A. M. Vermaseren, A. Vogt, S. Moch, The third-order QCD corrections to deep-inelastic scattering by photon exchange, Nucl. Phys. B724 (2005) 3–182. arXiv:hep-ph/0504242, doi:10.1016/j.nuclphysb.2005.06.020.

[97] P. A. Baikov, K. G. Chetyrkin, New four loop results in QCD, Nucl. Phys. Proc. Suppl. 160 (2006) 76–79. doi:10.1016/j.nuclphysbps.2006.09.031.

[98] V. Velizhanin, Four loop anomalous dimension of the second moment of the non-singlet twist-2 operator in QCD, Nucl.Phys. B860 (2012) 288–294. arXiv:1112.3954, doi:10.1016/j.nuclphysb.2012.03.006.

[99] V. Velizhanin, Four loop anomalous dimension of the third and fourth moments of the non-singlet twist-2 operator in QCD arXiv:1411.1331.