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Towards QCD running in 5 loops: quark mass anomalous dimension

P. A. Baikov,
Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, 1(2), Leninskie gory, Moscow 119991, Russian Federation
E-mail: baikov@theory.sinp.msu.ru

K. G. Chetyrkin* **and J. H. Kühn,**
Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany
E-mails : konstantin.chetyrkin@kit.edu
Johann.kuehn@kit.edu

We report first results of an ongoing project devoted to the analytical calculation of the QCD β-function and the quark mass anomalous dimension at the five loop level.

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*Speaker.

1. Introduction

The method of the renormalization group (RG) [**?**, **?**, **?**] is of vital importance in modern quantum field theory. It is enough to recall that the famous idea of the asymptotic freedom is based on the RG concept of the running coupling constant. The RG functions — β-functions and various anomalous dimensions — serve as coefficients in the RG equations and are expressed in terms of Feynman Integrals (FI's). The complexity of these integrals grows drastically with the number of loops.

During last three decades or so there has been a tremendous progress in our ability to compute analytically the RG functions. The progress has been under way in, essentially, three directions.

i. General developments of our ability to deal with multiloop FI's. These have been thoroughly documented by Vladimir Smirnov in his bestseller books "Evaluating Feynman integrals" and "Feynman Integral Calculus" [**?**, **?**] (see also [**?**, **?**]). As a result two types of most relevant for RG calculations FI's, namely, massless propagators and (completely) massive tadpoles (p- and m-integrals correspondingly) can be calculated (completely analytically) at the four loop level.

ii. Invention of special tools for significant simplifications of RG calculations. These include Infrared Rearrangement [**?**, **?**, **?**] and R^*-operation [**?**].

iii. Continuous development of Computer Algebra Systems, with FORM [**?**] as most prominent and indispensable tool.

The current state of art of (analytical) RG calculations can be summarized as follows: generic four-loop RG calculations are now possible (see, e.g. [**?**, **?**, **?**, **?**, **?**, **?**, **?**]) and five-loop ones are gradually getting "feasible" [**?**, **?**].

In this talk we give the results of the first **complete** calculations of **some of** QCD RG functions at five loops. These are the quark mass anomalous dimension and the anomalous dimension of the ghost field. The latter is one ingredient (among three) necessary for the construction of the five-loop contribution to the QCD β-function.

The precise evaluation of the quark mass anomalous dimension has important implications. The Higgs boson decay rate into charm and bottom quarks is proportional to the square of respective quark mass at the scale of m_H and the uncertainty from the presently unknown 5-loop terms in the running of the quark mass is of order 10^{-3}. This is comparable to the precision advocated for experiments e.g. at TLEP [**?**]. Similarly, the issue of Yukawa unification is affected by precise predictions for the anomalous quark mass dimension.

2. Preliminaries

Our starting point is the QCD Lagrangian with n_f quark flavours written in terms of renormalized fields, coupling constant g and quark massess m_f:

$$
\begin{aligned}
\mathcal{L}_0 = & -\frac{1}{4} Z_3 \, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} g Z_1^{3g} \, (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) \, (A_\mu \times A_\nu)^a \\
& - \frac{1}{4} g^2 Z_1^{4g} \, (A_\mu \times A_\nu)^2 - \frac{1}{2\xi_L} (\partial_\nu A_\mu)^2 + Z_3^c \, \partial_\nu \bar{c} \, (\partial_\nu c) + g Z_1^{ccg} \, \partial^\mu \bar{c} \, (A \times c) \\
& + Z_2 \sum_{f=1}^{n_f} \bar{\psi}^f \mathrm{i} \not{\partial} \, \psi^f + g Z_1^{\psi\psi g} \sum_{f=1}^{n_f} \bar{\psi}^f \not{A} \, \psi^f - Z_{\psi\psi} \sum_{f=1}^{n_f} m_f \, \bar{\psi}^f \, \psi^f ,
\end{aligned}
\tag{2.1}
$$

with bare gluon, quark and ghost fields related to the renormalized ones as follows:

$$A_0^{a\mu} = \sqrt{Z_3}\, A^{a\mu}, \ \ \psi_0^f = \sqrt{Z_2}\, \psi_0^f, \ \ c_0^a = \sqrt{Z_3^c}\, c^a. \tag{2.2}$$

The vertex Renormalization Constants (RCs)

$$Z_1^V, \quad V \in \{3\text{g},\ 4\text{g},\ \text{ccg},\ \psi\psi\text{g}\} \tag{2.3}$$

are to be chosen to renormalize 3-gluon, 4-gluon, ghost-ghost-gluon, quark-quark-gluon vertex functions respectively. The Slavnov-Taylor identities allows one to express all vertex RCs in terms of wave function RCs and an independent charge RC, $Z_g = \frac{g_0}{g}$:

$$Z_\xi = Z_3, \tag{2.4}$$

$$Z_g = \sqrt{Z_1^{4g}}\,(Z_3)^{-1}, \tag{2.5}$$

$$Z_g = Z_1^{3g}(Z_3)^{-3/2}, \tag{2.6}$$

$$Z_g = Z_1^{ccg}(Z_3)^{-1/2}(Z_3^c)^{-1}, \tag{2.7}$$

$$Z_g = Z_1^{\psi\psi g}(Z_3)^{-1/2}(Z_2)^{-1}. \tag{2.8}$$

Within the commonly accepted $\overline{\text{MS}}$ scheme RCs are independent of dimensional parameters (masses and momenta) and can be represented as follows

$$Z(h) = 1 + \sum_{i,j}^{1\le j\le i} Z_{ij} \frac{h^i}{\varepsilon^j}, \tag{2.9}$$

where $h = g^2/(16\pi^2)$ and the parameter ε is related to the continuous space time dimension D via $D = 4-2\varepsilon$. Given a RC $Z(h)$, the corresponding anomalous dimension is defined as

$$\gamma(h) = -\mu^2 \frac{\mathrm{d}\log Z(h)}{\mathrm{d}\mu^2} = \sum_{n=1}^{\infty} Z_{n,1}\, n\, h^n = -\sum_{n=0}^{\infty} (\gamma)_n\, h^{n+1}. \tag{2.10}$$

The anomalous dimension of the charge h is conventionally referred to as "QCD β-function"; equations (2.5-2.8) imply that all four expressions in the Table below

$\beta(h) =$			
$2\gamma_1^{ccg} - 2\,\gamma_3^c - \gamma_3$	$2\gamma_1^{\psi\psi g} - 2\,\gamma_2 - \gamma_3$	$2\,\gamma_1^{3g} - 3\,\gamma_3$	$\gamma_1^{4g} - 2\,\gamma_3$

Table 1: Four different representation the QCD β-function.

can be used to find the QCD β-function $\beta(h)$. In real calculations only the first or the second possibility is usually employed.

To calculate the quark mass anomalous dimension, γ_m, one needs to find the so-called quark mass renormalization constant, Z_m, which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m_f^0}{m_f} = \frac{Z_{\psi\psi}}{Z_2}. \tag{2.11}$$

The final formula for γ_m reads

$$\gamma_m = \gamma_{\psi\psi} - \gamma_2. \tag{2.12}$$

3. Five-loop running of the ghost field

As a first step towards five-loop QCD β-function we have computed the anomalous dimension of the ghost field

$$\gamma_3^c = -\sum_{i=0}^{\infty} (\gamma_3^c)_i h^{i+1}. \tag{3.1}$$

The anomalous dimension is known up to four loops from the works [**?**, **?**]. The new five-loop coefficient reads (in the Feynman gauge):

$$
\begin{aligned}
(\gamma_3^c)_4 = & -\frac{193301287}{2048} - \frac{19562145}{128}\zeta_3 - \frac{2060829}{128}\zeta_3^2 + \frac{1101573}{16}\zeta_4 \\
& + \frac{66632427}{128}\zeta_5 - \frac{36327825}{256}\zeta_6 - \frac{140900823}{512}\zeta_7 \\
& + n_f\left[\frac{633704171}{27648} + \frac{5166473}{144}\zeta_3 + \frac{233519}{64}\zeta_3^2 - \frac{764949}{32}\zeta_4 \right.\\
& \left. - \frac{32902291}{384}\zeta_5 + \frac{4123825}{128}\zeta_6 + \frac{14425075}{384}\zeta_7\right] \\
& + n_f^2\left[-\frac{1326547}{3456} - \frac{1739167}{864}\zeta_3 - \frac{2659}{6}\zeta_3^2 + \frac{13485}{8}\zeta_4 + \frac{8074}{9}\zeta_5 - \frac{16775}{12}\zeta_6\right] \\
& + n_f^3\left[-\frac{342895}{7776} - \frac{1211}{18}\zeta_3 - \frac{5}{2}\zeta_4 + \frac{284}{3}\zeta_5\right] + n_f^4\left[\frac{65}{108} + \frac{20}{27}\zeta_3 - \frac{4}{3}\zeta_4\right]
\end{aligned}
$$

Numerically ($a_s \equiv \frac{\alpha_s}{\pi} \equiv 4\,h$):

$$\gamma_3^c(n_f = 3) = \frac{3}{8}\left(a_s + 2.4375\,a_s^2 + 4.8867\,a_s^3 + 19.980\,a_s^4 + 122.246\,a_s^5\right).$$

For generic n_f:

$$
\begin{aligned}
\gamma_3^c = & \frac{3}{8}\left\{a_s + a_s^2\left(3.063 - 0.208 n_f\right) + a_s^3\left(10.556 - 1.768 n_f - 0.0405 n_f^2\right)\right. \\
& + a_s^4\left(49.325 - 10.957 n_f + 0.36562 n_f^2 + 0.0087 n_f^3\right) \\
& \left. + a_s^5\left(283.632 - 70.979 n_f + 5.498 n_f^2 + 0.0769 n_f^3 - 0.000128038 n_f^4\right)\right\}.
\end{aligned}
$$

It is instructive to observe that significant cancellations between n_f^0 and n_f^1 terms for the values of n_f around 3 or so persist also at five-loop order.

4. Five-loop quark mass anomalous dimension

The quark mass anomalous dimension is known to four loops from the works [**?**, **?**]. Our result for the five-loop coefficient in

$$\gamma_m = -\sum_{i=0}^{\infty} (\gamma_m)_i \, h^{i+1} \tag{4.1}$$

reads (note that γ^m is a gauge independent quantity):

$$
\begin{aligned}
(\gamma_m)_4 = \; & \frac{99512327}{162} + \frac{46402466}{243}\zeta_3 + 96800\,\zeta_3^2 - \frac{698126}{9}\zeta_4 \\
& - \frac{231757160}{243}\zeta_5 + 242000\,\zeta_6 + 412720\,\zeta_7 \\
+ \; & n_f \left[-\frac{150736283}{1458} - \frac{12538016}{81}\zeta_3 - \frac{75680}{9}\zeta_3^2 + \frac{2038742}{27}\zeta_4 \right. \\
& \left. + \frac{49876180}{243}\zeta_5 - \frac{638000}{9}\zeta_6 - \frac{1820000}{27}\zeta_7 \right] \\
+ \; & n_f^2 \left[\frac{1320742}{729} + \frac{2010824}{243}\zeta_3 + \frac{46400}{27}\zeta_3^2 - \frac{166300}{27}\zeta_4 - \frac{264040}{81}\zeta_5 + \frac{92000}{27}\zeta_6 \right] \\
+ \; & \boxed{ n_f^3 \left[\tfrac{91865}{1458} + \tfrac{12848}{81}\zeta_3 + \tfrac{448}{9}\zeta_4 - \tfrac{5120}{27}\zeta_5 \right] + n_f^4 \left[-\tfrac{260}{243} - \tfrac{320}{243}\zeta_3 + \tfrac{64}{27}\zeta_4 \right] }.
\end{aligned}
$$

Note that the boxed terms are in full agreement with predictions made on the basis of the $1/n_f$ method in [?, ?].

In numerical form γ_m reads

$$
\begin{aligned}
\gamma_m = & - a_s - a_s^2 (4.20833 - 0.138889 n_f) \\
& - a_s^3 \left(19.5156 - 2.28412 n_f - 0.0270062 n_f^2\right) \\
& - a_s^4 \left(98.9434 - 19.1075 n_f + 0.276163 n_f^2 + 0.00579322 n_f^3\right) \\
& - a_s^5 \left(559.71 - 143.6\, n_f + 7.4824 n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4\right)
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
\gamma_m \underset{n_f=3}{==} & - a_s - 3.79167\, a_s^2 - 12.4202\, a_s^3 - 44.2629\, a_s^4 - 198.907\, a_s^5, \\
g_m \underset{n_f=4}{==} & - a_s - 3.65278\, a_s^2 - 9.94704\, a_s^3 - 27.3029\, a_s^4 - 111.59\, a_s^5, \\
g_m \underset{n_f=5}{==} & - a_s - 3.51389\, a_s^2 - 7.41986\, a_s^3 - 11.0343\, a_s^4 - 41.8205\, a_s^5, \\
\gamma_m \underset{n_f=6}{==} & - a_s - 3.375\, a_s^2 - 4.83867\, a_s^3 + 4.50817\, a_s^4 + 9.76016\, a_s^5.
\end{aligned}
\tag{4.3}
$$

Inspection of eqs. (4.3) shows quite moderate growth of the series in a_s appearing in the quark mass anomalous dimension at various values of active quark flavours (recall that even for scales as small as 2 GeV $a_s \equiv \frac{\alpha_s}{\pi} \approx 0.1$).

5. Technical tools

As is well-known evaluation of any L-loop anomalous dimension in the $\overline{\text{MS}}$-scheme can be reduced, with the help of the R^*-operation [?], to the evaluation of some $L-1$-loop massless propagators [?]. In our case $L=5$ and we need to be able effectively compute a host of four-loop massless propagators (that is p-integrals). These, in turn, can be reduced to 28 master integrals. The reduction is based on evaluating sufficiently many terms of the $1/D$ expansion [?] of the corresponding coefficient functions [?]. The master integrals are known analytically from [?, ?].

Note that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using parallel MPI-based [**?**] as well as thread-based [**?**] versions of FORM [**?**].

6. Conclusions

Unfortunately, at the moment it is not possible to take self-consistently into account our five-loop result for γ_m for the quark mass running: this requires the knowledge of the five-loop QCD β-function. The latter problem is under calculation in our group.

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