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Anomalous magnetic moment with heavy virtual leptons

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Abstract

We compute the contributions to the electron and muon anomalous magnetic moment induced by heavy leptons up to four-loop order. Asymptotic expansion is applied to obtain three analytic expansion terms which show rapid convergence.

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1 Introduction

Since decades the anomalous magnetic moments of electron and muon, a_e and a_μ, are used to perform precision tests of QED.[1] In fact, in the case of the electron the experimental measurements and theoretical predictions have reached a precision which allows for the most precise extraction of the fine structure constant α. In contrast to a_e there is a sizable hadronic contribution to a_μ which involves as input measurements of the total cross section $\sigma(e^+e^- \to \text{hadrons})$ at low energies. Although all ingredients are measured and computed to high precision there is a discrepancy of about 3σ between the measured and predicted value for a_μ [4–6]. In this context it is interesting to mention that this difference is of the same order of magnitude as the four-loop QED contribution which to date has only been computed by one group [7]. Thus, it is important to provide an independent cross check for this ingredient. First results have been obtained in Ref. [8–10]. In particular, in Ref. [10] the contribution from Feynman diagrams containing two or three closed electron loops have been computed. In this letter we provide a further step

[1] See Refs. [1–3] for comprehensive reviews.

towards the full four-loop QED corrections to a_e and a_μ and compute the part induced by heavy leptons. In the case of the muon this means that Feynman diagrams have to be considered which contain closed tau loops and both closed muon and tau loops are present for a_e. Such contributions appear for the first time at two-loop order (cf. Fig. 1) and have been computed in Refs. [11–13]. Also the three-loop result is known in analytic form for arbitrary lepton masses [14–19] (see also [12, 13]). At four loops, however, only numerical results are available [7, 20–22]. We want to cross-check these results using a different method which leads to analytic results for a_e and a_μ. It is based on asymptotic expansion [23] in the ratio of the light and heavy lepton mass, M_l and M_h, which leads to a factorization of the two-scale integrals into simpler ones with at most one mass scale. The latter can be computed analytically. We have computed three terms of the expansion in M_l^2/M_h^2.

For the perturbative expansion of the QED corrections to a_e and a_μ we take over the commonly used notation from Refs. [7, 22] and write ($l = e, \mu$)

$$a_l = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_l^{(2n)}, \tag{1}$$

where $a_l^{(2n)}$ can be written in the form

$$\begin{aligned} a_e^{(2n)} &= A_{1,e}^{(2n)} + A_{2,e}^{(2n)}(M_e/M_\mu) + A_{2,e}^{(2n)}(M_e/M_\tau) + A_{3,e}^{(2n)}(M_e/M_\mu, M_e/M_\tau)\,, \\ a_\mu^{(2n)} &= A_{1,\mu}^{(2n)} + A_{2,\mu}^{(2n)}(M_\mu/M_e) + A_{2,\mu}^{(2n)}(M_\mu/M_\tau) + A_{3,\mu}^{(2n)}(M_\mu/M_e, M_\mu/M_\tau)\,. \end{aligned} \tag{2}$$

In this paper we compute $A_{2,e}^{(2n)}(M_e/M_\mu)$, $A_{2,e}^{(2n)}(M_e/M_\tau)$, $A_{3,e}^{(2n)}(M_e/M_\mu, M_e/M_\tau)$ and $A_{2,\mu}^{(2n)}(M_\mu/M_\tau)$ to four-loop order. We have also computed the corresponding two- and three-loop results and found complete agreement with the literature.

Before starting the actual calculation let us consider the parametrical size of our corrections. Actually, the heavy-lepton contribution decouples in the limit $M_h \to \infty$ and leads to a M_l^2/M_h^2 suppression. Thus the four-loop corrections to a_μ have the form $(\alpha/\pi)^4 \times M_\mu^2/M_\tau^2$ where M_μ^2/M_τ^2 is of order 10^{-3}. On the other hand we have $\alpha/\pi \approx 2 \cdot 10^{-3}$ which is of the same order of magnitude. Thus, from the parametric point of view the four-loop corrections induced by heavy leptons could be of the same order as the five-loop results obtained in Ref. [7]. Note, however, that in practice the contributions involving electron loops are large ($A_{2,e}^{(10)}(M_e/M_\mu)$ is of order 10^3) whereas the heavy-lepton contributions have coefficients which are at most of order 10.

In the case of a_e the ratio of the lepton masses is much smaller than for a_μ (Note that $M_e^2/M_\mu^2 = \mathcal{O}(10^{-5})$, $M_e^2/M_\tau^2 = \mathcal{O}(10^{-8})$) and thus the corresponding corrections are less relevant. Nevertheless, for completeness we provide also those results.

The remainder of the paper is structured as follows: In the next Section we briefly discuss some technical details which are important for our calculation. Section 3 is devoted to the presentation and discussion of the results. In particular, we compare to the numerical

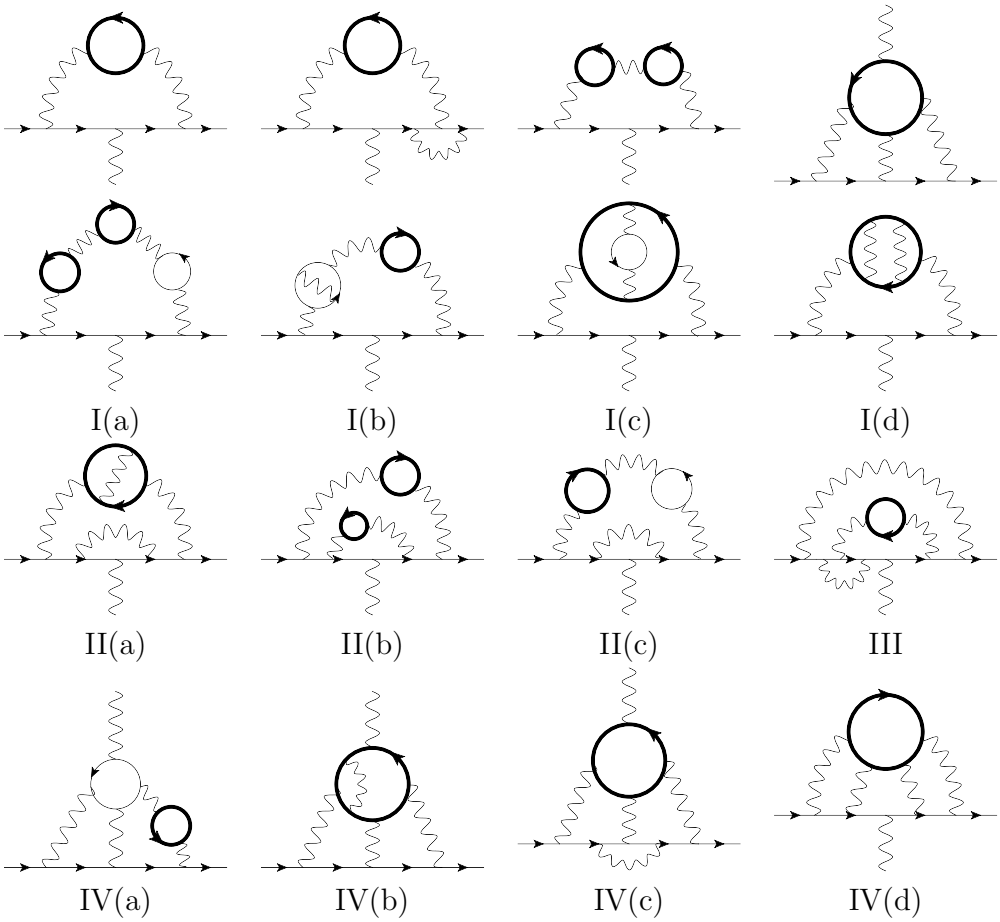


Figure 1: Sample Feynman diagrams contributing to the electron and muon $(g-2)$ containing a heavy lepton at two, three and four loops. Thin and thick solid lines represent light and heavy leptons, respectively, and wavy lines denote photons. The symbols below the four-loop diagrams label the individual diagram classes and are taken over from Refs. [7, 22].

results of Refs. [7, 22]. We conclude in Section 4. In the Appendix we present results for the on-shell counterterms for the fine structure constant, the lepton mass and the lepton wave function.

2 Some technical details

Typical Feynman diagrams to be considered for the heavy-lepton contribution of a_e and a_μ are shown in Fig. 1. At two-loop order only one diagram has to be considered.[2] At three loop-order 60 and at four loops 1169 Feynman diagrams are generated. In the following discussion we denote the heavy lepton mass by M_h and the light one by M_l.

For the generation of the diagrams we use `QGRAF` [25] and transform the amplitudes with the help of `q2e` [26, 27] to a `FORM` [28] readable output.

In a next step we apply `exp` [26, 27] to perform an asymptotic expansion for $M_h \gg M_l$. At two-loop order [see Fig. 2(a)] this leads to two so-called sub-diagrams which have to be Taylor-expanded in their external momenta. The first sub-diagram is given by the whole two-loop diagram which, after expansion, leads to two-loop vacuum integrals. The second contribution consists of a product of two one-loop diagrams. After expanding the one-loop vacuum integral in the external momentum one has to insert the result in the remaining one-loop on-shell integral and integrate over the second loop momentum. The described procedure is illustrated in the second line of Fig. 2(a). Fig. 2(b) shows a four-loop example which demonstrates the typical situation at this order: the original four-loop two-scale integral is transformed to a sum of products of N-loop vacuum integrals with scale M_h and $(4-N)$-loop on-shell integrals with $q^2 = M_l^2$ where $N = 1, 2, 3$ or 4 and q is the momentum flowing through the external lepton line. All integrals only contain one mass scale and are thus significantly simpler than the original one.

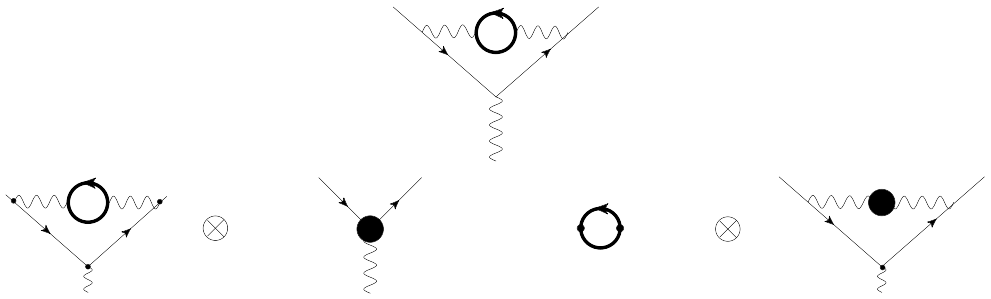
Both vacuum and on-shell integrals are reduced to master integrals with the help of `FIRE` [29, 30].[3] The master integrals are all known analytically and are taken from Refs. [31–40] and Refs. [41–43], respectively.

We renormalize our results in the on-shell scheme. For this purpose we need the counterterm for the fine structure constant, the (light) lepton mass and lepton wave function to three loops. The corresponding analytic results for the case of a massless second lepton loop can be found in Ref. [44], Refs. [42, 45–47] and [47, 48], respectively. In our case the opposite limit of a heavy lepton is needed which we computed ourselves using the rules of asymptotic expansion as described above. The analytic expressions are presented in the Appendix for completeness. Our results for the leading term of the lepton mass counterterm agrees with Ref. [49] and the one for the charge counterterm is easily obtained from the general expression presented in Ref. [10]. To our knowledge the three-loop result for the on-shell wave function renormalization constant is new.

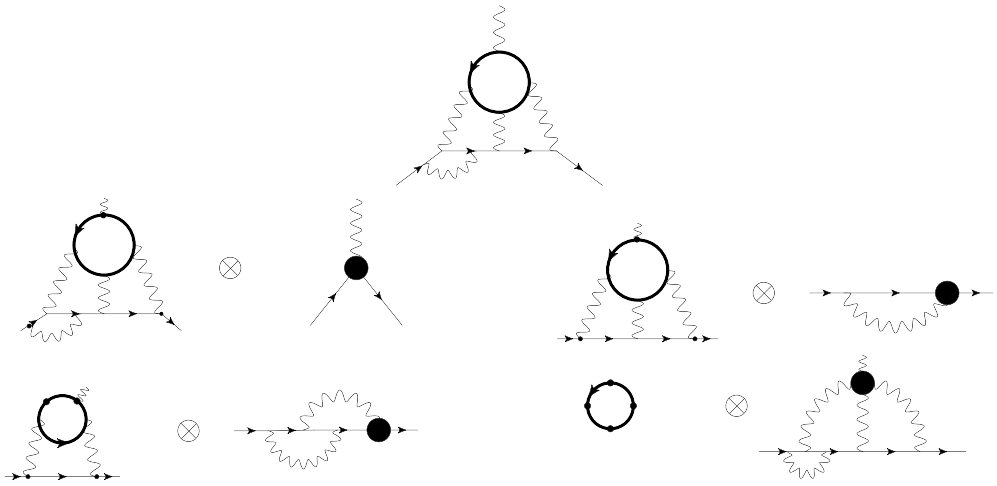
In addition, the heavy-lepton mass has to be renormalized in the two- and three-loop expression. The corresponding two-loop counterterm can be found in Refs. [50, 51]. Note that the two-loop counterterm which has to be inserted into the two-loop vertex diagram

[2] Note that the contribution where the photon couples to the closed lepton loop vanishes due to Furry's theorem [24].

[3] We thank A.V. Smirnov and V.A. Smirnov for allowing us to use the unpublished `C++` version of `FIRE`.



(a)



(b)

Figure 2: Graphical examples for the application of the asymptotic expansion at two (a) and four (b) loops. Thick solid, thin solid and wavy lines represent heavy and light leptons and photons, respectively. In (b) only four representative sub-diagrams are shown. Altogether there are eight contributions.

of Fig. 1 involves contributions with a closed light lepton loop. The expansion of this contribution in $M_l \ll M_h$ contains both even and odd powers in M_l/M_h which is the reason for the occurrence of odd expansion terms in $A_{2,\mu}^{(8)}$ (cf. Section 3).

There are several checks on the correctness of our result. Besides the obvious ones like finiteness we have performed two independent calculations. In particular, two independent routines for the decomposition of the scalar products in the numerator and the preparation of the `FIRE` input has been written. Furthermore for our calculation we have used general QED gauge parameter up to linear terms in ξ and have checked that the final result of the

leading term in the inverse heavy lepton expansion, i.e. the one proportional to M_l^2/M_h^2, is ξ-independent. Due to the complexity of the calculation we have used Feynman gauge for the higher order expansion terms.

3 Results and discussion

Let us in a first step present the analytic results of our calculation. The four-loop contribution to a_μ from Feynman diagrams involving a virtual tau lepton loop is given by

$$
\begin{aligned}
A_{2,\mu}^{(8)}(M_\mu/M_\tau) \;=\; & \left(\frac{M_\mu}{M_\tau}\right)^2 \left(\frac{37448693521}{2286144000} + \frac{89603}{16200}P_4 + \frac{52}{675}P_5 + \frac{4\pi^2\zeta_3}{15} + \frac{5771\ln(2)\pi^4}{32400}\right. \\
& \quad - \frac{3851\pi^2}{3600} - \frac{25307\zeta_5}{1440} - \frac{37600399\pi^4}{27216000} + \frac{35590996657\zeta_3}{508032000} \\
& \quad \left. + \ln\frac{M_\mu^2}{M_\tau^2}\left(-\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2}\right) + \frac{359}{1080}\ln^2\frac{M_\mu^2}{M_\tau^2}\right) \\
& + \left(\frac{M_\mu}{M_\tau}\right)^3 \frac{\pi^2}{90} \\
& + \left(\frac{M_\mu}{M_\tau}\right)^4 \left(\frac{392783023945426851403}{730774466976153600000} - \frac{3355249339331\pi^4}{2575112601600}\right. \\
& \quad + \frac{74184592369}{14306181120}P_4 + \frac{557}{9450}P_5 \\
& \quad - \frac{378681587\pi^2}{114307200} - \frac{652\ln(2)\pi^2}{1215} + \frac{26783\ln(2)\pi^4}{226800} \\
& \quad + \frac{7257500829155234 17\zeta_3}{10310750856806400} + \frac{66211\pi^2\zeta_3}{22680} - \frac{425983\zeta_5}{30240} \\
& \quad + \ln\frac{M_\mu^2}{M_\tau^2}\left(-\frac{1922512966823}{1229031014400} + \frac{47899\pi^2}{816480} + \frac{81782993\zeta_3}{123863040}\right) \\
& \quad \left. + \frac{193032971}{457228800}\ln^2\frac{M_\mu^2}{M_\tau^2} - \frac{24037}{362880}\ln^3\frac{M_\mu^2}{M_\tau^2}\right) \\
& + \left(\frac{M_\mu}{M_\tau}\right)^5 \left(\frac{2671\pi^2}{176400} + \frac{\pi^2}{140}\ln\frac{M_\mu^2}{M_\tau^2}\right) \\
& + \left(\frac{M_\mu}{M_\tau}\right)^6 \left(\frac{3262922004554663119 53239}{4974581098834034688000} + \frac{4785889811617\pi^2}{1234517760000}\right. \\
& \quad + \frac{989648650006997}{191294078976000}P_4 + \frac{7001}{207900}P_5 \\
& \quad - \frac{27903657664078117\pi^4}{11477644738560000} + \frac{711883\ln(2)\pi^4}{9979200} - \frac{148\ln(2)\pi^2}{315} \\
& \quad \left. + \frac{6446695611351419899\zeta_3}{66315280711680000} - \frac{18533\pi^2\zeta_3}{6048} + \frac{179971\zeta_5}{24192}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\frac{M_\mu^2}{M_\tau^2}\bigg(-\frac{2631561259843654279}{132735349555200000}+\frac{17955349\pi^2}{489888000}\\
&\qquad+\frac{314284167899\zeta_3}{19818086400}\bigg)\\
&+\frac{22710352067}{58786560000}\ln^2\frac{M_\mu^2}{M_\tau^2}-\frac{101799017}{979776000}\ln^3\frac{M_\mu^2}{M_\tau^2}\bigg)\\
&+\left(\frac{M_\mu}{M_\tau}\right)^7\left(\frac{79\pi^2}{15120}+\frac{\pi^2}{60}\ln\frac{M_\mu^2}{M_\tau^2}\right)+\mathcal{O}\left(\left(\frac{M_\mu}{M_\tau}\right)^8\right)\\
\approx\;& 0.0421670+0.0003257+0.0000015\,, \qquad (3)
\end{aligned}
$$

where $P_4 = 24a_4 + \ln^4(2) - \ln^2(2)\pi^2$, $P_5 = 120a_5 - \ln^5(2) + \frac{5}{3}\ln^3(2)\pi^2$, $a_n = \mathrm{Li}_n(1/2)$ and ζ_n is Riemann's zeta function. In the last line of Eq. (3) the analytic expression has been evaluated numerically using $M_\mu/M_\tau = 5.94649(54)\cdot 10^{-2}$ [52]. Furthermore the contributions from $(M_\mu/M_\tau)^n$ and $(M_\mu/M_\tau)^{n+1}$ $(n = 2, 4, 6)$ have been combined. One observes a rapid convergence of the series in M_μ/M_τ which suggests that with each additional order one gains two significant digits. To be conservative we take 10% of the last term in Eq. (3) as error estimate which leads to our final result

$$
A_{2,\mu}^{(8)}(M_\mu/M_\tau) \;\approx\; 0.0424941(2)(53)\,, \qquad (4)
$$

where the second uncertainty reflects the error in the input quantity M_μ/M_τ. The result in (4) agrees with the one from Ref. [7] $A_{2,\mu}^{(8)}(M_\mu/M_\tau) = 0.04234(12)$, however, our number is significantly more precise.

For completeness we also provide the numerical results for the two- and three-loop contributions which read[4]

$$
\begin{aligned}
A_{2,\mu}^{(4)}(M_\mu/M_\tau) &= 7.8079(14)\cdot 10^{-5}\,,\\
A_{2,\mu}^{(6)}(M_\mu/M_\tau) &= 3.6063(12)\cdot 10^{-4}\,. \qquad (5)
\end{aligned}
$$

It is interesting to note that the three-loop coefficient is only a factor of five larger than the two-loop one whereas $A_{2,\mu}^{(8)}(M_\mu/M_\tau)$ is about 100 times larger than $A_{2,\mu}^{(6)}(M_\mu/M_\tau)$. Using $\alpha = 1/137.035999174$ [22] one finally obtains for the τ-loop contribution to a_μ

$$
10^{11}\times a_\mu\Big|_{\tau\text{loops}} = 42.13 + 0.45 + 0.12\,, \qquad (6)
$$

where the numbers on the right-hand side correspond to the two-, three- and four-loop contribution. The numbers in Eq. (6) have to be compared with the universal contributions contained in $A_{1,\mu}$ which read [7]

$$
10^{11}\times a_\mu\Big|_{\text{univ.}} = 116\,140\,973.21 - 177\,230.51 + 1\,480.42 - 5.56 + 0.06\,, \qquad (7)
$$

group	$10^2 \cdot A_{2,\mu}^{(8)}(M_\mu/M_\tau)$	
	this work	[7]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

Table 1: Mass-dependent corrections to a_μ at four-loop order as obtained in this paper and the comparison with Refs. [7]. The uncertainties assigned to our numbers correspond to 10% of the highest available expansion terms, i.e., the ones of order $(M_\mu/M_\tau)^6$ and $(M_\mu/M_\tau)^7$. Uncertainties from the muon and tau lepton mass are not shown.

where the individual terms on the right-hand side represent the results from one to five loops.

The detailed comparison with Tab. I of Ref. [7] is shown in Tab. 1 where our result is split into eight different groups. In the first column the notation of [7] is used to indicate the contributions which have to be summed[5] in order to compare with our numbers. Within the numerical uncertainties we observe good agreement. Note, however, that our results based on asymptotic expansion provide at least two more significant digits.

Let us mention that the analytic result for the leading order expansion term of case IV(b) agrees with the result presented in Ref. [53] which has been obtained by transforming the result of Ref. [54] to QED.

Let us next turn to the anomalous magnetic moment of the electron. The numerical values for the two- and three-loop contributions read

$$\begin{aligned} A_{2,e}^{(4)}(M_e/M_\mu) &= 5.19738668(26) \cdot 10^{-7}\,, \\ A_{2,e}^{(6)}(M_e/M_\mu) &= -7.37394162(27) \cdot 10^{-6}\,, \\ A_{2,e}^{(4)}(M_e/M_\tau) &= 1.83798(33) \cdot 10^{-9}\,, \\ A_{2,e}^{(6)}(M_e/M_\tau) &= -6.5830(11) \cdot 10^{-8}\,, \end{aligned} \tag{8}$$

where $M_e/M_\mu = 4.83633166(12) \cdot 10^{-3}$ and $M_e/M_\tau = 2.87592(26) \cdot 10^{-4}$ [52] have been used. Inserting these values into Eq. (3) leads to the following four-loop results

$$A_{2,e}^{(8)}(M_e/M_\mu) \approx (9.161259603 + 0.000711078 + 2.2 \cdot 10^{-8}) \cdot 10^{-4}$$

[4]Since analytic expressions are available the uncertainties for the two- and three-loop results are due to the errors in the lepton masses.

[5]We add the uncertainties of Ref. [7] in quadrature when adding results from different groups.

group	$10^4 \cdot A_{2,e}^{(8)}(M_e/M_\mu)$	
	this work	[22]
I(a)	0.002264474414(6)	0.00226456(14)
I(b) + I(c) + II(b) + II(c)	−1.21390182678(6)	−1.21386(24)
I(d)	0.02472687590(2)	0.024725(7)
III	8.1715251555(1)	8.1792(95)
II(a) + IV(d)	−2.6414355180(7)	−2.642(12)
IV(a)	6.3578810372(3)	6.3583(44)
IV(b)	0.4157367168(5)	0.4105(93)
IV(c)	−1.954826212(2)	−1.897(64)

Table 2: Muon mass dependent corrections to a_e at four-loop order as obtained in this paper and the comparison with Refs. [22]. The uncertainties assigned to our numbers correspond to 10% of the highest available expansion terms, i.e., the ones of order $(M_e/M_\mu)^6$ and $(M_e/M_\mu)^7$. Uncertainties from the electron and muon mass are not shown.

$$\begin{aligned} &\approx 9.161970703(2)(372) \cdot 10^{-4}\,, \\ A_{2,e}^{(8)}(M_e/M_\tau) &\approx (7.42923268609971 + 2.75209424 \cdot 10^{-6} + 3.2 \cdot 10^{-13}) \cdot 10^{-6} \\ &\approx 7.42924(0)(118) \cdot 10^{-6}\,, \end{aligned} \tag{9}$$

where the uncertainty has again been estimated by 10% of the third term in the expansion and the parameter uncertainty is displayed separately. In Ref. [22] one finds the results[6] $A_{2,e}^{(8)}(M_e/M_\mu) = 9.222(66) \times 10^{-4}$ and $A_{2,e}^{(8)}(M_e/M_\tau) = 7.38(12) \times 10^{-6}$ which agree with our numerical values.

As far as the growth of the coefficients is concerned we observe the same pattern as for the muon: there is about one order of magnitude between two and three loops and a factor 100 between three and four loops. Note, however, that the three-loop result is negative for a_e.

In Tabs. 2 and 3 our results are shown for the individual classes of Feynman diagrams. Due to the smallness of the expansion parameters our method provides an accuracy of at least eight significant digits. The comparison with the results of Ref. [22] demonstrates good overall agreement. Note that we have applied the methods of Refs. [55–57], where four- and five-loop contributions to a_μ from polarization function insertions have been computed, to cross check our result for case I(d).

The quantity $A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau)$ has a more complicated structure since two different heavy masses are present. However, due to the strong hierarchy $M_\tau \gg M_\mu \gg M_e$ it is possible to apply the asymptotic expansion successively which again leads to one-scale

[6]Note that the entry for $A_{2,e}^{(8)}(M_e/M_\tau)$ in Tab. I of Ref. [22] should be multiplied by a factor 1/100. This misprint has been confirmed by the authors of Ref. [22].

group	$10^6 \cdot A_{2,e}^{(8)}(M_e/M_\tau)$	
	this work	[22]
I(a)	0.0008024665425029(1)	0.00080233(5)
I(b) + I(c) + II(b) + II(c)	−0.9458168451136621(8)	−0.94506(25)
I(d)	0.0087455060010553(1)	0.008744(1)
III	6.059301961911502(2)	6.061(12)
II(a) + IV(d)	−1.372489352896281(9)	−1.3835(30)
IV(a)	4.510496216222387(2)	4.5117(69)
IV(b)	0.147081582099596(4)	0.1431(95)
IV(c)	−0.97888609657284(3)	−1.02(11)

Table 3: Tau lepton mass dependent corrections to a_e at four-loop order as obtained in this paper and the comparison with Refs. [22]. The uncertainties assigned to our numbers correspond to 10% of the highest available expansion terms, i.e., the ones of order $(M_e/M_\tau)^6$ and $(M_e/M_\tau)^7$. Uncertainties from the electron and tau lepton mass are not shown. The result of Ref. [22] for the contribution I(d) has been multiplied by 1/100 [see footnote after Eq. (9)].

vacuum and on-shell integrals. Our final result reads

$$
\begin{aligned}
A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau) \;=\; & \frac{M_e^2}{M_\tau^2}\Bigg(-\frac{3123671}{1458000} - \frac{\pi^2}{270} + \frac{\pi^4}{30} - \frac{19\zeta_3}{45} \\
& \quad + \ln\frac{M_\mu^2}{M_\tau^2}\left(\frac{271073}{291600} - \frac{3\zeta_3}{2}\right) + \frac{89}{810}\ln^2\frac{M_\mu^2}{M_\tau^2}\Bigg) \\
& + \frac{M_e^2 M_\mu}{M_\tau^3}\frac{\pi^2}{90} \\
& + \frac{M_e^2 M_\mu^2}{M_\tau^4}\Bigg(-\frac{1213316893}{5834430000} + \frac{\pi^4}{3150} + \frac{1294\zeta_3}{3675} - \frac{3}{280}\ln^3\frac{M_\mu^2}{M_\tau^2} \\
& \quad + \ln\frac{M_\mu^2}{M_\tau^2}\left(-\frac{9573107}{18522000} + \frac{\zeta_3}{70}\right) + \frac{130813}{1058400}\ln^2\frac{M_\mu^2}{M_\tau^2}\Bigg) \\
& + \frac{M_e^4}{M_\mu^2 M_\tau^2}\Bigg(\frac{3304933}{14580000} + \frac{88\pi^2}{6075} - \frac{107\zeta_3}{360} + \frac{2533}{40500}\ln\frac{M_\mu^2}{M_\tau^2} \\
& \quad + \ln\frac{M_e^2}{M_\mu^2}\left(-\frac{3239}{121500} - \frac{79}{1350}\ln\frac{M_\mu^2}{M_\tau^2}\right) - \frac{7}{8100}\ln^2\frac{M_e^2}{M_\mu^2}\Bigg) \\
& + \frac{M_e^4}{M_\tau^4}\Bigg(-\frac{19009349146181}{10081895040000} - \frac{37877173\zeta_3}{76204800} - \frac{79\pi^2}{58800} \\
& \quad - \frac{373}{40320}P_4 + \frac{280111\pi^4}{14515200} \\
& \quad + \ln\frac{M_e^2}{M_\mu^2}\Bigg(\frac{441068819}{1714608000} - \frac{33487}{2721600}\ln\frac{M_\mu^2}{M_\tau^2}
\end{aligned}
$$

$$+\frac{1423}{38880}\ln^2\frac{M_\mu^2}{M_\tau^2}-\frac{\pi^2}{420}\Bigg)$$
$$+\ln\frac{M_\mu^2}{M_\tau^2}\left(\frac{767814079}{750141000}-\frac{\pi^2}{420}-\frac{61849\zeta_3}{80640}\right.$$
$$\left.-\frac{3034811}{38102400}\ln^2\frac{M_\mu^2}{M_\tau^2}+\frac{1181}{40824}\ln^3\frac{M_\mu^2}{M_\tau^2}\right)$$
$$+\frac{M_e^2M_\mu^3}{M_\tau^5}\frac{\pi^2}{90}+\frac{M_e^4M_\mu}{M_\tau^5}\left(\frac{79\pi^2}{19600}+\frac{\pi^2}{140}\ln\frac{M_e^2}{M_\tau^2}\right). \qquad (10)$$

After inserting numerical values for the lepton masses one obtains

$$\begin{aligned} A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau) &\approx (7.4426 + 0.0261)\cdot 10^{-7} \\ &\approx 7.4687(26)(10)\cdot 10^{-7}, \end{aligned} \qquad (11)$$

which has to be compared with $A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau) = 7.465(18)\cdot 10^{-7}$ as obtained in Ref. [22]. Again good agreement is found, however, our analytic result is more precise by about an order of magnitude.

It is interesting to note that the three-loop coefficient which is given by

$$A_{3,e}^{(6)}(M_e/M_\mu, M_e/M_\tau) = 1.90982(34)\cdot 10^{-13}, \qquad (12)$$

is more than six orders of magnitude smaller than the four-loop one which is due to the fact that the leading term is suppressed by $M_e^4/(M_\mu^2M_\tau^2)$ whereas at four loops the suppression factor is only M_e^2/M_τ^2. Note, however, that the overall contribution is very small.

Similarly to the three-loop expression also the leading term of the four-loop contribution where three one-loop heavy lepton bubbles are inserted into the photon propagator (see class I(a) in Fig. 1) is of order $\mathcal{O}(M_e^2/(M_\mu^2M_\tau^2))$. Thus we compute for this contribution also the next term of the hard-mass procedure. It is given by

$$\delta A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau)\Big|_{I(a),M^6} =$$
$$\frac{M_e^4M_\mu^2}{M_\tau^6}\Bigg(-\frac{1032407}{187535250}+\frac{1303}{297675}\ln\frac{M_\mu^2}{M_\tau^2}+\frac{4}{945}\ln^2\frac{M_\mu^2}{M_\tau^2}$$
$$+\ln\frac{M_e^2}{M_\mu^2}\left(-\frac{1039}{297675}+\frac{4}{945}\ln\frac{M_\mu^2}{M_\tau^2}\right)\Bigg)$$
$$+\frac{M_e^6}{M_\mu^4M_\tau^2}\left(\frac{204569}{30870000}+\frac{166}{18375}\ln\frac{M_e^2}{M_\mu^2}+\frac{1}{350}\ln^2\frac{M_e^2}{M_\mu^2}\right)$$
$$+\frac{M_e^6}{M_\mu^2M_\tau^4}\left(\frac{959}{90000}+\frac{31}{5250}\ln\frac{M_e^2}{M_\mu^2}+\frac{1}{350}\ln^2\frac{M_e^2}{M_\mu^2}\right)$$

group	$10^7 \cdot A_{3,e}^{(8)}(M_e/M_\mu, M_e/M_\tau)$	
	this work	ref.
I(a)	0.00001199558(2)	0.000011994(1)
I(b) + I(c)	0.172910(24)	0.172874(21)
II(b) + II(c)	−1.64747(17)	−1.64866(67)
IV(a)	8.9432(25)	8.941(17)

Table 4: Lepton mass dependent corrections to a_e at four-loop order induced by diagrams which contain at the same time the muon and tau lepton. The results obtained in this paper are compared to the ones of Refs. [22]. The uncertainties assigned to our numbers correspond to 10% of the highest available expansion terms and uncertainties from the lepton masses are not shown.

$$
\begin{aligned}
&+\frac{M_e^6}{M_\tau^6}\left(\ln\frac{M_e^2}{M_\mu^2}\left(\frac{2735573}{187535250}-\frac{199}{297675}\ln\frac{M_\mu^2}{M_\tau^2}+\frac{2}{945}\ln^2\frac{M_\mu^2}{M_\tau^2}\right)\right.\\
&\qquad+\frac{8\zeta_3}{315}-\frac{118286321}{19691201250}+\frac{676036}{31255875}\ln\frac{M_\mu^2}{M_\tau^2}\\
&\qquad\left.+\frac{394}{99225}\ln^2\frac{M_\mu^2}{M_\tau^2}+\frac{2}{945}\ln^3\frac{M_\mu^2}{M_\tau^2}\right). \qquad (13)
\end{aligned}
$$

This term is included in the numerical values shown in Table 4 where our results are compared to the ones of Ref. [22]. The quality of the agreement is as in the previous cases.

4 Conclusions

Four-loop corrections induced by a heavy lepton to the anomalous magnetic moment of the electron and the muon have been computed. This includes tau lepton contributions to a_μ and contributions with virtual muons and tau leptons to a_e. With the help of an asymptotic expansion in the mass ratios we obtained analytic results. Their numerical evaluation leads to full agreement with the results of Refs. [7,22] which have been obtained with numerical methods. However, our results are more precise. Actually, the uncertainty is of the order of or even smaller than the one originating from the imprecise knowledge of the lepton masses. Due to the decoupling of heavy particles the heavy-lepton contributions are numerically quite small.

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Appendix: On-shell counterterms

In this appendix we provide analytic results for the on-shell counterterms for α, M_l and ψ_l where the latter stands for the lepton wave function. We concentrate on the contributions relevant for our calculation, i.e., the corrections originating from closed heavy lepton loops.

In the formulae below we use the notation $L_x = \ln(\mu^2/M_x^2)$ with $x = e, \mu, \tau$ and $a_4 = \mathrm{Li}_4(1/2)$ and mark the contributions from closed electron, muon and tau loops by the labels $n_e = 1$, $n_\mu = 1$ and $n_\tau = 1$.

In our calculation we renormalize the coupling constant in a first step in the $\overline{\mathrm{MS}}$ scheme and switch to the on-shell scheme after having obtained a finite result. The relation between the fine structure constant defined in the $\overline{\mathrm{MS}}$ scheme, $\bar{\alpha}(\mu) \equiv \bar{\alpha}$, and the corresponding on-shell quantity reads

$$
\begin{aligned}
\frac{\bar{\alpha}}{\alpha} &= 1 + \frac{\alpha}{\pi} \sum_{i=e,\mu,\tau} \frac{L_i n_i}{3} + \left(\frac{\alpha}{\pi}\right)^2 \left[\left(\sum_{i=e,\mu,\tau} \frac{L_i n_i}{3} \right)^2 + \sum_{i=e,\mu,\tau} \left(\frac{15}{16} + \frac{L_i}{4} \right) n_i \right] \\
&\quad + \left(\frac{\alpha}{\pi}\right)^3 \left[\left(\sum_{i=e,\mu,\tau} \frac{L_i n_i}{3} \right)^3 + \sum_{\substack{i,\,j\,=\,e,\,\mu,\,\tau \\ M_i\,<\,M_j}} n_i n_j \left(-\frac{311}{1296} - \frac{\pi^2}{18} + L_i \left(\frac{15}{16} + \frac{5 L_j}{12} \right) \right.\right. \\
&\qquad + \frac{23 L_j}{144} + \frac{\pi^2}{6} \left(\frac{M_i}{M_j}\right) + \left(-\frac{167}{150} + \frac{L_i}{45} - \frac{L_j}{45} \right) \left(\frac{M_i}{M_j}\right)^2 + \frac{\pi^2}{6} \left(\frac{M_i}{M_j}\right)^3 \\
&\qquad + \left(-\frac{23353331}{37044000} - \frac{29 L_i^2}{420} + L_i \left(-\frac{28967}{88200} + \frac{29 L_j}{210} \right) \right. \\
&\qquad\quad \left. + \frac{28967 L_j}{88200} - \frac{29 L_j^2}{420} - \frac{\pi^2}{18} \right) \left(\frac{M_i}{M_j}\right)^4 \\
&\qquad \left. \left. + \left(\frac{5288963}{62511750} + \frac{2 L_i^2}{315} + L_i \left(\frac{4609}{99225} - \frac{4 L_j}{315} \right) - \frac{4609 L_j}{99225} + \frac{2 L_j^2}{315} \right) \left(\frac{M_i}{M_j}\right)^6 \right) \right] \\
&\quad + \sum_{i=e,\mu,\tau} n_i \left(\frac{77}{576} - \frac{L_i}{32} + \frac{5\pi^2}{24} - \frac{\ln(2)\pi^2}{3} + \frac{\zeta_3}{192} \right)
\end{aligned}
$$

$$+\sum_{i=e,\mu,\tau} n_i^2 \left(-\frac{695}{648}+\frac{79L_i}{144}+\frac{5L_i^2}{24}+\frac{\pi^2}{9}+\frac{7\zeta_3}{64}\right)\Bigg], \tag{14}$$

where terms up to $\mathcal{O}(M_i^8/M_j^8)$ are included. In the case of the heavy lepton contributions to a_e this formula can immediately be applied, in the case of a_μ one has to set $n_e = 0$.

The bare and on-shell renormalized lepton mass and wave function are related by

$$\begin{aligned} M_l^{\rm bare} &= Z_{m,l}^{\rm OS}\, M_l\,, \\ \psi_l^{\rm bare} &= Z_{2,l}^{\rm OS}\, \psi_l\,, \end{aligned} \tag{15}$$

where the renormalization constants for the muon mass and wave function are given by

$$\begin{aligned} Z_{m,\mu}^{\rm OS} &= 1+\frac{\bar\alpha}{\pi}\Bigg[-1-\frac{3}{4\epsilon}-\frac{3L_\mu}{4}+\epsilon\left(-2-L_\mu-\frac{3L_\mu^2}{8}-\frac{\pi^2}{16}\right) \\ &\quad +\epsilon^2\left(-4-\frac{L_\mu^2}{2}-\frac{L_\mu^3}{8}-\frac{\pi^2}{12}+L_\mu\left(-2-\frac{\pi^2}{16}\right)+\frac{\zeta_3}{4}\right) \\ &\quad +\epsilon^3\Bigg(-8-\frac{L_\mu^3}{6}-\frac{L_\mu^4}{32}-\frac{\pi^2}{6}-\frac{3\pi^4}{640}+L_\mu^2\left(-1-\frac{\pi^2}{32}\right) \\ &\qquad +L_\mu\left(-4-\frac{\pi^2}{12}+\frac{\zeta_3}{4}\right)+\frac{\zeta_3}{3}\Bigg)\Bigg] \\ &\quad +\left(\frac{\bar\alpha}{\pi}\right)^2\Bigg[\frac{1}{\epsilon^2}\left(\frac{9}{32}-\frac{n_\mu}{8}-\frac{n_\tau}{8}\right)+\frac{1}{\epsilon}\left(\frac{45}{64}+\frac{9L_\mu}{16}+\frac{5n_\mu}{48}+\frac{5n_\tau}{48}\right) \\ &\quad +\frac{199}{128}+\frac{45L_\mu}{32}+\frac{9L_\mu^2}{16}-\frac{17\pi^2}{64}+\frac{\ln(2)\pi^2}{2}-\frac{3\zeta_3}{4} \\ &\quad +n_\mu\left(\frac{143}{96}+\frac{13L_\mu}{24}+\frac{L_\mu^2}{8}-\frac{\pi^2}{6}\right) \\ &\quad +n_\tau\Bigg(-\frac{89}{288}+\frac{13L_\tau}{24}+\frac{L_\mu L_\tau}{4}-\frac{L_\tau^2}{8}+\left(\frac{19}{150}+\frac{L_\mu}{15}-\frac{L_\tau}{15}\right)\left(\frac{M_\mu}{M_\tau}\right)^2 \\ &\quad +\left(\frac{1389}{78400}+\frac{9L_\mu}{560}-\frac{9L_\tau}{560}\right)\left(\frac{M_\mu}{M_\tau}\right)^4+\left(\frac{997}{198450}+\frac{2L_\mu}{315}-\frac{2L_\tau}{315}\right)\left(\frac{M_\mu}{M_\tau}\right)^6 \\ &\quad +\left(\frac{1229}{627264}+\frac{5L_\mu}{1584}-\frac{5L_\tau}{1584}\right)\left(\frac{M_\mu}{M_\tau}\right)^8\Bigg) \\ &\quad +\epsilon\Bigg(\frac{677}{256}-12a_4+\frac{45L_\mu^2}{32}+\frac{3L_\mu^3}{8}-\frac{\ln^4(2)}{2}-\frac{205\pi^2}{128}+3\ln(2)\pi^2 \\ &\qquad -\ln^2(2)\pi^2+\frac{7\pi^4}{40}-\frac{135\zeta_3}{16}+L_\mu\left(\frac{199}{64}-\frac{17\pi^2}{32}+\ln(2)\pi^2-\frac{3\zeta_3}{2}\right) \\ &\qquad +n_\mu\left(\frac{1133}{192}+\frac{17L_\mu^2}{24}+\frac{L_\mu^3}{8}-\frac{227\pi^2}{288}+\ln(2)\pi^2+L_\mu\left(\frac{175}{48}-\frac{5\pi^2}{16}\right)-\frac{7\zeta_3}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&+n_\tau\left(\frac{869}{1728}+\frac{7L_\tau}{144}+\frac{L_\mu^2L_\tau}{8}+\frac{3L_\tau^2}{8}-\frac{L_\tau^3}{8}+\frac{13\pi^2}{288}+L_\mu\left(\frac{L_\tau}{3}+\frac{L_\tau^2}{8}+\frac{\pi^2}{48}\right)\right.\\
&+\left(-\frac{701}{3375}+\frac{L_\mu^2}{30}+L_\mu\left(\frac{1}{30}+\frac{L_\tau}{15}\right)+\frac{11L_\tau}{50}-\frac{L_\tau^2}{10}\right)\left(\frac{M_\mu}{M_\tau}\right)^2\\
&+\left(\frac{20481}{10976000}+\frac{9L_\mu^2}{1120}+L_\mu\left(\frac{27}{1120}+\frac{9L_\tau}{560}\right)+\frac{111L_\tau}{9800}-\frac{27L_\tau^2}{1120}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&+\left(\frac{584149}{125023500}+\frac{L_\mu^2}{315}+L_\mu\left(\frac{5}{378}+\frac{2L_\tau}{315}\right)-\frac{631L_\tau}{198450}-\frac{L_\tau^2}{105}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\\
&\left.+\left(\frac{176625767}{60857153280}+\frac{5L_\mu^2}{3168}+L_\mu\left(\frac{1}{126}+\frac{5L_\tau}{1584}\right)-\frac{8821L_\tau}{2195424}-\frac{5L_\tau^2}{1056}\right)\left(\frac{M_\mu}{M_\tau}\right)^8\right)\right)\Bigg]\\
&+\left(\frac{\bar\alpha}{\pi}\right)^3\left[\frac{1}{\epsilon^3}\left(-\frac{9}{128}+\frac{3n_\mu}{32}-\frac{n_\mu^2}{36}-\frac{n_\mu n_\tau}{18}+\frac{3n_\tau}{32}-\frac{n_\tau^2}{36}\right)\right.\\
&+\frac{1}{\epsilon^2}\left(-\frac{63}{256}-\frac{27L_\mu}{128}+\left(-\frac{5}{192}+\frac{3L_\mu}{32}\right)n_\mu+\frac{5n_\mu^2}{216}\right.\\
&\left.+\frac{5n_\mu n_\tau}{108}+\left(-\frac{5}{192}+\frac{3L_\mu}{32}\right)n_\tau+\frac{5n_\tau^2}{216}\right)\\
&+\frac{1}{\epsilon}\left(-\frac{457}{512}-\frac{189L_\mu}{256}-\frac{81L_\mu^2}{256}+\frac{35n_\mu^2}{1296}\right.\\
&+\frac{35n_\mu n_\tau}{648}+\frac{35n_\tau^2}{1296}+\frac{111\pi^2}{512}-\frac{3\ln(2)\pi^2}{8}+\frac{9\zeta_3}{16}\\
&+n_\tau\left(\frac{79}{128}+\frac{3L_\mu^2}{64}+L_\mu\left(\frac{3}{64}-\frac{3L_\tau}{16}\right)-\frac{13L_\tau}{32}+\frac{3L_\tau^2}{32}+\frac{\pi^2}{128}-\frac{\zeta_3}{4}\right.\\
&+\left(-\frac{19}{200}-\frac{L_\mu}{20}+\frac{L_\tau}{20}\right)\left(\frac{M_\mu}{M_\tau}\right)^2+\left(-\frac{4167}{313600}-\frac{27L_\mu}{2240}+\frac{27L_\tau}{2240}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&\left.+\left(-\frac{997}{264600}-\frac{L_\mu}{210}+\frac{L_\tau}{210}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\right)\\
&\left.+n_\mu\left(-\frac{281}{384}-\frac{23L_\mu}{64}-\frac{3L_\mu^2}{64}+\frac{17\pi^2}{128}-\frac{\zeta_3}{4}\right)\right)\\
&-\frac{14225}{3072}-3a_4-\frac{567L_\mu^2}{512}-\frac{81L_\mu^3}{256}-\frac{\ln^4(2)}{8}-\frac{6037\pi^2}{3072}+5\ln(2)\pi^2+\frac{5\ln^2(2)\pi^2}{4}\\
&-\frac{73\pi^4}{480}+\frac{5\zeta_5}{8}+\frac{153\zeta_3}{128}-\frac{\pi^2\zeta_3}{16}+L_\mu\left(-\frac{1371}{512}+\frac{333\pi^2}{512}-\frac{9\ln(2)\pi^2}{8}+\frac{27\zeta_3}{16}\right)\\
&+n_\tau\left(\frac{6367}{2304}-4a_4+\frac{L_\mu^3}{64}+L_\mu^2\left(\frac{3}{128}-\frac{9L_\tau}{32}\right)-\frac{9L_\tau^2}{32}+\frac{3L_\tau^3}{32}\right.\\
&\left.-\frac{\ln^4(2)}{6}-\frac{23\pi^2}{768}+\frac{\ln^2(2)\pi^2}{6}+\frac{11\pi^4}{360}-\frac{29\zeta_3}{16}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+L_\mu\left(\frac{415}{384}-\frac{21L_\tau}{32}-\frac{\pi^2}{128}\right)+L_\tau\left(-\frac{1}{64}+\frac{5\pi^2}{24}-\frac{\ln(2)\pi^2}{3}-\frac{\zeta_3}{4}\right)\\
&+\left(\frac{M_\mu}{M_\tau}\right)^2\left(\frac{8153}{12150}-\frac{31L_\mu^2}{360}+L_\mu\left(-\frac{67}{4050}+\frac{L_\tau}{45}\right)\right.\\
&\qquad\left.-\frac{4349L_\tau}{16200}+\frac{23L_\tau^2}{360}+\frac{2\pi^2}{135}-\frac{77\zeta_3}{144}\right)\\
&+\left(\frac{M_\mu}{M_\tau}\right)^4\left(\frac{13231711}{98784000}-\frac{17L_\mu^2}{1120}+L_\mu\left(\frac{1907}{44800}-\frac{13L_\tau}{2240}\right)\right.\\
&\qquad\left.-\frac{517L_\tau}{6272}+\frac{47L_\tau^2}{2240}+\frac{\pi^2}{105}-\frac{147\zeta_3}{1024}\right)\\
&+\left(\frac{M_\mu}{M_\tau}\right)^6\left(\frac{3752184623}{90016920000}-\frac{8L_\mu^2}{2025}+L_\mu\left(\frac{925261}{35721000}-\frac{181L_\tau}{28350}\right)\right.\\
&\qquad\left.\left.-\frac{664523L_\tau}{17860500}+\frac{293L_\tau^2}{28350}+\frac{32\pi^2}{6075}-\frac{119\zeta_3}{1920}\right)\right)\\
&+n_\tau^2\left(\frac{1685}{7776}+\frac{31L_\tau}{108}-\frac{13L_\tau^2}{72}-\frac{L_\mu L_\tau^2}{12}+\frac{L_\tau^3}{18}-\frac{7\zeta_3}{18}\right.\\
&+\left(\frac{23}{324}-\frac{19L_\tau}{225}-\frac{2L_\mu L_\tau}{45}+\frac{2L_\tau^2}{45}\right)\left(\frac{M_\mu}{M_\tau}\right)^2\\
&+\left(-\frac{119}{24000}+L_\mu\left(-\frac{1}{100}-\frac{3L_\tau}{280}\right)-\frac{71L_\tau}{39200}+\frac{3L_\tau^2}{280}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&+\left.\left(-\frac{1594}{496125}+L_\mu\left(-\frac{1}{175}-\frac{4L_\tau}{945}\right)+\frac{704L_\tau}{297675}+\frac{4L_\tau^2}{945}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\right)\\
&+n_\mu n_\tau\left(-\frac{1327}{3888}-\frac{13L_\mu L_\tau}{36}-\frac{L_\mu^2 L_\tau}{12}+\frac{L_\tau^3}{36}+L_\tau\left(-\frac{5}{8}+\frac{\pi^2}{9}\right)+\frac{2\zeta_3}{9}\right.\\
&+\left(-\frac{1541}{3375}-\frac{11L_\mu}{45}-\frac{L_\mu^2}{45}+\frac{4L_\tau}{25}+\frac{L_\tau^2}{45}+\frac{4\pi^2}{135}\right)\left(\frac{M_\mu}{M_\tau}\right)^2\\
&+\left(-\frac{1833259}{16464000}-\frac{9L_\mu^2}{560}+L_\mu\left(-\frac{3551}{39200}+\frac{3L_\tau}{140}\right)+\frac{193L_\tau}{2450}-\frac{3L_\tau^2}{560}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&+\left.\left(-\frac{1997398}{93767625}-\frac{2L_\mu^2}{315}+L_\mu\left(-\frac{8021}{297675}+\frac{8L_\tau}{945}\right)+\frac{7024L_\tau}{297675}-\frac{2L_\tau^2}{945}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\right)\\
&+n_\mu\left(-\frac{5257}{2304}+\frac{8a_4}{3}-\frac{117L_\mu^2}{128}-\frac{11L_\mu^3}{64}+\frac{\ln^4(2)}{9}-\frac{1327\pi^2}{6912}+\frac{37\zeta_3}{96}\right.\\
&+\left.\frac{5\ln(2)\pi^2}{36}-\frac{\ln^2(2)\pi^2}{9}+\frac{91\pi^4}{2160}+L_\mu\left(-\frac{1145}{384}+\frac{221\pi^2}{384}-\frac{\ln(2)\pi^2}{3}-\frac{\zeta_3}{4}\right)\right)\\
&\left.+n_\mu^2\left(-\frac{9481}{7776}-\frac{13L_\mu^2}{72}-\frac{L_\mu^3}{36}+\frac{4\pi^2}{135}+L_\mu\left(-\frac{197}{216}+\frac{\pi^2}{9}\right)+\frac{11\zeta_3}{18}\right)\right],
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
Z_{2,\mu}^{\rm OS} &= 1+\frac{\bar{\alpha}}{\pi}\Bigg[-1-\frac{3}{4\epsilon}-\frac{3L_\mu}{4}+\epsilon\left(-2-L_\mu-\frac{3L_\mu^2}{8}-\frac{\pi^2}{16}\right)\\
&\quad+\epsilon^2\left(-4-\frac{L_\mu^2}{2}-\frac{L_\mu^3}{8}-\frac{\pi^2}{12}+L_\mu\left(-2-\frac{\pi^2}{16}\right)+\frac{\zeta_3}{4}\right)\\
&\quad+\epsilon^3\Bigg(-8-\frac{L_\mu^3}{6}-\frac{L_\mu^4}{32}-\frac{\pi^2}{6}-\frac{3\pi^4}{640}+L_\mu^2\left(-1-\frac{\pi^2}{32}\right)\\
&\qquad+L_\mu\left(-4-\frac{\pi^2}{12}+\frac{\zeta_3}{4}\right)+\frac{\zeta_3}{3}\Bigg)\Bigg]\\
&\quad+\left(\frac{\bar{\alpha}}{\pi}\right)^2\Bigg[\frac{9}{32\epsilon^2}+\frac{1}{\epsilon}\left(\frac{51}{64}+\frac{9L_\mu}{16}+\left(\frac{1}{16}+\frac{L_\mu}{4}\right)n_\mu+\left(\frac{1}{16}+\frac{L_\tau}{4}\right)n_\tau\right)\\
&\qquad+\frac{433}{128}+\frac{51L_\mu}{32}+\frac{9L_\mu^2}{16}-\frac{49\pi^2}{64}+\ln(2)\pi^2-\frac{3\zeta_3}{2}\\
&\qquad+n_\mu\left(\frac{947}{288}+\frac{11L_\mu}{24}+\frac{3L_\mu^2}{8}-\frac{5\pi^2}{16}\right)\\
&\qquad+n_\tau\Bigg(\frac{\pi^2}{48}-\frac{5}{96}+\frac{11L_\tau}{24}+\frac{L_\mu L_\tau}{4}+\frac{L_\tau^2}{8}+\frac{1}{15}\left(\frac{M_\mu}{M_\tau}\right)^2\\
&\qquad\quad+\left(-\frac{129}{78400}-\frac{9L_\mu}{560}+\frac{9L_\tau}{560}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&\qquad\quad+\left(-\frac{367}{99225}-\frac{4L_\mu}{315}+\frac{4L_\tau}{315}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\\
&\qquad\quad+\left(-\frac{569}{209088}-\frac{5L_\mu}{528}+\frac{5L_\tau}{528}\right)\left(\frac{M_\mu}{M_\tau}\right)^8\Bigg)\\
&\qquad+\epsilon\Bigg(\frac{211}{256}+\frac{51L_\mu^2}{32}+\frac{3L_\mu^3}{8}-\ln^4(2)-\frac{339\pi^2}{128}+\frac{23\ln(2)\pi^2}{4}-2\ln^2(2)\pi^2\\
&\qquad\quad+L_\mu\left(\frac{433}{64}-\frac{49\pi^2}{32}+2\ln(2)\pi^2-3\zeta_3\right)-\frac{297\zeta_3}{16}+\frac{7\pi^4}{20}-24a_4\\
&\qquad\quad+n_\mu\Bigg(\frac{17971}{1728}+\frac{5L_\mu^2}{8}+\frac{7L_\mu^3}{24}-\frac{445\pi^2}{288}+2\ln(2)\pi^2\\
&\qquad\qquad+L_\mu\left(\frac{1043}{144}-\frac{29\pi^2}{48}\right)-\frac{85\zeta_3}{12}\Bigg)\\
&\qquad\quad+n_\tau\Bigg(\frac{89}{576}+\frac{L_\mu^2 L_\tau}{8}+\frac{7L_\tau^2}{24}+\frac{L_\tau^3}{24}+\frac{11\pi^2}{288}-\frac{\zeta_3}{12}+L_\tau\left(\frac{9}{16}+\frac{\pi^2}{24}\right)\\
&\qquad+L_\mu\left(\frac{L_\tau}{3}+\frac{L_\tau^2}{8}+\frac{\pi^2}{48}\right)+\left(-\frac{7}{75}+\frac{2L_\tau}{15}\right)\left(\frac{M_\mu}{M_\tau}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{49659}{10976000}-\frac{9L_\mu^2}{1120}+L_\mu\left(-\frac{27}{1120}-\frac{9L_\tau}{560}\right)+\frac{51L_\tau}{2450}+\frac{27L_\tau^2}{1120}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&+\left(-\frac{71329}{62511750}-\frac{2L_\mu^2}{315}+L_\mu\left(-\frac{5}{189}-\frac{4L_\tau}{315}\right)+\frac{1891L_\tau}{99225}+\frac{2L_\tau^2}{105}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\\
&+\Bigg(-\frac{55373867}{20285717760}-\frac{5L_\mu^2}{1056}+L_\mu\left(-\frac{1}{42}-\frac{5L_\tau}{528}\right)\\
&\quad+\frac{13441L_\tau}{731808}+\frac{5L_\tau^2}{352}\Bigg)\left(\frac{M_\mu}{M_\tau}\right)^8\Bigg)\Bigg)\Bigg]\\
&+\left(\frac{\bar\alpha}{\pi}\right)^3\Bigg[-\frac{9}{128\epsilon^3}+\frac{1}{\epsilon^2}\Bigg(-\frac{81}{256}-\frac{27L_\mu}{128}+\left(-\frac{7}{192}-\frac{3L_\mu}{16}\right)n_\mu+\frac{n_\mu^2}{72}\\
&\qquad+\frac{n_\mu n_\tau}{36}+\left(-\frac{7}{192}-\frac{3L_\tau}{16}\right)n_\tau+\frac{n_\tau^2}{72}\Bigg)\\
&+\frac{1}{\epsilon}\Bigg(-\frac{1039}{512}-\frac{243L_\mu}{256}-\frac{81L_\mu^2}{256}+\frac{303\pi^2}{512}-\frac{3\ln(2)\pi^2}{4}+\frac{9\zeta_3}{8}\\
&\quad+\left(-\frac{5}{432}-\frac{L_\mu^2}{12}\right)n_\mu^2+\left(-\frac{5}{216}-\frac{L_\mu L_\tau}{6}\right)n_\mu n_\tau+\left(-\frac{5}{432}-\frac{L_\tau^2}{12}\right)n_\tau^2\\
&\quad+n_\tau\Bigg(\frac{85}{128}+L_\mu\left(-\frac{3}{64}-\frac{3L_\tau}{8}\right)-\frac{13L_\tau}{32}-\frac{3L_\tau^2}{32}-\frac{\pi^2}{64}\\
&\qquad-\frac{1}{20}\left(\frac{M_\mu}{M_\tau}\right)^2+\left(\frac{387}{313600}+\frac{27L_\mu}{2240}-\frac{27L_\tau}{2240}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&\qquad+\left(\frac{367}{132300}+\frac{L_\mu}{105}-\frac{L_\tau}{105}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\Bigg)\\
&\quad+n_\mu\left(-\frac{707}{384}-\frac{29L_\mu}{64}-\frac{15L_\mu^2}{32}+\frac{15\pi^2}{64}\right)\Bigg)\\
&-\frac{10823}{3072}-\frac{729L_\mu^2}{512}-\frac{81L_\mu^3}{256}-\frac{5\ln^4(2)}{12}-\frac{58321\pi^2}{9216}-10a_4\\
&+\frac{685\ln(2)\pi^2}{48}+3\ln^2(2)\pi^2-\frac{41\pi^4}{120}-\frac{739\zeta_3}{128}+\frac{\pi^2\zeta_3}{8}\\
&+L_\mu\left(-\frac{3117}{512}+\frac{909\pi^2}{512}-\frac{9\ln(2)\pi^2}{4}+\frac{27\zeta_3}{8}\right)-\frac{5\zeta_5}{16}\\
&+n_\tau^2\Bigg(-\frac{35}{2592}-\frac{11L_\tau^2}{72}-\frac{L_\mu L_\tau^2}{12}-\frac{L_\tau^3}{12}-\frac{L_\tau\pi^2}{72}-\frac{2L_\tau}{45}\left(\frac{M_\mu}{M_\tau}\right)^2\\
&\quad+\left(-\frac{121}{24000}+L_\mu\left(\frac{1}{100}+\frac{3L_\tau}{280}\right)-\frac{349L_\tau}{39200}-\frac{3L_\tau^2}{280}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&\quad+\left(\frac{353}{496125}+L_\mu\left(\frac{2}{175}+\frac{8L_\tau}{945}\right)-\frac{2668L_\tau}{297675}-\frac{8L_\tau^2}{945}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+n_\mu n_\tau\Bigg(-\frac{35}{1296} - \frac{L_\mu^2 L_\tau}{4} + L_\tau\left(-\frac{481}{216} + \frac{5\pi^2}{24}\right)\\
&\quad + L_\mu\left(-\frac{11L_\tau}{36} - \frac{L_\tau^2}{12} - \frac{\pi^2}{72}\right) + \left(-\frac{37}{90} - \frac{2L_\mu}{45} + \frac{2\pi^2}{45}\right)\left(\frac{M_\mu}{M_\tau}\right)^2\\
&\quad + \left(-\frac{317689}{16464000} - \frac{3L_\mu}{1120} + \frac{3L_\mu^2}{560} + \frac{37L_\tau}{9800} - \frac{3L_\tau^2}{560}\right)\left(\frac{M_\mu}{M_\tau}\right)^4\\
&\quad + \left(-\frac{159337}{18753525} + \frac{151L_\mu}{14175} + \frac{4L_\mu^2}{945} - \frac{2437L_\tau}{297675} - \frac{4L_\tau^2}{945}\right)\left(\frac{M_\mu}{M_\tau}\right)^6\Bigg)\\
&+n_\mu^2\left(-\frac{8425}{2592} - \frac{11L_\mu^2}{72} - \frac{L_\mu^3}{6} + \frac{2\pi^2}{45} + L_\mu\left(-\frac{481}{216} + \frac{5\pi^2}{24}\right) + \frac{7\zeta_3}{3}\right)\\
&+n_\tau\Bigg(\frac{4285}{2304} + L_\mu^2\left(-\frac{3}{128} - \frac{3L_\tau}{8}\right) - \frac{L_\tau^2}{8} - \frac{L_\tau^3}{32} - \frac{29\pi^2}{768} + \frac{3\zeta_3}{16}\\
&\quad + L_\mu\left(\frac{87}{128} - \frac{31L_\tau}{32} - \frac{3L_\tau^2}{16} - \frac{\pi^2}{32}\right) + L_\tau\left(-\frac{133}{192} + \frac{95\pi^2}{192} - \frac{2\ln(2)\pi^2}{3} + \zeta_3\right)\\
&\quad + \left(\frac{M_\mu}{M_\tau}\right)^2\left(-\frac{461}{1296} - \frac{71L_\mu}{360} + \frac{17L_\tau}{360} + \frac{\pi^2}{45}\right)\\
&\quad + \left(\frac{M_\mu}{M_\tau}\right)^4\Bigg(-\frac{2229623}{8232000} + \frac{3L_\mu^2}{80} + L_\mu\left(-\frac{28607}{940800} - \frac{87L_\tau}{2240}\right)\\
&\qquad\qquad + \frac{3209L_\tau}{94080} + \frac{3L_\tau^2}{2240} + \frac{\pi^2}{105} + \frac{147\zeta_3}{1024}\Bigg)\\
&\quad + \left(\frac{M_\mu}{M_\tau}\right)^6\Bigg(-\frac{15665719421}{90016920000} + \frac{1913L_\mu^2}{56700} + L_\mu\left(-\frac{1587179}{71442000} - \frac{1103L_\tau}{28350}\right)\\
&\qquad\qquad + \frac{2181719L_\tau}{71442000} + \frac{293L_\tau^2}{56700} + \frac{16\pi^2}{6075} + \frac{119\zeta_3}{960}\Bigg)\Bigg)\\
&+n_\mu\Bigg(-\frac{76897}{6912} + 12a_4 - \frac{143L_\mu^2}{128} - \frac{19L_\mu^3}{32} + \frac{\ln^4(2)}{2} - \frac{11551\pi^2}{20736} + \frac{7\ln(2)\pi^2}{18}\\
&\quad - \frac{\ln^2(2)\pi^2}{2} + \frac{31\pi^4}{720} + \frac{1763\zeta_3}{288} + L_\mu\left(-\frac{2891}{384} + \frac{233\pi^2}{192} - \frac{2\ln(2)\pi^2}{3} + \zeta_3\right)\Bigg)\Bigg],
\end{aligned}
\tag{17}
$$

where we include terms up to $\mathcal{O}(1/M_\tau^8)$.

The mass and wave function renormalization constants for the electron can be constructed from the above results by replacing M_μ by M_e, L_μ by L_e and n_μ by n_e. Moreover the terms proportional to n_τ and n_τ^2 have to be duplicated and afterwards the replacements $n_\tau \to n_\mu$, $M_\tau \to M_\mu$ and $L_\tau \to L_\mu$ have to be performed in one of the expressions. Furthermore, one has to add the contributions involving simultaneously virtual muon and

tau loops which are given by

$$
\begin{aligned}
\delta Z_{m,e}^{\mathrm{OS}} &= \left(\frac{\bar{\alpha}}{\pi}\right)^3 \left[-\frac{1}{18\epsilon^3} + \frac{5}{108\epsilon^2} + \frac{35}{648\epsilon} - \frac{1327}{3888} + \frac{2\zeta_3}{9} \right. \\
&\quad + \frac{31L_\tau}{54} - \frac{13L_\mu L_\tau}{36} - \frac{L_e L_\mu L_\tau}{6} + \frac{L_\mu^2 L_\tau}{12} + \frac{L_\tau^3}{36} \\
&\quad + \frac{M_\mu^2}{M_\tau^2}\left(-\frac{47}{150} - \frac{L_\mu}{5} + \frac{L_\tau}{5}\right) + \frac{M_e^2}{M_\mu^2}\left(-\frac{19L_\tau}{225} - \frac{2L_e L_\tau}{45} + \frac{2L_\mu L_\tau}{45}\right) \\
&\quad + \frac{M_e^2}{M_\tau^2}\left(\frac{1937}{10125} - \frac{2L_\mu}{45} - \frac{2L_e L_\mu}{45} + \frac{L_\mu^2}{45} - \frac{L_\tau}{25} + \frac{L_\tau^2}{45}\right) \\
&\quad + \frac{M_e^2 M_\mu^2}{M_\tau^4}\left(-\frac{107}{3675} - \frac{L_\mu}{35} + \frac{L_\tau}{35}\right) + \frac{M_e^4}{M_\mu^2 M_\tau^2}\left(-\frac{33}{1000} - \frac{L_e}{50} + \frac{L_\mu}{50}\right) \\
&\quad + \frac{M_\mu^4}{M_\tau^4}\left(-\frac{224261}{2058000} - \frac{529L_\mu}{9800} - \frac{3L_\mu^2}{280} + \frac{529L_\tau}{9800} + \frac{3L_\mu L_\tau}{140} - \frac{3L_\tau^2}{280}\right) \\
&\quad + \frac{M_e^4}{M_\mu^4}\left(-\frac{463L_\tau}{39200} - \frac{3L_e L_\tau}{280} + \frac{3L_\mu L_\tau}{280}\right) \\
&\quad \left. + \frac{M_e^4}{M_\tau^4}\left(\frac{39379}{1029000} - \frac{9L_\mu}{1120} - \frac{3L_e L_\mu}{280} + \frac{3L_\mu^2}{560} - \frac{37L_\tau}{9800} + \frac{3L_\tau^2}{560}\right)\right], \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\delta Z_{2,e}^{\mathrm{OS}} &= \left(\frac{\bar{\alpha}}{\pi}\right)^3 \left[\frac{1}{36\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{5}{216} - \frac{L_\mu L_\tau}{6}\right) - \frac{35}{1296} - \frac{11L_\mu L_\tau}{36} \right. \\
&\quad - \frac{L_e L_\mu L_\tau}{6} - \frac{L_\mu^2 L_\tau}{12} - \frac{L_\mu L_\tau^2}{12} - \frac{L_\mu \pi^2}{72} - \frac{L_\tau \pi^2}{72} \\
&\quad - \frac{2L_\tau}{45}\frac{M_e^2}{M_\mu^2} - \frac{2L_\mu}{45}\frac{M_e^2}{M_\tau^2} + \left(\frac{13}{1000} + \frac{L_e}{50} - \frac{L_\mu}{50}\right)\frac{M_e^4}{M_\mu^2 M_\tau^2} \\
&\quad + \frac{M_e^4}{M_\mu^4}\left(\frac{43L_\tau}{39200} + \frac{3L_e L_\tau}{280} - \frac{3L_\mu L_\tau}{280}\right) \\
&\quad \left. + \frac{M_e^4}{M_\tau^4}\left(-\frac{39379}{1029000} - \frac{3L_\mu}{1120} + \frac{3L_e L_\mu}{280} - \frac{3L_\mu^2}{560} + \frac{37L_\tau}{9800} - \frac{3L_\tau^2}{560}\right)\right]. \qquad (19)
\end{aligned}
$$

These formulae include only terms up to quartic order in the inverse heavy mass since the corresponding contributions to a_e are only computed up to this order.

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