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Analytic results for planar three-loop four-point integrals from a Knizhnik-Zamolodchikov equation

Johannes M. Henn[a] Alexander V. Smirnov[b,d] Vladimir A. Smirnov[c]

[a] *Institute for Advanced Study, Princeton, NJ 08540, USA*
[b] *Scientific Research Computing Center, Moscow State University, 119992 Moscow, Russia*
[c] *Skobeltsyn Institute of Nuclear Physics of Moscow State University, 119992 Moscow, Russia*
[d] *Institut für Theoretische Teilchenphysik, KIT, 76128 Karlsruhe, Germany*

E-mail: jmhenn@ias.edu, asmirnov80@gmail.com, smirnov@theory.sinp.msu.ru

Abstract: We apply a recently suggested new strategy to solve differential equations for master integrals for families of Feynman integrals. After a set of master integrals has been found using the integration-by-parts method, the crucial point of this strategy is to introduce a new basis where all master integrals are pure functions of uniform transcendentality. In this paper, we apply this method to all planar three-loop four-point massless on-shell master integrals. We explicitly find such a basis, and show that the differential equations are of the Knizhnik-Zamolodchikov type. We explain how to solve the latter to all orders in the dimensional regularization parameter ϵ, including all boundary constants, in a purely algebraic way. The solution is expressed in terms of harmonic polylogarithms. We explicitly write out the Laurent expansion in ϵ for all master integrals up to weight six.

Keywords: scattering amplitudes, gauge theory, NLO computations, multiloop Feynman integrals, dimensional regularization, harmonic polylogarithms

Contents

1 Introduction

The method of differential equations (DE) suggested in [1, 2] is one of the most powerful modern methods of evaluating multiloop Feynman integrals. It was presented in a systematic form in [3–6] where it was successfully applied to the evaluation of four-point two-loop massless Feynman integrals with one leg off shell. In this formulation, DE are applied to the evaluation of *master integrals* whose number is always finite [7]. This approach supposes that one has a solution of integration by parts (IBP) relations [8] at hand, i.e. an algorithm which expresses any Feynman integral of a given family as a linear combination of the master integrals.[1] There are several public codes to solve IBP relations [9–14] and many private codes. In the present work, we applied the `c++` version of `FIRE` [10, 11].

The idea of the method is to take derivatives of a given master integral with respect to kinematical invariants and masses. Then the result of this differentiation is written in terms of Feynman integrals of the given family and, according to the known IBP reduction, in terms of the master integrals. In this way, one obtains a system of first-order differential

[1] We use the term *family* of Feynman integrals to refer to a set of integrals sharing the same denominator factors, and possibly having numerators. In this terminology, an integral with all propagators present can be thought of as the parent integral, and integrals with missing propagators as descendants.

equations for the master integrals, and can then try to solve this system with appropriate boundary conditions. The method of DE was successfully applied in many calculations. For reviews, see [15, 16], and [17, 18] for some recent examples.

Despite its power and generality, one can encounter practical problems when using this method for complicated families of Feynman integrals. One difficulty can lie in the fact that the class of integral functions appropriate to describe the solution is complicated, and it only becomes apparent in the course of the calculation which class of functions is needed. Another difficulty can arise when there are several master integrals that satisfy coupled differential equations. These can turn out rather cumbersome to solve in practice. Also, the results for the master integrals are often rather lengthy and their structure is not particularly transparent.

Quite recently a new strategy of solving DE for master integrals was suggested [19] by one the authors of the present paper. When applicable, it overcomes the problems indicated above. The key ingredient of this strategy is to choose a convenient basis of master integrals having desirable properties. The goal is to choose all master integrals such that they are *pure* functions of uniform *weight*, i.e uniform degree of transcendentality. For generalized polylogarithms [20, 21] that are defined through iterated integrals over logarithmic differential forms, the weight of a function is defined as the number of integrations needed to define it. A linear combination of such functions has uniform (i.e. homogeneous) weight if all its summands have the same weight. Finally, a function is called pure if the weight of its differential is lowered by one unit. This last property is motivated by the fact that such functions satisfy simple differential equations. This will be important in the following. In the remainder of this paper, we will use the terms weight and (degree of) transcendentality without distinction.

The fact that certain loop integrals have uniform transcendentality was observed in many calculations, especially in supersymmetric theories, see e.g. [22–26], and more recently in [27, 28].[2] Certainly results for generic scattering amplitudes in QCD do not appear to have simple transcendentality properties, at least in the way they are conventionally presented. One may ask, however, whether such results can be written in terms of a finite number of building blocks that have the properties discussed above. Reference [19] suggests that all master integrals can indeed be chosen to be pure functions of uniform transcendentality, including the integrals needed for QCD, and provides criteria for finding such a basis.

Suppose that for a given family the set of master integrals has already been identified, using IBP relations. The main point of the strategy of [19] is then to turn to a new basis of the master integrals which all have uniform transcendentality. This transition is given by a linear transformation in the space of master integrals and the corresponding matrix is rational with respect to dimension and usually algebraic w.r.t. kinematic invariants.

As explained in [19] one can use various strategies to reveal uniformly transcendental master integrals. One efficient method is to replace propagators by delta functions and

[2]The concept of transcendentality also played an important role in a different context, at the level of anomalous dimensions of composite operators, where the anomalous dimensions in $N = 4$ SUSY Yang–Mills theory may be obtained from the leading-transcendentality contributions in QCD [29].

analyze whether the resulting expression is uniformly transcendental. In other cases, explicit integral representations can be derived, using Feynman parameters or other means [30], to make the transcendental properties of the answer manifest. We also wish to mention related work in the mathematical literature [31].

Let us denote the kinematical variables by $x = (x_1, \ldots, x_n)$, the set of N basis integrals by $f = (f_1, \ldots, f_N)$, and let us work in $D = 4-2\epsilon$ dimensions. The general set of differential equations takes the form

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,, \tag{1.1}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

The existence of a basis of master integrals with the above properties is closely related to the possibility to obtain a much simpler system of differential equations, as conjectured in [19],

$$\partial_i f(\epsilon, x) = \epsilon\, A_i(x) f(\epsilon, x)\,. \tag{1.2}$$

The essential difference w.r.t. (1.1) is that the matrix in the equation is just proportional to ϵ. As a result such a system of equations can be solved in a very easy and natural way. There is no general proof that, for any family of Feynman integrals, one can turn from (1.1) to (1.2). However, we are going to provide non-trivial examples of Feynman integrals where this is possible and thereby arrive at new results.

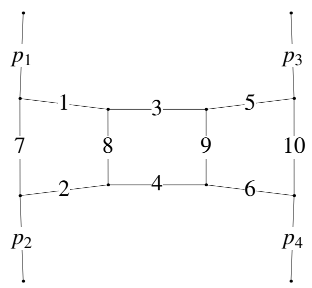
In [19] it was shown that this strategy can successfully be applied to all the on-shell massless two-loop Feynman integrals, and previous results, in particular, for the two double box integrals of this family [22, 32], can be reproduced.

The goal of the present paper is to derive new results with the strategy of [19]. We will consider the two families of planar three-loop massless on-shell integrals corresponding to the ladder (i.e. triple box) and the tennis court graph shown in Fig. 1. (The notation A and E for the families of master integrals follows that of [33]. Other letters stand for non-planar integrals.) These integrals have fifteen indices: we associate the first ten of them to the edges of these graphs, as shown in Fig. 1, and the last five to numerators. Explicitly, we have

$$\begin{aligned} F^A_{a_1,\ldots,a_{15}}(s,t;D) = &\int\!\!\int\!\!\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_1+p_2+k_1)^2]^{a_2}(-k_2^2)^{a_3}} \\ &\times \frac{[-(k_1-p_3)^2]^{-a_{11}}[-(p_1+k_2)^2]^{-a_{12}}[-(k_2-p_3)^2]^{-a_{13}}}{[-(p_1+p_2+k_2)^2]^{a_4}(-k_3^2)^{a_5}[-(p_1+p_2+k_3)^2]^{a_6}[-(p_1+k_1)^2]^{a_7}} \\ &\times \frac{[-(p_1+k_3)^2]^{-a_{14}}[-(k_1-k_3)^2]^{-a_{15}}}{[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\,, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} F^E_{a_1,\ldots,a_{15}}(s,t;D) = &\int\!\!\int\!\!\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{[-(k_1-k_3)^2]^{a_1}[-(p_1+k_1)^2]^{a_2}[-(p_1+p_2+k_1)^2]^{a_3}} \\ &\times \frac{[-(p_1+p_2+k_3)^2]^{-a_{11}}[-(p_1+k_2)^2]^{-a_{12}}[-(k_1-p_3)^2]^{-a_{13}}}{[-(p_1+p_2+k_2)^2]^{a_4}[-(k_2-p_3)^2]^{a_5}[-(k_2-k_3)^2]^{a_6}[-(k_1-k_2)^2]^{a_7}} \end{aligned}$$



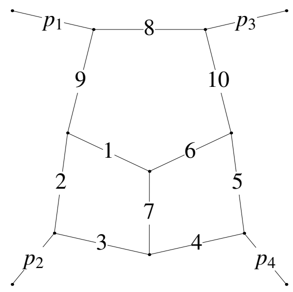


Figure 1. The triple box (A) and tennis court diagrams (E). Latin numbers refer to propagators associated to line parameters a_i, cf. eqs. (1.3) and (1.4). Lines associated to possible numerators are not shown in the figures.

$$\times \frac{(-k_1^2)^{-a_{14}}(-k_2^2)^{-a_{15}}}{(-k_3^2)^{a_8}[-(p_1+k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\,. \tag{1.4}$$

Here $s=(p_1+p_2)^2$ and $t=(p_1+p_3)^2$ denote the Mandelstam invariants. For later use, we note that $u=(p_2+p_3)^2=-s-t$.

As we explain presently, the master integrals for these two families represent all master integrals needed to evaluate any massless planar on-shell three-loop four-point scattering amplitude. We explicitly find a basis where all master integrals have uniform transcendentality, and show that the differential equations are of the Knizhnik-Zamolodchikov type [34]. We explain how to solve the latter to all orders in the dimensional regularization parameter ϵ, including all boundary constants, in a purely algebraic way, for *all* master integrals. The solution is expressed in terms of harmonic polylogarithms. We explicitly write out the Laurent expansion in ϵ for all master integrals up to weight six. Up to now, two analytical results for integrals of this family were known: for the triple box without numerator [23] and for the tennis court diagram with a special numerator [24].

We would also like mention a perhaps surprising outcome of our analysis. As a by-product of our calculation, we also obtained analytic results for single-scale integrals appearing in form factors. Naïvely, the DE method cannot be applied to these cases, since their scale dependence is trivially fixed by their engineering dimension. However, they are a part of the system of differential equations for the more general four-point integrals, where they enter as boundary values. The latter, however, are greatly constrained by the finiteness of planar integrals in the u-channel as $u\to 0$. As we will discuss in more detail below, these consistency conditions fix all boundary constants, up to trivial propagator-type integrals. In this way, one obtains results for non-trivial single-scale integrals, to any order in ϵ. One may verify agreement with the planar form factor integrals computed in references [25, 35–39]. We find this way of computing these integrals rather elegant.

Let us now explain why the master integrals computed above are sufficient to describe *all* the families of three-loop four-point planar on-shell massless diagrams (which have fifteen indices, with the number of positive indices being lower or equal to ten.) To see this, let us first observe that we can construct integrals with the maximal number of positive indices by building graphs with trivalent vertices. A quartic vertex can always be obtained as a special case, with one index being zero. Let us then observe than the triple box and the tennis court are the only graphs composed of cubic vertices with no triangles as subgraphs. So, any other graph has at least one triangle subgraph. In this case, one can use the presence of such a triangle and apply IBP relations to reduce an index, either internal or external, of this triangle to zero starting from positive values [8]. In graph-theoretical language, this means shrinking the corresponding line to a point. By analyzing various graphs obtained by this procedure we can see that the resulting reduced graphs can be also obtained, in some way, either from the triple box or/and from the tennis court.

This paper is organized as follows. In section 2, we explain the strategy we use for finding integrals that give rise to pure functions of uniform transcendentality, providing several examples. We then present our basis choice for the master integrals. In section 3, we present the differential equations satisfied by the latter, and explain how to solve them in the ϵ expansion. We also discuss physical boundary conditions. We analyze the structure of the solution. Explicit results for the ten-propagator integrals are relegated to Appendix B, and for all integrals to the ancillary files `resultA.m` and `resultE.m`. For convenience, we also present in these files the corresponding matrices appearing in the differential equations. We conclude in section 4.

2 Choice of integral basis

An important part of the result of this paper is to provide a basis of master integrals for the families of Feynman integrals A and E where each basis element is a pure function of uniform weight. Ideas for how to construct such a basis where outlined in ref. [19]. In practice, these lead to very useful criteria for choosing master integrals. Here we wish to explain the criteria that we found most useful in the present context.

When constructing good candidate integrals at $(L+1)$ loops, it is very convenient to have a solution of the problem at L loops at hand, as one can often infer from this which integrals to choose at the next loop order. We will see this in more detail in the following examples. In the present case, the solution at two loops was presented in [19].

2.1 Example 1: massless bubble subintegrals

Many of the three-loop integrals we are interested in have bubble subintegrals (we will also sometimes refer to these as propagator-type subintegrals), i.e. they are lower-loop integrals with certain bubble insertions. In fact, the integrals of Fig. 3 and Fig. 5 are all of this type. For definiteness, let us consider the specific case of integral f_{19}^{A} of Fig. 3.

It is clear that we can always integrate out propagator subintegrals and obtain a lower-loop integral, albeit with some power(s) shifted by ϵ. More concretely, we have

$$\int \frac{d^D k}{[-k^2]^{a_1}[-(k+p)^2]^{a_2}} = \frac{\Gamma(a-D/2)\Gamma(D/2-a_1)\Gamma(D/2-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a)} \frac{i\pi^{D/2}}{(-p^2)^{a-D/2}}\,, \tag{2.1}$$

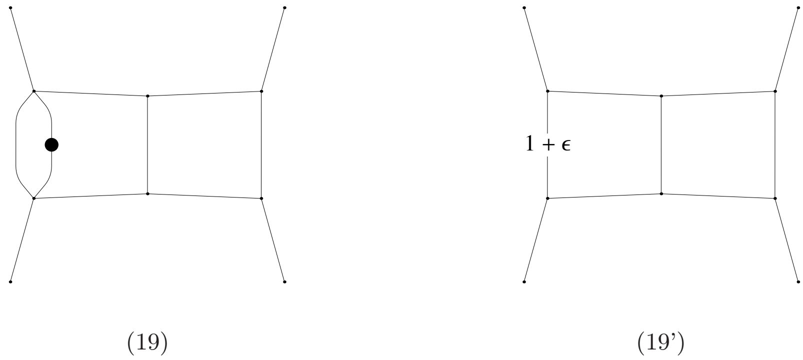


Figure 2. Integrating out propagator subintegrals related the basis choice at $(L+1)$ loops to the corresponding choice at L loops, up to some trivial prefactors, and indices shifted by ϵ.

where $a = a_1 + a_2$. In particular, if the indices a_1 and a_2 are equal to one and two, as in the present case, we see that after integrating out the bubble subintegral, we obtain, up to some inessential prefactor, a double box integral with one index shifted from 1 to $1+\epsilon$, cf. Fig. 2.

One might be worried about the effect of the shift of the power by ϵ. In fact, experience shows that in most cases the shifts in ϵ can be ignored for the purposes of uniform transcendentality. A qualitative explanation, which is applicable to many cases, is the following. Consider the integral

$$I(x,\epsilon) := \int_0^1 \frac{1}{x+t} t^\epsilon \, dt \,. \tag{2.2}$$

For $\epsilon = 0$, this evaluates to a logarithm, and hence has degree one. The full integral has a Taylor expansion in ϵ. It is easy to see that the coefficient of ϵ^n has weight $(n+1)$. Assigning weight -1 to ϵ, we see that $I(x,\epsilon)$ is a function for which each term in the expansion in ϵ has uniform weight one. We see that the presence of the factor t^ϵ had was inessential as far as the transcendental weight of the integral was concerned.

We see that this reasoning motivates the choice for the master integrals shown in Figs. 3,5. Similarly, in the case of triangle subintegrals, explicit parametrizations can be useful. In particular, whenever there is a triangle integral with an on-shell corner, a well-known trick is to use Feynman parameters to combine the two propagators adjacent to the on-shell leg. In this way, one obtains a one-fold integral over a configuration with a propagator subintegral, which was discussed above.

2.2 Example 2: leading singularities, (generalized) unitarity cuts

A more general method is to study leading singularities or the closely related (generalized) unitarity cuts of loop integrals. In particular, a very useful cut can be done whenever we have a box subintegral. In this case, we can consider the same integral with the four propagators of the box cut, i.e. replaced by delta functions. Alternatively, we may view this as replacing the integration by contours in the complex plane around the poles of the

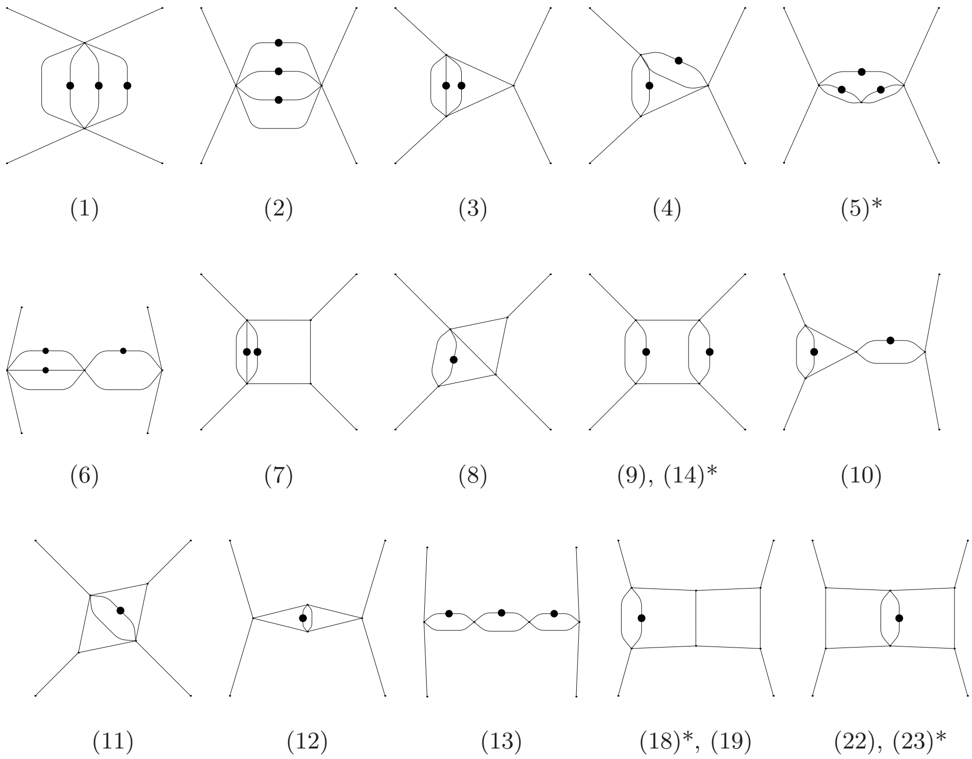


Figure 3. Master integrals for integral family A that have bubble subintegrals. Dots denote doubled propagators. An asterisk indicates that there are numerator factors not shown in the figure.

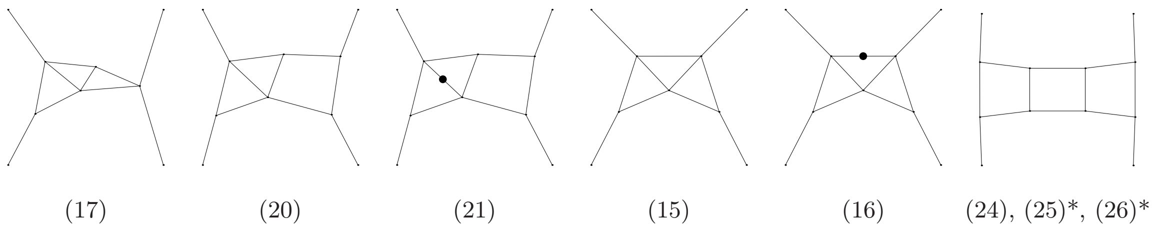


Figure 4. Master integrals for integral family A without bubble subintegrals. Dots denote doubled propagators. An asterisk indicates that there are numerator factors not shown in the figure.

propagators. As a result, the subintegral is completely localized and can be easily evaluated. In this way, we relate the $(L+1)$-loop integral to an L-loop integral. The strategy is then to choose the integrals such that the resulting lower-loop integrals that can be obtained by cutting lines have uniform transcendentality.

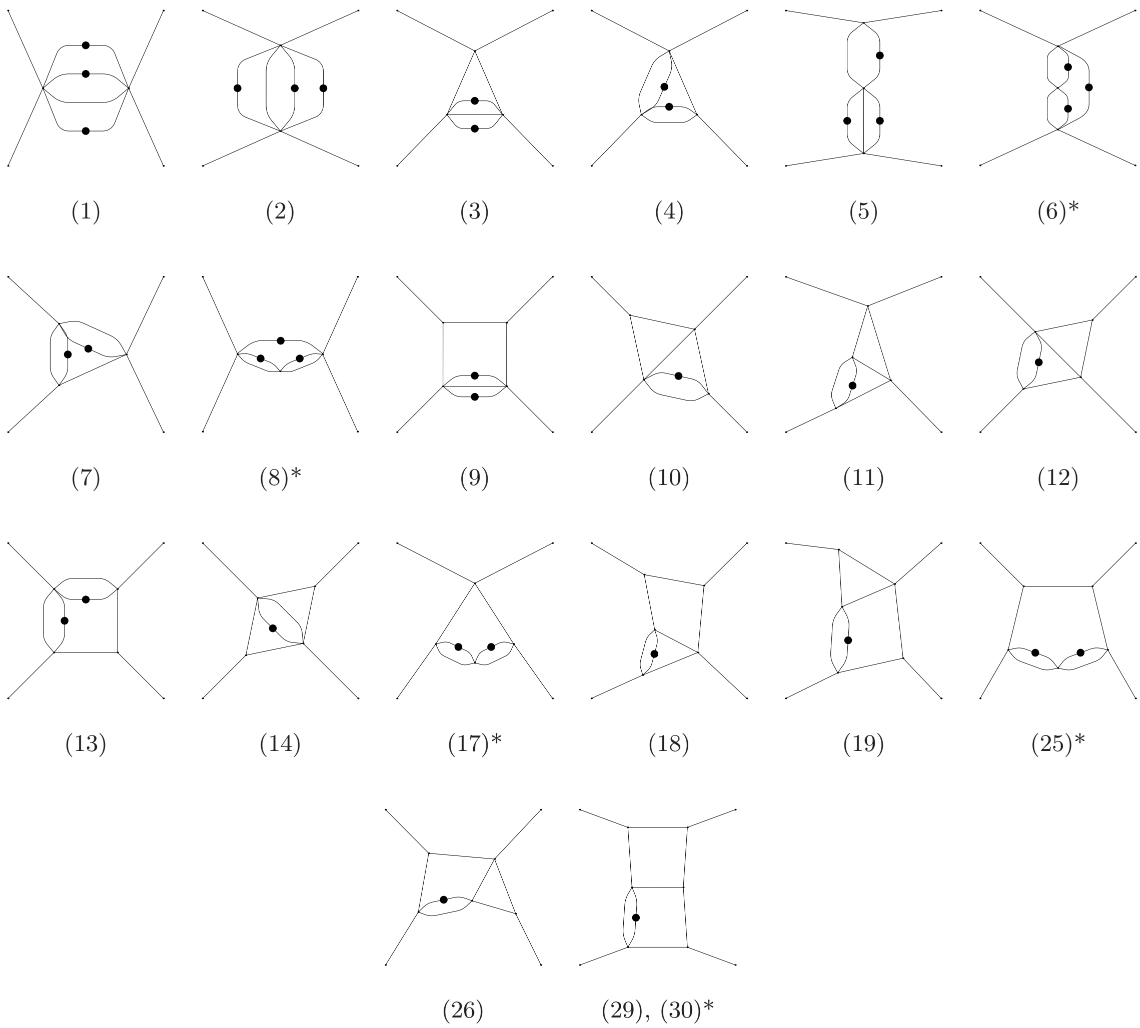


Figure 5. Master integrals for integral family E that have bubble subintegrals. Dots denote doubled propagators. An asterisk indicates that there are numerator factors not shown in the figure.

2.3 General comments

In summary, we can use these rules to generate candidate integrals that are expected to be pure functions of uniform transcendentality. One can then use the IBP reduction to determine how many of the candidate integrals are linearly independent and can hence be used as master integrals. One then proceeds by writing out the system of differential equations in the new basis. As we discuss in the following section, this provides an immediate test of the basis choice – when successful, the transcendentality properties of the basis functions are made manifest by the differential equations. Before presenting our choice of integral basis, we make a number of general comments on the strategy of finding such a basis.

The discussion of unitarity cuts in the examples was four-dimensional. Of course, in principle one can also analyze these cuts in $4-2\epsilon$ dimensions This is closely related to massive integrals. In practice, we have found that in most cases, the naïve four-dimensional integrand analysis is sufficient. See also the related discussion for example 1.

We would also like to mention another fact that makes this approach extremely efficient in practice: for a given family of integrals, one can start working in sectors with fewer propagators (i.e. number of positive indices), which restricts the size of the basis. This allows to verify the properties of the basis choice by looking at a small number of integrals at a time, by inspecting the resulting differential equations.

In some cases, it can happen that the candidate integrals one selects using the above criteria do not yet have the desired properties, e.g. if not enough cuts were considered, or if there are subtle effects that invalidate a naïve four-dimensional analysis. Often, a small modification to the candidate integral(s) is then sufficient to obtain the desired properties. Exactly how to modify the integrals can be deduced by inspecting the differential equations. An example of such a case are integrals f^E_{34} and f^E_{36} given below.

Finally, it should be obvious from the above discussion that the ideas for finding convenient basis elements do not rely on planarity, massless particles, four-point kinematics, etc., although all of those features lead to technical simplifications. More generally, we would also expect that generalizations of the 'd-log' representations of ref. [30] can give insight into transcendentality properties of loop integrals. For example, in the slightly simpler setting of heavy quark effective theory (i.e. Wilson line) integrals, such representations were used successfully, see [40].

2.4 Integral basis for integral classes A and E

In the way explained above we straightforwardly arrived at the basis choice depicted in the Figs. 3,4,5 and 6. There are 26 master integrals in family A, and 41 in family E. 7 integrals are shared between the two families, so that we have a total of 60 inequivalent integrals. (Some further integrals can be obtained from interchanging s and t.)

In formulas, we define

$$f_i^A = \epsilon^3 \, (-s)^{3\epsilon} \, \frac{e^{3\epsilon\gamma_{\rm E}}}{(i\pi^{D/2})^3} \; g_i^A \, . \tag{2.3}$$

This formula has three prefactors that we explain presently. The factor $(-s)^{3\epsilon}$ is there to make the basis functions f_i^A dimensionless. The factor ϵ^3 ensures that all basis functions admit a Taylor expansion around $\epsilon = 0$. Finally, we have pulled out a standard conventional normalization factor for three-loop integrals. The functions g_i^A are defined as

$$g_1^A = tF^A_{0,0,0,0,0,0,2,2,2,1,0,0,0,0,0}, \qquad g_2^A = sF^A_{0,2,0,0,1,0,0,2,2,0,0,0,0,0,0}, \tag{2.4}$$
$$g_3^A = \epsilon sF^A_{0,0,0,0,1,1,2,2,1,0,0,0,0,0,0}, \qquad g_4^A = \epsilon sF^A_{0,0,0,1,2,0,2,1,1,0,0,0,0,0,0}, \tag{2.5}$$
$$g_5^A = sF^A_{0,1,2,-1,0,1,0,2,2,0,0,0,0,0,0}, \qquad g_6^A = s^2 F^A_{0,2,2,0,2,1,0,1,0,0,0,0,0,0,0}, \tag{2.6}$$
$$g_7^A = \epsilon st F^A_{0,0,0,0,1,1,2,2,1,1,0,0,0,0,0}, \qquad g_8^A = \epsilon^2(s+t) F^A_{0,0,0,1,1,0,2,1,1,1,0,0,0,0,0}, \tag{2.7}$$
$$g_9^A = \epsilon st F^A_{0,0,1,1,0,0,2,1,1,2,0,0,0,0,0}, \qquad g_{10}^A = \epsilon s^2 F^A_{0,0,1,1,2,1,2,1,0,0,0,0,0,0,0}, \tag{2.8}$$
$$g_{11}^A = \epsilon^2(s+t) F^A_{0,1,0,0,1,0,1,1,2,1,0,0,0,0,0}, \qquad g_{12}^A = -\epsilon(2\epsilon-1) sF^A_{1,1,0,0,1,1,0,2,1,0,0,0,0,0,0}, \tag{2.9}$$
$$g_{13}^A = s^3 F^A_{2,1,2,1,2,1,0,0,0,0,0,0,0,0,0}, \qquad g_{14}^A = \epsilon sF^A_{0,0,1,1,0,0,2,1,1,2,0,0,-1,0,0}, \tag{2.10}$$

$$g^A_{15} = \epsilon^3 t F^A_{0,1,1,0,0,1,1,1,1,1,0,0,0,0,0}, \qquad g^A_{16} = \epsilon^2 s^2 F^A_{0,1,2,0,0,1,1,1,1,1,0,0,0,0,0}, \tag{2.11}$$
$$g^A_{17} = \epsilon^3 s F^A_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0}, \qquad g^A_{18} = \epsilon^2 s^2 F^A_{0,0,1,1,1,1,2,1,1,1,0,0,-1,0,0}, \tag{2.12}$$
$$g^A_{19} = \epsilon^2 s^2 t F^A_{0,0,1,1,1,1,2,1,1,1,0,0,0,0,0}, \qquad g^A_{20} = \epsilon^3 s(s+t) F^A_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0}, \tag{2.13}$$
$$g^A_{21} = \epsilon^2 s^2 t F^A_{0,1,1,0,1,1,1,2,1,1,0,0,0,0,0}, \qquad g^A_{22} = \epsilon^2 s^2 t F^A_{1,1,0,0,1,1,1,2,1,1,0,0,0,0,0}, \tag{2.14}$$
$$g^A_{23} = \epsilon^2 s^2 F^A_{1,1,0,0,1,1,1,2,1,1,-1,0,0,0,0}, \qquad g^A_{24} = \epsilon^3 s^3 t F^A_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0}, \tag{2.15}$$
$$g^A_{25} = \epsilon^3 s^3 F^A_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}, \qquad g^A_{26} = \epsilon^3 s^3 F^A_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,0} \tag{2.16}$$

For integral family E, we have (2.3) with 'A' replaced by 'E', and

$$g^E_{1} = s\, F^E_{0,0,1,0,0,2,2,2,0,0,0,0,0,0,0}, \qquad g^E_{2} = t F^E_{0,1,0,0,0,2,2,0,0,2,0,0,0,0,0}, \tag{2.17}$$
$$g^E_{3} = -2\epsilon t F^E_{0,0,1,0,0,2,2,0,1,1,0,0,0,0,0}, \qquad g^E_{4} = -2\epsilon t F^E_{0,0,1,0,1,2,2,0,1,0,0,0,0,0,0}, \tag{2.18}$$
$$g^E_{5} = t^2 F^E_{0,2,0,0,2,0,1,0,2,1,0,0,0,0,0}, \qquad g^E_{6} = t F^E_{0,1,0,0,2,2,2,0,1,0,0,-1,0,0,0}, \tag{2.19}$$
$$g^E_{7} = -2\epsilon s F^E_{0,1,0,1,0,2,2,1,0,0,0,0,0,0,0}, \qquad g^E_{8} = +s F^E_{1,0,2,2,0,1,0,2,0,0,-1,0,0,0,0}, \tag{2.20}$$
$$g^E_{9} = -2\epsilon s t F^E_{0,0,1,0,0,2,2,1,1,1,0,0,0,0,0}, \qquad g^E_{10} = 4\epsilon^2 (s+t) F^E_{0,0,2,0,1,1,1,1,1,0,0,0,0,0,0}, \tag{2.21}$$
$$g^E_{11} = 4\epsilon^2 t F^E_{0,2,0,1,0,1,1,0,1,1,0,0,0,0,0}, \qquad g^E_{12} = 4\epsilon^2 (s+t) F^E_{0,2,0,1,0,1,1,1,0,1,0,0,0,0,0}, \tag{2.22}$$
$$g^E_{13} = -2\epsilon s t F^E_{0,2,0,1,1,2,1,1,0,0,0,0,0,0,0}, \qquad g^E_{14} = 4\epsilon^2 (s+t) F^E_{0,1,1,0,0,2,1,1,0,1,0,0,0,0,0}, \tag{2.23}$$
$$g^E_{15} = 4\epsilon^2 t F^E_{1,0,1,0,1,1,1,0,2,0,0,0,0,0,0}, \qquad g^E_{16} = 4\epsilon^2 s F^E_{1,0,1,0,2,1,1,1,0,0,0,0,0,0,0}, \tag{2.24}$$
$$g^E_{17} = -2\epsilon t F^E_{2,0,1,2,0,1,0,0,1,1,-1,0,0,0,0}, \qquad g^E_{18} = 4\epsilon^2 s t F^E_{0,2,0,1,0,1,1,1,1,1,0,0,0,0,0}, \tag{2.25}$$
$$g^E_{19} = 4\epsilon^2 s t F^E_{0,2,0,1,1,1,1,1,1,0,0,0,0,0,0}, \qquad g^E_{20} = -8\epsilon^3 t F^E_{0,1,1,0,1,1,1,0,1,1,0,0,0,0,0}, \tag{2.26}$$
$$g^E_{21} = -8\epsilon^3 s F^E_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0}, \qquad g^E_{22} = 4\epsilon^2 t^2 F^E_{0,1,1,0,2,1,1,1,1,0,0,0,0,0,0}, \tag{2.27}$$
$$g^E_{23} = -8\epsilon^3 (s+t) F^E_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0}, \qquad g^E_{24} = 4\epsilon^2 s t F^E_{1,0,2,0,1,1,1,1,1,0,0,0,0,0,0}, \tag{2.28}$$
$$g^E_{25} = -2\epsilon s t F^E_{2,0,1,2,0,1,0,1,1,1,-1,0,0,0,0}, \qquad g^E_{26} = 4\epsilon^2 s t F^E_{1,0,2,1,1,1,0,1,1,0,0,0,0,0,0}, \tag{2.29}$$
$$g^E_{27} = -8\epsilon^3 t F^E_{1,1,1,1,1,1,0,1,0,0,0,0,0,0,0}, \qquad g^E_{28} = 4\epsilon^2 s^2 F^E_{1,1,1,1,1,1,0,2,0,0,0,0,0,0,0}, \tag{2.30}$$
$$g^E_{29} = 4\epsilon^2 s t^2 F^E_{0,2,0,1,1,1,1,1,1,1,0,0,0,0,0}, \qquad g^E_{30} = 4\epsilon^2 t^2 F^E_{0,2,0,1,1,1,1,1,1,1,-1,0,0,0,0}, \tag{2.31}$$
$$g^E_{31} = -8\epsilon^3 t(s+t) F^E_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0}, \qquad g^E_{32} = 4\epsilon^2 s t^2 F^E_{0,1,1,0,1,1,2,1,1,1,0,0,0,0,0}, \tag{2.32}$$
$$g^E_{33} = 4 s t F^E_{1,2,0,1,1,1,1,1,0,1,0,0,-1,0,0} \epsilon^2, \qquad g^E_{35} = 4\epsilon^2 s t F^E_{1,1,1,1,1,1,1,2,0,0,-1,0,0,0,0}, \tag{2.33}$$
$$g^E_{34} = 8\epsilon^3 t \left(\frac{t}{s+t} F^E_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0,0} - F^E_{1,1,0,1,1,1,1,1,0,1,0,0,0,0,0,-1} \right), \tag{2.34}$$
$$g^E_{36} = -8\epsilon^3 s \left(\frac{t}{s+t} F^E_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0,0} + F^E_{1,1,1,1,1,1,1,1,0,0,0,-1,0,0,0} \right), \tag{2.35}$$
$$g^E_{37} = -8\epsilon^3 t^2 F^E_{1,1,1,1,1,1,1,0,1,1,-1,0,0,0,0}, \qquad g^E_{38} = -8\epsilon^3 s t F^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,0}, \tag{2.36}$$
$$g^E_{39} = -8\epsilon^3 s t^2 F^E_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}, \qquad g^E_{40} = -8\epsilon^3 t^2 F^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,0,0}, \tag{2.37}$$
$$g^E_{41} = -8\epsilon^3 s t F^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0}. \tag{2.38}$$

Having found a convenient set of master integrals, let us now study the system of differential equations they satisfy. We will find that the ladder indeed make all the properties that we were looking for manifest.

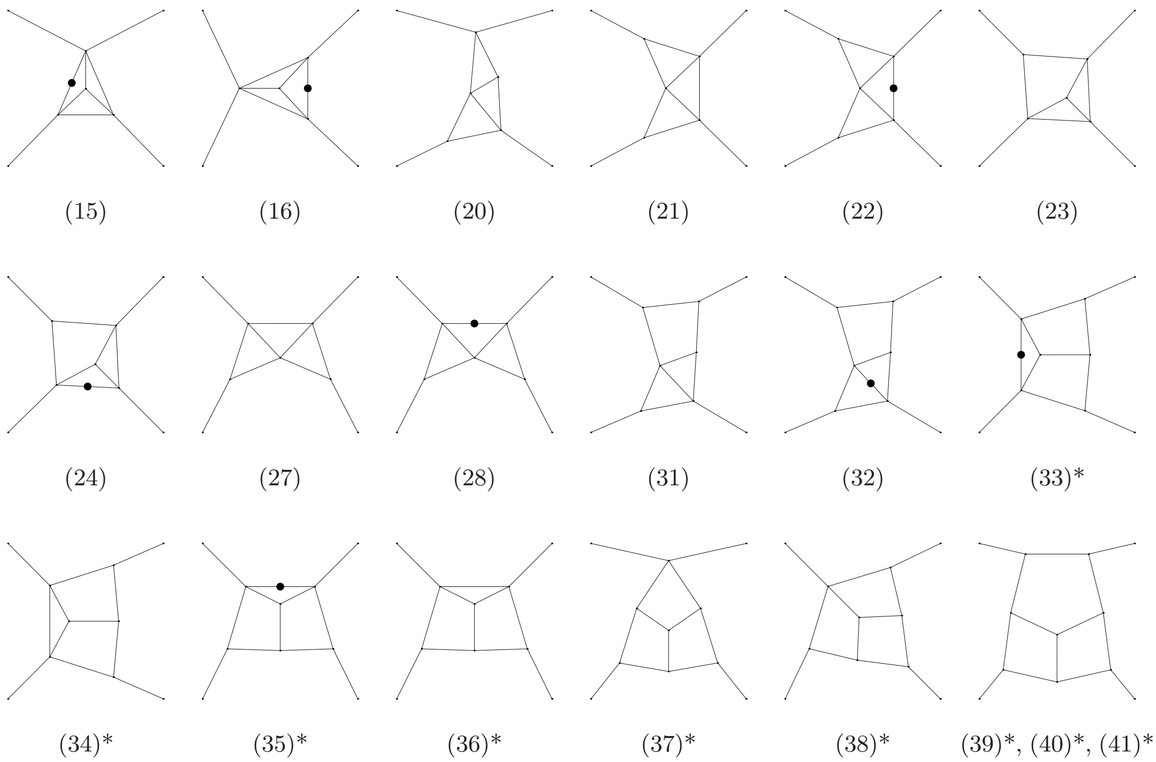


Figure 6. Master integrals for integral family E without bubble subintegrals. Dots denote doubled propagators. An asterisk indicates that there are numerator factors not shown in the figure. Integrals 34 and 36 involve an admixture of integral 23, see eqs. in the main text.

3 Knizhnik-Zamolodchikov equation for four-point integrals

Here we study the differential equations satisfied by the master integrals. We find that with the above choice of basis, the differential equations take the form predicted in ref. [19]. The basis integrals discussed in the previous section were normalized to be dimensionless, and hence only depend on the ratio $x = t/s$. In this variable, the differential equations take the following form,

$$\partial_x \, f(x, \epsilon) = \epsilon \, \left[\frac{a}{x} + \frac{b}{1+x} \right] \, f(x, \epsilon) \, . \tag{3.1}$$

This is a specialization of eq. (1.2) to one variable, with a specific form of the matrix $A(x)$. Here a and b are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E. Explicit exressions for these matrices are presented in Appendix A. We obtain this system of equations for both the triple ladder and the tennis court family of integrals.

We wish to emphasize that the size of the system does not pose any problems when solving the equations, since the solution is obtained in a completely algebraic way.

We see that equation (3.1) has three regular singularities, at $x = 0$, $x = -1$, and $x = \infty$. These three points correspond to the limits $s = 0$, $u = 0$, and $t = 0$, respectively. The absence of singularities of planar integrals as $u \to 0$ will provide an important boundary condition, as discussed in the next section. We remark that equation (3.1) is a particular case of the Knizhnik-Zamolodchikov equations [34]. It can also be described as a Fuchsian system of differential equations with three regular singular points.

Let us now discuss the solution of those equations. The normalization of the master integrals in eq. (2.17) was chosen such that functions f_i are finite as $\epsilon \to 0$. We are interested in a solution near $D \approx 4$ dimensions, so we parametrize, e.g. for family A,

$$f_i^A(x,\epsilon) = \sum_{j=0}^{6} \epsilon^j f_i^{A,j}(x) + \mathcal{O}(\epsilon^7)\,. \tag{3.2}$$

From eq. (3.1) it is clear that the iterative solution in ϵ for all functions f_i can be expressed in terms of harmonic polylogarithms [41] of argument x and with indices drawn from $0, -1$. Equation (3.1) determines the solution up to boundary constants. We will determine the latter in the next section. Here we would already like to mention that the boundary constants have the property of uniform weight, and this, together with the structure of eq. (3.1), implies that all basis functions are pure functions of uniform weight, as anticipated.

3.1 Boundary conditions

For planar graphs we expect the limit $u \to 0$, i.e. $x \to -1$ to be finite. Another condition that we can impose it that the solution be real for $x > 0$, i.e. when s and t have the same sign. For planar graphs, this is obvious from the Feynman parametrization. As we will see, these assumptions fix almost all of the boundary constants in this problem, except for some elementary propagator-type integrals.

As can be seen from (3.1), the entries $1/(1 + x)$ can lead to terms singular as $x \to -1$, and the regularity at $x \to -1$ therefore imposes constraints on the integration constants. For example, at order ϵ, this condition means that $H_1(x) = \log(1 + x)$ must come with zero coefficient, and this imposes constraints on the integration constants at order ϵ^0. The absence of the function Li_2 at order ϵ^2 in our results can be understand in this way. Given these constraints, one might wonder how one can obtain functions different from logarithms. The answer is the following. At higher orders, there can be an interplay between boundary constants at different orders, as the following example shows,

$$\pi^2 \int_{-1+\delta}^{x} d\log(1+y) - \int_{-1+\delta}^{x} \log^2 y \, d\log(1+y)\,, \tag{3.3}$$

which is finite as $\delta \to 0$, and hence there can be finite combinations of HPLs with indices -1.

In practice, we found that when computing up to order ϵ^n, considering the consistency condition with $x \to -1$ at order $\epsilon^{(n+1)}$ and $\epsilon^{(n+2)}$ gives all constraints. These constraints are very powerful. We found that, together with condition that the solution be real for $x > 0$, they determine most boundary conditions.

The only additional information needed can easily be obtained from the propagator-type integral f_1, which can be expressed in terms of Γ functions,

$$
\begin{aligned}
f_1^A &= e^{3\epsilon\gamma_{\rm E}}\Gamma^4(1-\epsilon)\Gamma(1+3\epsilon)/\Gamma(1-4\epsilon)\\
&= 1-\epsilon^2\,\frac{\pi^2}{4}-29\epsilon^3\,\zeta_3-\epsilon^4\,\frac{71}{160}\pi^4+\epsilon^5\left(\frac{29}{4}\pi^2\zeta_3-\frac{1263}{5}\zeta_5\right)\\
&\quad+\epsilon^6\left(-\frac{11539}{24192}\pi^6+\frac{841}{2}\zeta_3^2\right)+\mathcal{O}(\epsilon^7)\,.
\end{aligned}
\tag{3.4}
$$

3.2 Summary and explicit results

In summary, the equations (3.1), together with finiteness at $x\to-1$, reality of the solution in the region $x>0$, and the exact result for the trivial integral (3.4) determines all basis functions to all orders in ϵ. The solution can be obtained in an algebraic way. At each order ϵ^n, it is given by a linear combination of HPLs. The transcendental weight of each term is n. In Appendix B, we present explicit results for the ten-propagator integrals, up to order ϵ^6, i.e. transcendental weight 6. Explicit results for all integrals, and up to weight 6, can be found in the ancillary files `resultA.m` and `resultE.m`.

We performed a series of analytical and numerical checks of our results. The highest poles in ϵ were evaluated using the general Mellin-Barnes representations derived in refs. [23, 24]. The two known analytical results for the triple box without numerator [23], i.e. f_{24}^A and for the tennis court diagram with a special numerator [24] i.e. f_{39}^E also served as important checks. All the master integrals (except for the ten-propagator integrals of family E) were also numerically checked with `FIESTA` [42, 43] with sufficient accuracy.

Finally, we wish to mention that the *symbol* [44, 45] of the terms in the solution can be obtained in an even more straightforward way, and in that case the only information required in addition to eq. (3.1) is the value of the first term in the ϵ expansion. The latter follows from the boundary conditions, as explained above, but we give it here for convenience. We have

$$
\begin{aligned}
f^{A,0}=&\{1,1,-\frac{1}{9},-\frac{1}{6},1,-1,\frac{16}{9},0,1,\frac{1}{4},0,0,1,-\frac{1}{4},0,-\frac{1}{4},0,-\frac{4}{9},\frac{49}{36},0,\frac{7}{3},\frac{25}{9},\\
&-\frac{16}{9},\frac{16}{9},-\frac{49}{36},-\frac{4}{9}\}\,,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
f^{E,0}=&\{1,1,\frac{2}{9},\frac{1}{3},-1,1,\frac{1}{3},1,-\frac{32}{9},0,0,0,-\frac{8}{3},0,0,\frac{1}{9},\frac{2}{9},-\frac{8}{9},-\frac{8}{9},0,0,-1,0,-\frac{4}{9},-\frac{32}{9},\\
&-\frac{8}{9},0,-1,\frac{77}{9},-\frac{16}{3},0,\frac{28}{3},-\frac{49}{9},0,-\frac{49}{9},0,-\frac{2}{9},-\frac{14}{9},\frac{128}{9},-\frac{98}{9},-\frac{56}{9}\}\,.
\end{aligned}
\tag{3.6}
$$

This, together with the differential equations (3.1) and the explicit form of the matrices a and b given in eqs. (A.1) - (A.4) completely specifies the symbol of the answer, to any order in ϵ.

4 Discussion and outlook

In this work, we computed the master integrals for planar massless four-point integrals. Via IBP, they are sufficient to compute all integrals relevant for virtual corrections to $2\to2$

scattering at that order. We wrote out results in the small ϵ expansion up to weight six, and using the information provided here, higher-order results can be obtained at will.

It is interesting to note that as a by-product of our analysis, we also obtained result for three-loop single scale integrals that naïrely cannot be obtained from differential equations. We found that they were entirely determined from consistency of the system of differential equations with the physical boundary conditions.

We focused on the phenomenologically relevant expansion of the master integrals for $\epsilon \to 0$, and solved this problem in principle to all orders in ϵ. It is interesting to ask if one can write down a solution for the master integrals valid for finite ϵ. The Knizhnik-Zamolodchikov equations should be a good starting point for such an analysis. See for example ref. [46] and references therein for cases where the solution can be expressed in terms of (generalized) hypergeometric functions.

An obvious future direction is to apply this method to previously unknown non-planar integrals at three loops. The latter are required in order to evaluate the three-loop non-planar contributions to supersymmetric Yang-Mills and supergravity theories, where explicit representations in terms of loop integrals are available, see [33] and references therein.

The knowledge that certain integrals are pure functions of uniform transcendentality, can also be of practical advantage independently of the differential equations methods. Apart from serving as an important check of calculations, this property simplifies very much the application of the so-called `PSLQ` algorithm [47] because one then needs to consider only transcendental numbers of a given weight, and not numbers of lower weights. Another characteristic example of uniform transcendentality is within the method suggested in ref. [48], where the dependence of the coefficient at the n-th term of a Taylor series is revealed from the information about finite number of terms and the uniform transcendentality essentially restricts the number of terms in the corresponding Ansatz.

It would be interesting to understand further criteria for integrals to be pure functions of uniform transcendentally. It is possible that this might also be of interest for mathematicians, who have been investigating transcendental properties of Feynman integrals, see e.g. [31, 49] and references therein, albeit usually for particular classes of single-scale off-shell integrals in strictly four dimensions.

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A Matrices in Knizhnik-Zamolodchikov equation

The non-zero matrix elements of a and b in (3.1) for both cases are given by the following relations:

$$\begin{aligned}
a^A_{1,1} &= -3, a^A_{7,1} = 4/3, a^A_{7,7} = -3, a^A_{8,1} = -1/6, a^A_{8,4} = -1, a^A_{8,8} = -3, a^A_{9,1} = 1,\\
a^A_{9,9} &= -3, a^A_{11,1} = -1/3, a^A_{11,2} = 1/3, a^A_{11,11} = -3, a^A_{14,1} = -1/4, a^A_{14,9} = 1/2,\\
a^A_{15,15} &= 3, a^A_{16,1} = 1/3, a^A_{16,4} = -8, a^A_{16,5} = -8/3, a^A_{16,15} = 12, a^A_{16,16} = -3,\\
a^A_{18,1} &= 2, a^A_{18,3} = -2/3, a^A_{18,4} = -40/9, a^A_{18,7} = -1, a^A_{18,8} = -24, a^A_{18,9} = -2,\\
a^A_{18,10} &= 4/3, a^A_{18,14} = -8/3, a^A_{18,18} = 1, a^A_{18,19} = 2/3, a^A_{19,1} = -2, a^A_{19,4} = 8,\\
a^A_{19,7} &= 3/2, a^A_{19,8} = 24, a^A_{19,9} = 2, a^A_{19,19} = -3, a^A_{20,1} = 23/27, a^A_{20,2} = 17/54,\\
a^A_{20,3} &= -1/6, a^A_{20,4} = -56/9, a^A_{20,5} = -14/9, a^A_{20,6} = 1/6, a^A_{20,7} = -1, a^A_{20,8} = -20/3,\\
a^A_{20,11} &= -2, a^A_{20,15} = 8/3, a^A_{20,16} = -2, a^A_{20,17} = -2/3, a^A_{20,20} = 1, a^A_{20,21} = 1/3,\\
a^A_{21,1} &= -4/3, a^A_{21,2} = -4/3, a^A_{21,7} = 3, a^A_{21,11} = 12, a^A_{21,21} = -3, a^A_{22,1} = -4/3,\\
a^A_{22,2} &= -4/3, a^A_{22,7} = 3, a^A_{22,11} = 12, a^A_{22,22} = -3, a^A_{23,1} = 20/9, a^A_{23,2} = 19/9,\\
a^A_{23,3} &= -2, a^A_{23,7} = -3, a^A_{23,11} = -20, a^A_{23,12} = 1, a^A_{23,22} = 2, a^A_{23,23} = 1,\\
a^A_{24,19} &= 4, a^A_{24,21} = -4, a^A_{24,22} = 2, a^A_{24,24} = -3, a^A_{25,1} = -8/3, a^A_{25,2} = 41/18,\\
a^A_{25,3} &= -7/2, a^A_{25,4} = 68/3, a^A_{25,5} = 14/9, a^A_{25,6} = 7/2, a^A_{25,8} = 48, a^A_{25,9} = 4,\\
a^A_{25,10} &= 3, a^A_{25,11} = -12, a^A_{25,12} = 3, a^A_{25,13} = 1, a^A_{25,17} = -2, a^A_{25,19} = -6,\\
a^A_{25,21} &= 6, a^A_{25,22} = -2, a^A_{25,24} = 2, a^A_{25,25} = 1, a^A_{26,1} = -28/9, a^A_{26,2} = -7/6,\\
a^A_{26,3} &= 9/2, a^A_{26,4} = 20/3, a^A_{26,5} = 22/9, a^A_{26,6} = 3/2, a^A_{26,7} = 3, a^A_{26,8} = 16,\\
a^A_{26,10} &= 3, a^A_{26,11} = 12, a^A_{26,13} = 1, a^A_{26,15} = -16, a^A_{26,16} = 4,\\
a^A_{26,17} &= 6, a^A_{26,19} = -2, a^A_{26,20} = -12, a^A_{26,21} = 2, a^A_{26,22} = -3,\\
a^A_{26,23} &= -3, a^A_{26,24} = 1, a^A_{26,25} = 1\,,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
b^A_{7,1} &= -4/3, b^A_{7,3} = 4, b^A_{7,7} = 1, b^A_{8,8} = 2, b^A_{9,1} = -1, b^A_{9,9} = 2, b^A_{9,14} = 4,\\
b^A_{11,11} &= 3, b^A_{14,1} = 1/4, b^A_{14,9} = -1/2, b^A_{14,14} = -1, b^A_{15,1} = -1/12, b^A_{15,4} = 2,\\
b^A_{15,5} &= 2/3, b^A_{15,8} = 2, b^A_{15,15} = -3, b^A_{15,16} = 1, b^A_{16,1} = -1/3, b^A_{16,4} = 8,\\
b^A_{16,5} &= 8/3, b^A_{16,15} = -12, b^A_{16,16} = 4, b^A_{18,1} = -2, b^A_{18,3} = 2/3, b^A_{18,4} = 40/9,\\
b^A_{18,7} &= 1, b^A_{18,8} = 24, b^A_{18,9} = 2, b^A_{18,10} = 2/3, b^A_{18,14} = 8/3, b^A_{18,18} = -1,\\
b^A_{18,19} &= -2/3, b^A_{19,1} = 2, b^A_{19,3} = 4, b^A_{19,4} = -40/3, b^A_{19,7} = -3/2, b^A_{19,8} = -24,\\
b^A_{19,9} &= -2, b^A_{19,10} = -2, b^A_{19,18} = 3, b^A_{19,19} = 2, b^A_{20,20} = 1, b^A_{21,1} = -16/9,\\
b^A_{21,2} &= 13/9, b^A_{21,3} = 7, b^A_{21,4} = 40/3, b^A_{21,5} = 4, b^A_{21,6} = 2, b^A_{21,8} = 16,\\
b^A_{21,15} &= -16, b^A_{21,16} = 4, b^A_{21,17} = 4, b^A_{21,20} = -12, b^A_{21,21} = 1, b^A_{22,1} = 4/3,\\
b^A_{22,2} &= 5/3, b^A_{22,3} = 6, b^A_{22,7} = -3, b^A_{22,11} = -12, b^A_{22,12} = -3, b^A_{22,22} = 3,\\
b^A_{22,23} &= 3, b^A_{23,1} = -20/9, b^A_{23,2} = -10/9, b^A_{23,7} = 3, b^A_{23,11} = 20, b^A_{23,12} = 2,
\end{aligned}$$

$$
\begin{aligned}
b^A_{23,22} &= -2, b^A_{23,23} = -2, b^A_{24,2} = -17/9, b^A_{24,3} = 7, b^A_{24,4} = -40/3, b^A_{24,5} = -28/9,\\
b^A_{24,6} &= -7, b^A_{24,10} = -6, b^A_{24,12} = -6, b^A_{24,13} = -2, b^A_{24,17} = 4, b^A_{24,19} = -4,\\
b^A_{24,21} &= 4, b^A_{24,22} = -2, b^A_{24,24} = 3, b^A_{24,25} = 2, b^A_{24,26} = 2, b^A_{25,1} = 52/9,\\
b^A_{25,2} &= -1/2, b^A_{25,3} = -5/2, b^A_{25,4} = -100/3, b^A_{25,5} = -22/9, b^A_{25,6} = 3/2, b^A_{25,7} = -3,\\
b^A_{25,8} &= -64, b^A_{25,9} = -4, b^A_{25,10} = -1, b^A_{25,12} = 3, b^A_{25,13} = 1, b^A_{25,15} = 16,\\
b^A_{25,16} &= -4, b^A_{25,17} = -6, b^A_{25,18} = 6, b^A_{25,19} = 6, b^A_{25,20} = 12, b^A_{25,21} = -4,\\
b^A_{25,22} &= 2, b^A_{25,24} = -2, b^A_{25,25} = -1, b^A_{25,26} = -2, b^A_{26,1} = 28/9, b^A_{26,2} = 7/6,\\
b^A_{26,3} &= -9/2, b^A_{26,4} = -20/3, b^A_{26,5} = -22/9, b^A_{26,6} = 3/2,\\
b^A_{26,7} &= -3, b^A_{26,8} = -16, b^A_{26,10} = 3, b^A_{26,11} = -12, b^A_{26,13} = 1, b^A_{26,15} = 16,\\
b^A_{26,16} &= -4, b^A_{26,17} = -6, b^A_{26,19} = 2, b^A_{26,20} = 12, b^A_{26,21} = -2, b^A_{26,22} = 3,\\
b^A_{26,23} &= 3, b^A_{26,24} = -1, b^A_{26,25} = -1\,,
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
a^E_{2,2} &= -3, a^E_{3,3} = -3, a^E_{4,4} = -3, a^E_{5,5} = -3, a^E_{6,6} = -3, a^E_{9,3} = -4, a^E_{9,9} = -1,\\
a^E_{10,1} &= 2/3, a^E_{10,4} = -2, a^E_{10,10} = -2, a^E_{11,11} = -3, a^E_{12,2} = -2/3, a^E_{12,7} = 2,\\
a^E_{12,12} &= -3, a^E_{13,4} = -4, a^E_{13,13} = -2, a^E_{14,1} = 4/3, a^E_{14,2} = -4/3, a^E_{14,14} = -3,\\
a^E_{15,15} &= -3, a^E_{17,17} = -3, a^E_{18,7} = -8/3, a^E_{18,11} = -4, a^E_{18,12} = 4, a^E_{18,18} = -1,\\
a^E_{19,4} &= -8, a^E_{19,6} = 16/3, a^E_{19,13} = 2, a^E_{19,19} = -3, a^E_{20,20} = -3, a^E_{21,1} = -2/3,\\
a^E_{21,4} &= -8, a^E_{21,6} = 16/3, a^E_{21,10} = 4, a^E_{21,21} = -3, a^E_{21,22} = 2, a^E_{22,22} = -4,\\
a^E_{23,1} &= 1/3, a^E_{23,4} = 8/3, a^E_{23,7} = 4, a^E_{23,10} = -4, a^E_{23,13} = 1, a^E_{23,15} = -10/3,\\
a^E_{23,16} &= -3, a^E_{23,23} = -3, a^E_{23,24} = -1, a^E_{24,24} = -3, a^E_{25,17} = -4, a^E_{25,25} = -1,\\
a^E_{26,26} &= -3, a^E_{27,27} = 3, a^E_{28,2} = 4/3, a^E_{28,7} = 16, a^E_{28,8} = -32/3, a^E_{28,27} = -6,\\
a^E_{28,28} &= -3, a^E_{29,2} = -4, a^E_{29,4} = 16, a^E_{29,5} = -6, a^E_{29,6} = -16/3, a^E_{29,29} = -3,\\
a^E_{29,30} &= -2, a^E_{30,5} = 6, a^E_{30,30} = -3, a^E_{31,1} = 184/27, a^E_{31,2} = 68/27, a^E_{31,3} = 2/3,\\
a^E_{31,4} &= 224/9, a^E_{31,5} = 4/3, a^E_{31,6} = -112/9, a^E_{31,9} = 4, a^E_{31,10} = -40/3, a^E_{31,14} = -4,\\
a^E_{31,20} &= 2/3, a^E_{31,21} = -8/3, a^E_{31,22} = -4, a^E_{31,31} = -5, a^E_{31,32} = 2/3, a^E_{32,1} = 112/9,\\
a^E_{32,2} &= -4/9, a^E_{32,3} = 14, a^E_{32,4} = 80/3, a^E_{32,5} = -8, a^E_{32,6} = -16, a^E_{32,9} = 6,\\
a^E_{32,10} &= -16, a^E_{32,14} = -12, a^E_{32,20} = 2, a^E_{32,21} = -8, a^E_{32,22} = -4, a^E_{32,31} = -6,\\
a^E_{32,32} &= -1, a^E_{33,2} = 38/3, a^E_{33,4} = 22/3, a^E_{33,6} = -24, a^E_{33,15} = 25/3, a^E_{33,22} = -4,\\
a^E_{33,24} &= 11/2, a^E_{33,33} = -3, a^E_{33,34} = 5, a^E_{34,34} = -3, a^E_{35,2} = -8/3, a^E_{35,4} = 8/3,\\
a^E_{35,15} &= 20/3, a^E_{35,24} = 4, a^E_{35,35} = -2, a^E_{36,1} = 29/6, a^E_{36,2} = 12/5, a^E_{36,4} = 40/3,\\
a^E_{36,7} &= 28, a^E_{36,8} = -40/3, a^E_{36,10} = -18, a^E_{36,12} = -8, a^E_{36,13} = 27/5, a^E_{36,15} = -8/3,\\
a^E_{36,16} &= -3/2, a^E_{36,24} = -8/5, a^E_{36,27} = -24/5, a^E_{36,28} = -4, a^E_{36,35} = -2/5, a^E_{36,36} = 1,\\
a^E_{37,37} &= -3, a^E_{38,2} = -4, a^E_{38,4} = -12, a^E_{38,6} = 16, a^E_{38,15} = -10, a^E_{38,22} = 4, a^E_{38,24} = -5,\\
a^E_{38,34} &= -10, a^E_{38,35} = 2, a^E_{38,38} = -3, a^E_{39,1} = -224/9, a^E_{39,2} = -64/9, a^E_{39,3} = -28,\\
a^E_{39,4} &= 104/3, a^E_{39,5} = -8, a^E_{39,6} = -64/3, a^E_{39,9} = -12, a^E_{39,10} = 32, a^E_{39,14} = 24,
\end{aligned}
$$

$$
\begin{aligned}
a^E_{39,15} &= 20, a^E_{39,20} = -4, a^E_{39,21} = 16, a^E_{39,24} = 10, a^E_{39,30} = -8, a^E_{39,31} = 12,\\
a^E_{39,32} &= -4, a^E_{39,34} = 20, a^E_{39,39} = -3, a^E_{39,40} = -2, a^E_{40,1} = 224/9, a^E_{40,2} = 136/9,\\
a^E_{40,3} &= 28, a^E_{40,4} = -32/3, a^E_{40,5} = 8, a^E_{40,6} = -32/3, a^E_{40,9} = 12, a^E_{40,10} = -32,\\
a^E_{40,14} &= -24, a^E_{40,20} = 4, a^E_{40,21} = -16, a^E_{40,22} = -8, a^E_{40,30} = 8, a^E_{40,31} = -12,\\
a^E_{40,32} &= 4, a^E_{40,40} = -3, a^E_{41,1} = -26/9, a^E_{41,2} = 124/3, a^E_{41,3} = -28, a^E_{41,4} = 208/3,\\
a^E_{41,5} &= 4, a^E_{41,6} = -32, a^E_{41,7} = 208/3, a^E_{41,8} = 128/9, a^E_{41,10} = 40, a^E_{41,12} = -140,\\
a^E_{41,13} &= 46, a^E_{41,14} = -18, a^E_{41,15} = -4/3, a^E_{41,16} = 2, a^E_{41,17} = 16, a^E_{41,18} = -6,\\
a^E_{41,19} &= -16, a^E_{41,20} = -4, a^E_{41,21} = -8, a^E_{41,22} = -12, a^E_{41,23} = 2, a^E_{41,24} = -3,\\
a^E_{41,25} &= -4, a^E_{41,26} = -2, a^E_{41,27} = 4, a^E_{41,28} = 4, a^E_{41,29} = -4,\\
a^E_{41,30} &= -4, a^E_{41,31} = 6, a^E_{41,32} = 2, a^E_{41,33} = -4, a^E_{41,35} = 2,\\
a^E_{41,37} &= 1, a^E_{41,38} = -1, a^E_{41,39} = 1, a^E_{41,40} = 1\,,
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
b^E_{9,1} &= 8/3, b^E_{9,3} = 4, b^E_{9,9} = 1, b^E_{10,10} = 2, b^E_{12,12} = 2, b^E_{13,4} = 4,\\
b^E_{13,7} &= 4, b^E_{13,13} = 1, b^E_{14,14} = 3, b^E_{18,7} = 8/3, b^E_{18,11} = 4, b^E_{18,12} = -4,\\
b^E_{18,18} &= 1, b^E_{19,4} = 8, b^E_{19,6} = -16/3, b^E_{19,13} = -2, b^E_{19,19} = 3, b^E_{21,1} = 2/3,\\
b^E_{21,4} &= 8, b^E_{21,6} = -16/3, b^E_{21,10} = -4, b^E_{21,21} = -3, b^E_{21,22} = -2, b^E_{22,1} = -4/3,\\
b^E_{22,4} &= -16, b^E_{22,6} = 32/3, b^E_{22,21} = 6, b^E_{22,22} = 4, b^E_{23,23} = 5, b^E_{24,1} = -2,\\
b^E_{24,2} &= -4/3, b^E_{24,4} = 16/3, b^E_{24,7} = 8, b^E_{24,10} = -8, b^E_{24,12} = 4, b^E_{24,15} = -8/3,\\
b^E_{24,16} &= -6, b^E_{24,23} = 4, b^E_{24,24} = 1, b^E_{25,8} = 8/3, b^E_{25,17} = 4, b^E_{25,25} = 1,\\
b^E_{26,7} &= 8, b^E_{26,8} = -16/3, b^E_{26,13} = -2, b^E_{26,26} = 3, b^E_{27,2} = 2/3, b^E_{27,7} = 8,\\
b^E_{27,8} &= -16/3, b^E_{27,12} = -4, b^E_{27,27} = -3, b^E_{27,28} = -2, b^E_{28,2} = -4/3, b^E_{28,7} = -16,\\
b^E_{28,8} &= 32/3, b^E_{28,27} = 6, b^E_{28,28} = 4, b^E_{29,2} = 4, b^E_{29,4} = -16, b^E_{29,5} = 6,\\
b^E_{29,6} &= 16/3, b^E_{29,18} = 3, b^E_{29,19} = 2, b^E_{29,29} = 2, b^E_{29,30} = 2, b^E_{30,2} = -2,\\
b^E_{30,5} &= -3, b^E_{30,6} = 8/3, b^E_{30,7} = -4, b^E_{30,11} = -6, b^E_{30,12} = 6, b^E_{30,13} = 2,\\
b^E_{30,18} &= -3, b^E_{30,19} = -4, b^E_{30,29} = -1, b^E_{30,30} = -1, b^E_{31,31} = 1, b^E_{32,1} = -64/9,\\
b^E_{32,2} &= 52/9, b^E_{32,3} = -14, b^E_{32,4} = -80/3, b^E_{32,5} = 8, b^E_{32,6} = 16, b^E_{32,10} = 16,\\
b^E_{32,20} &= -2, b^E_{32,21} = 8, b^E_{32,22} = 4, b^E_{32,31} = 6, b^E_{32,32} = 1, b^E_{33,1} = -2,\\
b^E_{33,2} &= -18, b^E_{33,4} = -18, b^E_{33,6} = 24, b^E_{33,7} = -16, b^E_{33,10} = 16, b^E_{33,12} = 16,\\
b^E_{33,13} &= -8, b^E_{33,15} = -3, b^E_{33,16} = 2, b^E_{33,22} = 4, b^E_{33,23} = -8, b^E_{33,24} = -3/2,\\
b^E_{33,33} &= 2, b^E_{33,34} = -5, b^E_{34,1} = -13/15, b^E_{34,2} = -16/5, b^E_{34,4} = -122/5, b^E_{34,6} = 40/3,\\
b^E_{34,7} &= -12, b^E_{34,10} = 28/5, b^E_{34,12} = 58/5, b^E_{34,13} = -16/5, b^E_{34,15} = 1/5, b^E_{34,16} = 11/5,\\
b^E_{34,21} &= 24/5, b^E_{34,22} = 4, b^E_{34,23} = -14/5, b^E_{34,24} = -3/5, b^E_{34,33} = 2/5, b^E_{34,34} = -1,\\
b^E_{35,1} &= -39/2, b^E_{35,2} = -14/3, b^E_{35,4} = -52/3, b^E_{35,7} = -24, b^E_{35,8} = 24, b^E_{35,10} = 22,\\
b^E_{35,12} &= 22, b^E_{35,13} = -11, b^E_{35,15} = 2/3, b^E_{35,16} = 3/2, b^E_{35,23} = -6, b^E_{35,24} = 3/2,\\
b^E_{35,28} &= 4, b^E_{35,35} = 2, b^E_{35,36} = -5, b^E_{36,1} = -9/2, b^E_{36,2} = -12/5, b^E_{36,4} = -32/3,
\end{aligned}
$$

$$
\begin{aligned}
b^E_{36,7} &= -24, b^E_{36,8} = 40/3, b^E_{36,10} = 14, b^E_{36,12} = 8, b^E_{36,13} = -22/5, b^E_{36,15} = -2/3,\\
b^E_{36,16} &= -3/2, b^E_{36,23} = 2, b^E_{36,24} = 3/5, b^E_{36,27} = 24/5, b^E_{36,28} = 4, b^E_{36,35} = 2/5,\\
b^E_{36,36} &= -1, b^E_{38,2} = 4, b^E_{38,4} = 12, b^E_{38,6} = -16, b^E_{38,15} = 10, b^E_{38,22} = -4,\\
b^E_{38,24} &= 5, b^E_{38,34} = 10, b^E_{38,35} = -2, b^E_{38,38} = 3, b^E_{39,1} = 68/9, b^E_{39,2} = -8/9,\\
b^E_{39,3} &= 28, b^E_{39,4} = -56, b^E_{39,5} = 8, b^E_{39,6} = 64/3, b^E_{39,12} = 8, b^E_{39,13} = -12,\\
b^E_{39,14} &= -12, b^E_{39,15} = -52/3, b^E_{39,16} = -4, b^E_{39,18} = 12, b^E_{39,20} = 4, b^E_{39,21} = -16,\\
b^E_{39,23} &= -4, b^E_{39,24} = -8, b^E_{39,25} = 8, b^E_{39,26} = 4, b^E_{39,30} = 8, b^E_{39,31} = -12,\\
b^E_{39,32} &= 4, b^E_{39,34} = -20, b^E_{39,38} = 2, b^E_{39,39} = 3, b^E_{39,40} = 2, b^E_{39,41} = 2,\\
b^E_{40,1} &= -68/9, b^E_{40,2} = -64/9, b^E_{40,3} = -28, b^E_{40,4} = 32, b^E_{40,5} = -8, b^E_{40,6} = 32/3,\\
b^E_{40,12} &= -8, b^E_{40,13} = 12, b^E_{40,14} = 12, b^E_{40,15} = -8/3, b^E_{40,16} = 4, b^E_{40,18} = -12,\\
b^E_{40,20} &= -4, b^E_{40,21} = 16, b^E_{40,22} = 8, b^E_{40,23} = 4, b^E_{40,24} = -2, b^E_{40,25} = -8,\\
b^E_{40,26} &= -4, b^E_{40,30} = -8, b^E_{40,31} = 12, b^E_{40,32} = -4, b^E_{40,35} = 4, b^E_{40,37} = 1,\\
b^E_{40,38} &= -6, b^E_{40,39} = -2, b^E_{40,40} = -1, b^E_{40,41} = -2, b^E_{41,1} = 26/9, b^E_{41,2} = -332/9,\\
b^E_{41,3} &= 28, b^E_{41,4} = -208/3, b^E_{41,5} = -4, b^E_{41,6} = 32, b^E_{41,7} = -80, b^E_{41,10} = -40,\\
b^E_{41,12} &= 124, b^E_{41,13} = -30, b^E_{41,14} = 18, b^E_{41,15} = 4/3, b^E_{41,16} = -2, b^E_{41,17} = -16,\\
b^E_{41,18} &= 6, b^E_{41,19} = 16, b^E_{41,20} = 4, b^E_{41,21} = 8, b^E_{41,22} = 12, b^E_{41,23} = -2,\\
b^E_{41,24} &= 3, b^E_{41,25} = -4, b^E_{41,26} = -6, b^E_{41,29} = 4, b^E_{41,30} = 4, b^E_{41,31} = -6,\\
b^E_{41,32} &= -2, b^E_{41,33} = 4, b^E_{41,35} = -2, b^E_{41,37} = -1, b^E_{41,38} = 1,\\
b^E_{41,39} &= -1, b^E_{41,40} = -1\,.
\end{aligned}
\tag{A.4}
$$

B Explicit results up to weight six

Here are results for master integrals with ten propagators. We denote harmonic polylogarithms [41] by $H_{\vec{w}} = H_{\vec{w}}(x)$. All the other results can be found in the ancillary files `resultA.m` and `resultE.m`.

B.1 Triple ladder master integrals

$$
\begin{aligned}
f^A_{24}(x,\epsilon) = {} & \frac{16}{9} - \frac{11}{3}\epsilon H_0 + \epsilon^2\left(-\frac{3\pi^2}{2} + 6H_{0,0}\right) + \epsilon^3\left(-\frac{3}{2}\pi^2 H_{-1} + \frac{65}{12}\pi^2 H_0 - 3H_{-1,0,0}\right.\\
& \left. -3H_{0,0,0} - \frac{131\zeta_3}{9}\right) + \epsilon^4\left(-\frac{1411\pi^4}{1080} - \frac{3}{2}\pi^2 H_{-1,-1} + \frac{7}{2}\pi^2 H_{-1,0} + \frac{23}{2}\pi^2 H_{0,-1} - 19\pi^2 H_{0,0}\right.\\
& \left. -3H_{-1,-1,0,0} + 18H_{-1,0,0,0} + 23H_{0,-1,0,0} - 36H_{0,0,0,0} - 3H_{-1}\zeta_3 + \frac{82}{3}H_0\zeta_3\right)\\
& +\epsilon^5\left(-\frac{13}{8}\pi^4 H_{-1} + \frac{683}{160}\pi^4 H_0 - \frac{3}{2}\pi^2 H_{-1,-1,-1} + \frac{7}{2}\pi^2 H_{-1,-1,0} + \frac{35}{2}\pi^2 H_{-1,0,-1}\right.\\
& -\frac{55}{4}\pi^2 H_{-1,0,0} + \frac{47}{2}\pi^2 H_{0,-1,-1} - \frac{185}{6}\pi^2 H_{0,-1,0} - \frac{119}{2}\pi^2 H_{0,0,-1} + \frac{261}{4}\pi^2 H_{0,0,0}\\
& -3H_{-1,-1,-1,0,0} + 18H_{-1,-1,0,0,0} + 35H_{-1,0,-1,0,0} - 81H_{-1,0,0,0,0} + 47H_{0,-1,-1,0,0}\\
& -138H_{0,-1,0,0,0} - 119H_{0,0,-1,0,0} + 243H_{0,0,0,0,0} + \frac{73\pi^2\zeta_3}{4} - 3H_{-1,-1}\zeta_3 - 49H_{-1,0}\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+47H_{0,-1}\zeta_3-33H_{0,0}\zeta_3-\frac{301\zeta_5}{15}\Bigg)+\epsilon^6\Bigg(-\frac{624607\pi^6}{544320}-\frac{13}{8}\pi^4H_{-1,-1}+\frac{323}{120}\pi^4H_{-1,0}\\
&+\frac{641}{72}\pi^4H_{0,-1}-\frac{665}{48}\pi^4H_{0,0}-\frac{3}{2}\pi^2H_{-1,-1,-1,-1}+\frac{7}{2}\pi^2H_{-1,-1,-1,0}+\frac{35}{2}\pi^2H_{-1,-1,0,-1}\\
&-\frac{55}{4}\pi^2H_{-1,-1,0,0}+\frac{107}{2}\pi^2H_{-1,0,-1,-1}-\frac{317}{6}\pi^2H_{-1,0,-1,0}-\frac{151}{2}\pi^2H_{-1,0,0,-1}\\
&+51\pi^2H_{-1,0,0,0}+\frac{71}{2}\pi^2H_{0,-1,-1,-1}-\frac{353}{6}\pi^2H_{0,-1,-1,0}-\frac{247}{2}\pi^2H_{0,-1,0,-1}\\
&+\frac{427}{4}\pi^2H_{0,-1,0,0}-\frac{311}{2}\pi^2H_{0,0,-1,-1}+\frac{1025}{6}\pi^2H_{0,0,-1,0}+\frac{531}{2}\pi^2H_{0,0,0,-1}\\
&-\frac{441}{2}\pi^2H_{0,0,0,0}-3H_{-1,-1,-1,-1,0,0}+18H_{-1,-1,-1,0,0,0}+35H_{-1,-1,0,-1,0,0}\\
&-81H_{-1,-1,0,0,0,0}+107H_{-1,0,-1,-1,0,0}-210H_{-1,0,-1,0,0,0}-151H_{-1,0,0,-1,0,0}\\
&+324H_{-1,0,0,0,0,0}+71H_{0,-1,-1,-1,0,0}-282H_{0,-1,-1,0,0,0}-247H_{0,-1,0,-1,0,0}\\
&+621H_{0,-1,0,0,0,0}-311H_{0,0,-1,-1,0,0}+714H_{0,0,-1,0,0,0}+531H_{0,0,0,-1,0,0}\\
&-1134H_{0,0,0,0,0,0}-\frac{37}{12}\pi^2H_{-1}\zeta_3-\frac{220}{3}\pi^2H_0\zeta_3-3H_{-1,-1,-1}\zeta_3-49H_{-1,-1,0}\zeta_3\\
&+107H_{-1,0,-1}\zeta_3+138H_{-1,0,0}\zeta_3+71H_{0,-1,-1}\zeta_3+141H_{0,-1,0}\zeta_3-311H_{0,0,-1}\zeta_3\\
&-48H_{0,0,0}\zeta_3+\frac{167\zeta_3^2}{9}+57H_{-1}\zeta_5-\frac{444}{5}H_0\zeta_5\Bigg)+\mathcal{O}(\epsilon^7)\,. \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
f^A_{25}(x,\epsilon)=&-\frac{49}{36}+\frac{5}{2}\epsilon H_0+\epsilon^2\left(\frac{241\pi^2}{144}-3H_{0,0}\right)+\epsilon^3\Bigg(\frac{11}{4}\pi^2H_{-1}-\frac{47}{8}\pi^2H_0+\frac{11}{2}H_{-1,0,0}\\
&-\frac{9}{2}H_{0,0,0}+\frac{641\zeta_3}{36}\Bigg)+\epsilon^4\Bigg(\frac{847\pi^4}{640}+\frac{23}{4}\pi^2H_{-1,-1}-\frac{89}{12}\pi^2H_{-1,0}-\frac{63}{4}\pi^2H_{0,-1}+\frac{39}{2}\pi^2H_{0,0}\\
&+\frac{23}{2}H_{-1,-1,0,0}-33H_{-1,0,0,0}-\frac{63}{2}H_{0,-1,0,0}+54H_{0,0,0,0}+\frac{23}{2}H_{-1}\zeta_3-39H_0\zeta_3\Bigg)\\
&+\epsilon^5\Bigg(\frac{1609}{720}\pi^4H_{-1}-\frac{4141}{960}\pi^4H_0+\frac{35}{4}\pi^2H_{-1,-1,-1}-\frac{173}{12}\pi^2H_{-1,-1,0}-\frac{119}{4}\pi^2H_{-1,0,-1}\\
&+\frac{207}{8}\pi^2H_{-1,0,0}-\frac{171}{4}\pi^2H_{0,-1,-1}+\frac{183}{4}\pi^2H_{0,-1,0}+\frac{287}{4}\pi^2H_{0,0,-1}-\frac{513}{8}\pi^2H_{0,0,0}\\
&+\frac{35}{2}H_{-1,-1,-1,0,0}-69H_{-1,-1,0,0,0}-\frac{119}{2}H_{-1,0,-1,0,0}+\frac{297}{2}H_{-1,0,0,0,0}-\frac{171}{2}H_{0,-1,-1,0,0}\\
&+189H_{0,-1,0,0,0}+\frac{287}{2}H_{0,0,-1,0,0}-\frac{567}{2}H_{0,0,0,0,0}-\frac{3737\pi^2\zeta_3}{144}+\frac{35}{2}H_{-1,-1}\zeta_3+\frac{65}{2}H_{-1,0}\zeta_3\\
&-\frac{171}{2}H_{0,-1}\zeta_3+\frac{117}{2}H_{0,0}\zeta_3+\frac{1143\zeta_5}{20}\Bigg)+\epsilon^6\Bigg(\frac{3710783\pi^6}{4354560}+\frac{3181}{720}\pi^4H_{-1,-1}\\
&-\frac{4181}{720}\pi^4H_{-1,0}-\frac{185}{16}\pi^4H_{0,-1}+\frac{2111}{160}\pi^4H_{0,0}+\frac{47}{4}\pi^2H_{-1,-1,-1,-1}-\frac{257}{12}\pi^2H_{-1,-1,-1,0}\\
&-\frac{203}{4}\pi^2H_{-1,-1,0,-1}+\frac{395}{8}\pi^2H_{-1,-1,0,0}-\frac{371}{4}\pi^2H_{-1,0,-1,-1}+\frac{1085}{12}\pi^2H_{-1,0,-1,0}\\
&+\frac{531}{4}\pi^2H_{-1,0,0,-1}-\frac{177}{2}\pi^2H_{-1,0,0,0}-\frac{471}{4}\pi^2H_{0,-1,-1,-1}+\frac{499}{4}\pi^2H_{0,-1,-1,0}\\
&+\frac{755}{4}\pi^2H_{0,-1,0,-1}-\frac{1203}{8}\pi^2H_{0,-1,0,0}+\frac{923}{4}\pi^2H_{0,0,-1,-1}-\frac{2645}{12}\pi^2H_{0,0,-1,0}
\end{aligned}
$$

$$
-\frac{1179}{4}\pi^2 H_{0,0,0,-1}+\frac{837}{4}\pi^2 H_{0,0,0,0}+\frac{47}{2}H_{-1,-1,-1,-1,0,0}-105H_{-1,-1,-1,0,0,0}
-\frac{203}{2}H_{-1,-1,0,-1,0,0}+\frac{621}{2}H_{-1,-1,0,0,0,0}-\frac{371}{2}H_{-1,0,-1,-1,0,0}+357H_{-1,0,-1,0,0,0}
+\frac{531}{2}H_{-1,0,0,-1,0,0}-594H_{-1,0,0,0,0,0}-\frac{471}{2}H_{0,-1,-1,-1,0,0}+513H_{0,-1,-1,0,0,0}
+\frac{755}{2}H_{0,-1,0,-1,0,0}-\frac{1701}{2}H_{0,-1,0,0,0,0}+\frac{923}{2}H_{0,0,-1,-1,0,0}-861H_{0,0,-1,0,0,0}
-\frac{1179}{2}H_{0,0,0,-1,0,0}+1215H_{0,0,0,0,0,0}-\frac{703}{24}\pi^2 H_{-1}\zeta_3+93\pi^2 H_0\zeta_3+\frac{47}{2}H_{-1,-1,-1}\zeta_3
+\frac{149}{2}H_{-1,-1,0}\zeta_3-\frac{371}{2}H_{-1,0,-1}\zeta_3-137H_{-1,0,0}\zeta_3-\frac{471}{2}H_{0,-1,-1}\zeta_3-\frac{13}{2}H_{0,-1,0}\zeta_3
+\frac{923}{2}H_{0,0,-1}\zeta_3-\frac{9901\zeta_3^2}{72}+\frac{163}{2}H_{-1}\zeta_5-82H_0\zeta_5\Bigg)+\mathcal{O}(\epsilon^7)\,. \tag{B.2}
$$

$$
f_{26}^A(x,\epsilon)=-\frac{4}{9}+\frac{13\pi^2\epsilon^2}{36}+\frac{1}{2}\epsilon H_0+\epsilon^3\left(\frac{9}{4}\pi^2 H_{-1}-\frac{15}{8}\pi^2 H_0+\frac{9}{2}H_{-1,0,0}-\frac{9}{2}H_{0,0,0}-\frac{71\zeta_3}{18}\right)
+\epsilon^4\Bigg(\frac{61\pi^4}{720}+\frac{21}{4}\pi^2 H_{-1,-1}-\frac{25}{4}\pi^2 H_{-1,0}-\frac{21}{4}\pi^2 H_{0,-1}+\frac{25}{4}\pi^2 H_{0,0}+\frac{21}{2}H_{-1,-1,0,0}
-27H_{-1,0,0,0}-\frac{21}{2}H_{0,-1,0,0}+27H_{0,0,0,0}+\frac{21}{2}H_{-1}\zeta_3-2H_0\zeta_3\Bigg)
+\epsilon^5\Bigg(\frac{337}{240}\pi^4 H_{-1}-\frac{1217}{960}\pi^4 H_0+\frac{33}{4}\pi^2 H_{-1,-1,-1}-\frac{53}{4}\pi^2 H_{-1,-1,0}-\frac{93}{4}\pi^2 H_{-1,0,-1}
+\frac{165}{8}\pi^2 H_{-1,0,0}-\frac{33}{4}\pi^2 H_{0,-1,-1}+\frac{53}{4}\pi^2 H_{0,-1,0}+\frac{93}{4}\pi^2 H_{0,0,-1}-\frac{165}{8}\pi^2 H_{0,0,0}
+\frac{33}{2}H_{-1,-1,-1,0,0}-63H_{-1,-1,0,0,0}-\frac{93}{2}H_{-1,0,-1,0,0}+\frac{243}{2}H_{-1,0,0,0,0}-\frac{33}{2}H_{0,-1,-1,0,0}
+63H_{0,-1,0,0,0}+\frac{93}{2}H_{0,0,-1,0,0}-\frac{243}{2}H_{0,0,0,0,0}-\frac{859\pi^2\zeta_3}{72}+\frac{33}{2}H_{-1,-1}\zeta_3+\frac{27}{2}H_{-1,0}\zeta_3
-\frac{33}{2}H_{0,-1}\zeta_3-\frac{27}{2}H_{0,0}\zeta_3-\frac{1457\zeta_5}{30}\Bigg)+\epsilon^6\Bigg(\frac{2029\pi^6}{217728}+\frac{287}{80}\pi^4 H_{-1,-1}-\frac{311}{80}\pi^4 H_{-1,0}
-\frac{287}{80}\pi^4 H_{0,-1}+\frac{311}{80}\pi^4 H_{0,0}+\frac{45}{4}\pi^2 H_{-1,-1,-1,-1}-\frac{81}{4}\pi^2 H_{-1,-1,-1,0}-\frac{177}{4}\pi^2 H_{-1,-1,0,-1}
+\frac{353}{8}\pi^2 H_{-1,-1,0,0}-\frac{249}{4}\pi^2 H_{-1,0,-1,-1}+\frac{269}{4}\pi^2 H_{-1,0,-1,0}+\frac{377}{4}\pi^2 H_{-1,0,0,-1}
-\frac{135}{2}\pi^2 H_{-1,0,0,0}-\frac{45}{4}\pi^2 H_{0,-1,-1,-1}+\frac{81}{4}\pi^2 H_{0,-1,-1,0}+\frac{177}{4}\pi^2 H_{0,-1,0,-1}
-\frac{353}{8}\pi^2 H_{0,-1,0,0}+\frac{249}{4}\pi^2 H_{0,0,-1,-1}-\frac{269}{4}\pi^2 H_{0,0,-1,0}-\frac{377}{4}\pi^2 H_{0,0,0,-1}+\frac{135}{2}\pi^2 H_{0,0,0,0}
+\frac{45}{2}H_{-1,-1,-1,-1,0,0}-99H_{-1,-1,-1,0,0,0}-\frac{177}{2}H_{-1,-1,0,-1,0,0}+\frac{567}{2}H_{-1,-1,0,0,0,0}
-\frac{249}{2}H_{-1,0,-1,-1,0,0}+279H_{-1,0,-1,0,0,0}+\frac{377}{2}H_{-1,0,0,-1,0,0}-486H_{-1,0,0,0,0,0}
-\frac{45}{2}H_{0,-1,-1,-1,0,0}+99H_{0,-1,-1,0,0,0}+\frac{177}{2}H_{0,-1,0,-1,0,0}-\frac{567}{2}H_{0,-1,0,0,0,0}
+\frac{249}{2}H_{0,0,-1,-1,0,0}-279H_{0,0,-1,0,0,0}-\frac{377}{2}H_{0,0,0,-1,0,0}+486H_{0,0,0,0,0,0}-\frac{255}{8}\pi^2 H_{-1}\zeta_3
$$

$$+\frac{97}{4}\pi^2 H_0\zeta_3+\frac{45}{2}H_{-1,-1,-1}\zeta_3+\frac{111}{2}H_{-1,-1,0}\zeta_3-\frac{249}{2}H_{-1,0,-1}\zeta_3-99H_{-1,0,0}\zeta_3$$
$$-\frac{45}{2}H_{0,-1,-1}\zeta_3-\frac{111}{2}H_{0,-1,0}\zeta_3+\frac{249}{2}H_{0,0,-1}\zeta_3+99H_{0,0,0}\zeta_3+\frac{275\zeta_3^2}{18}$$
$$-\frac{15}{2}H_{-1}\zeta_5+\frac{351}{5}H_0\zeta_5\Big)+\mathcal{O}(\epsilon^7)\,. \tag{B.3}$$

B.2 Tennis court master integrals

$$f_{39}^E(x,\epsilon)=\frac{128}{9}-\frac{52}{3}\epsilon H_0+\epsilon^2\left(-\frac{38\pi^2}{3}+8H_{0,0}\right)+\epsilon^3\Big(-10\pi^2 H_{-1}+\frac{157}{9}\pi^2 H_0-20H_{-1,0,0}$$
$$+28H_{0,0,0}-\frac{964\zeta_3}{9}\Big)+\epsilon^4\Big(\frac{2429\pi^4}{810}-10\pi^2 H_{-1,-1}+\frac{50}{3}\pi^2 H_{-1,0}+6\pi^2 H_{0,-1}-4\pi^2 H_{0,0}$$
$$-20H_{-1,-1,0,0}+80H_{-1,0,0,0}+12H_{0,-1,0,0}-64H_{0,0,0,0}-20H_{-1}\zeta_3+\frac{328}{3}H_0\zeta_3\Big)$$
$$+\epsilon^5\Big(\frac{5}{18}\pi^4 H_{-1}-\frac{10913\pi^4 H_0}{1080}-10\pi^2 H_{-1,-1,-1}+\frac{50}{3}\pi^2 H_{-1,-1,0}+30\pi^2 H_{-1,0,-1}$$
$$-\frac{71}{3}\pi^2 H_{-1,0,0}-26\pi^2 H_{0,-1,-1}+\frac{82}{3}\pi^2 H_{0,-1,0}+70\pi^2 H_{0,0,-1}-\frac{227}{3}\pi^2 H_{0,0,0}$$
$$-20H_{-1,-1,-1,0,0}+80H_{-1,-1,0,0,0}+60H_{-1,0,-1,0,0}-172H_{-1,0,0,0,0}-52H_{0,-1,-1,0,0}$$
$$+112H_{0,-1,0,0,0}+140H_{0,0,-1,0,0}-140H_{0,0,0,0,0}+\frac{3257\pi^2\zeta_3}{27}-20H_{-1,-1}\zeta_3-20H_{-1,0}\zeta_3$$
$$-52H_{0,-1}\zeta_3+52H_{0,0}\zeta_3-\frac{3556\zeta_5}{5}\Big)+\epsilon^6\Big(\frac{1391417\pi^6}{408240}+\frac{5}{18}\pi^4 H_{-1,-1}+\frac{641}{90}\pi^4 H_{-1,0}$$
$$-\frac{1207}{90}\pi^4 H_{0,-1}+\frac{3163}{180}\pi^4 H_{0,0}-10\pi^2 H_{-1,-1,-1,-1}+\frac{50}{3}\pi^2 H_{-1,-1,-1,0}+30\pi^2 H_{-1,-1,0,-1}$$
$$-\frac{71}{3}\pi^2 H_{-1,-1,0,0}+126\pi^2 H_{-1,0,-1,-1}-66\pi^2 H_{-1,0,-1,0}-98\pi^2 H_{-1,0,0,-1}+66\pi^2 H_{-1,0,0,0}$$
$$-218\pi^2 H_{0,-1,-1,-1}+\frac{562}{3}\pi^2 H_{0,-1,-1,0}+270\pi^2 H_{0,-1,0,-1}-\frac{527}{3}\pi^2 H_{0,-1,0,0}$$
$$+358\pi^2 H_{0,0,-1,-1}-\frac{926}{3}\pi^2 H_{0,0,-1,0}-394\pi^2 H_{0,0,0,-1}+\frac{746}{3}\pi^2 H_{0,0,0,0}$$
$$-20H_{-1,-1,-1,-1,0,0}+80H_{-1,-1,-1,0,0,0}+60H_{-1,-1,0,-1,0,0}-172H_{-1,-1,0,0,0,0}$$
$$+252H_{-1,0,-1,-1,0,0}-144H_{-1,0,-1,0,0,0}-196H_{-1,0,0,-1,0,0}+296H_{-1,0,0,0,0,0}$$
$$-436H_{0,-1,-1,-1,0,0}+688H_{0,-1,-1,0,0,0}+540H_{0,-1,0,-1,0,0}-940H_{0,-1,0,0,0,0}$$
$$+716H_{0,0,-1,-1,0,0}-1136H_{0,0,-1,0,0,0}-788H_{0,0,0,-1,0,0}+1208H_{0,0,0,0,0,0}+\frac{269}{3}\pi^2 H_{-1}\zeta_3$$
$$-\frac{1916}{9}\pi^2 H_0\zeta_3-20H_{-1,-1,-1}\zeta_3-20H_{-1,-1,0}\zeta_3+252H_{-1,0,-1}\zeta_3+32H_{-1,0,0}\zeta_3$$
$$-436H_{0,-1,-1}\zeta_3+44H_{0,-1,0}\zeta_3+716H_{0,0,-1}\zeta_3-608H_{0,0,0}\zeta_3+\frac{788\zeta_3^2}{3}$$
$$-516H_{-1}\zeta_5+\frac{8432}{5}H_0\zeta_5\Big)+\mathcal{O}(\epsilon^7)\,. \tag{B.4}$$

$$f_{40}^E(x,\epsilon)=-\frac{98}{9}+\frac{50}{3}\epsilon H_0+\epsilon^2\left(\frac{755\pi^2}{54}-10H_{0,0}\right)+\epsilon^3\Big(28\pi^2 H_{-1}-\frac{635}{18}\pi^2 H_0+56H_{-1,0,0}$$

$$
\begin{aligned}
&-58H_{0,0,0}+122\zeta_3\Big)+\epsilon^4\Big(\frac{331\pi^4}{144}+84\pi^2H_{-1,-1}-\frac{244}{3}\pi^2H_{-1,0}-92\pi^2H_{0,-1}+\frac{463}{6}\pi^2H_{0,0}\\
&+168H_{-1,-1,0,0}-320H_{-1,0,0,0}-184H_{0,-1,0,0}+310H_{0,0,0,0}+168H_{-1}\zeta_3-238H_0\zeta_3\Big)\\
&+\epsilon^5\Big(\frac{197}{45}\pi^4H_{-1}+\frac{91}{80}\pi^4H_0+284\pi^2H_{-1,-1,-1}-\frac{748}{3}\pi^2H_{-1,-1,0}-276\pi^2H_{-1,0,-1}\\
&+\frac{554}{3}\pi^2H_{-1,0,0}-308\pi^2H_{0,-1,-1}+\frac{724}{3}\pi^2H_{0,-1,0}+236\pi^2H_{0,0,-1}-\frac{665}{6}\pi^2H_{0,0,0}\\
&+568H_{-1,-1,-1,0,0}-928H_{-1,-1,0,0,0}-552H_{-1,0,-1,0,0}+1096H_{-1,0,0,0,0}-616H_{0,-1,-1,0,0}\\
&+832H_{0,-1,0,0,0}+472H_{0,0,-1,0,0}-826H_{0,0,0,0,0}-\frac{839\pi^2\zeta_3}{6}+568H_{-1,-1}\zeta_3-392H_{-1,0}\zeta_3\\
&-616H_{0,-1}\zeta_3+370H_{0,0}\zeta_3+\frac{17818\zeta_5}{15}\Big)+\epsilon^6\Big(-\frac{393371\pi^6}{181440}+\frac{527}{45}\pi^4H_{-1,-1}+\frac{59}{45}\pi^4H_{-1,0}\\
&+\frac{319}{45}\pi^4H_{0,-1}-\frac{7927}{240}\pi^4H_{0,0}+948\pi^2H_{-1,-1,-1,-1}-\frac{2452}{3}\pi^2H_{-1,-1,-1,0}\\
&-908\pi^2H_{-1,-1,0,-1}+\frac{1630}{3}\pi^2H_{-1,-1,0,0}-988\pi^2H_{-1,0,-1,-1}+\frac{2092}{3}\pi^2H_{-1,0,-1,0}\\
&+692\pi^2H_{-1,0,0,-1}-220\pi^2H_{-1,0,0,0}-1052\pi^2H_{0,-1,-1,-1}+\frac{2380}{3}\pi^2H_{0,-1,-1,0}\\
&+740\pi^2H_{0,-1,0,-1}-\frac{970}{3}\pi^2H_{0,-1,0,0}+836\pi^2H_{0,0,-1,-1}-\frac{1444}{3}\pi^2H_{0,0,-1,0}\\
&-332\pi^2H_{0,0,0,-1}-\frac{1009}{6}\pi^2H_{0,0,0,0}+1896H_{-1,-1,-1,-1,0,0}-3008H_{-1,-1,-1,0,0,0}\\
&-1816H_{-1,-1,0,-1,0,0}+2968H_{-1,-1,0,0,0,0}-1976H_{-1,0,-1,-1,0,0}+2208H_{-1,0,-1,0,0,0}\\
&+1384H_{-1,0,0,-1,0,0}-2192H_{-1,0,0,0,0,0}-2104H_{0,-1,-1,-1,0,0}+2656H_{0,-1,-1,0,0,0}\\
&+1480H_{0,-1,0,-1,0,0}-1576H_{0,-1,0,0,0,0}+1672H_{0,0,-1,-1,0,0}-1216H_{0,0,-1,0,0,0}\\
&-664H_{0,0,0,-1,0,0}+118H_{0,0,0,0,0,0}-518\pi^2H_{-1}\zeta_3+\frac{2629}{6}\pi^2H_0\zeta_3+1896H_{-1,-1,-1}\zeta_3\\
&-1272H_{-1,-1,0}\zeta_3-1976H_{-1,0,-1}\zeta_3+592H_{-1,0,0}\zeta_3-2104H_{0,-1,-1}\zeta_3+1800H_{0,-1,0}\zeta_3\\
&+1672H_{0,0,-1}\zeta_3-238H_{0,0,0}\zeta_3-505\zeta_3^2+1272H_{-1}\zeta_5-2930H_0\zeta_5\Big)+\mathcal{O}(\epsilon^7)\,. \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
f_{41}^E(x,\epsilon)&=-\frac{56}{9}+4\epsilon H_0+\epsilon^2\Big(\frac{166\pi^2}{27}+4H_{0,0}\Big)+\epsilon^3\Big(8\pi^2H_{-1}-\frac{11}{3}\pi^2H_0+16H_{-1,0,0}\\
&-12H_{0,0,0}+\frac{200\zeta_3}{3}\Big)+\epsilon^4\Big(-\frac{151\pi^4}{36}+12\pi^2H_{-1,-1}-\frac{20}{3}\pi^2H_{-1,0}+20\pi^2H_{0,-1}-21\pi^2H_{0,0}\\
&+24H_{-1,-1,0,0}-16H_{-1,0,0,0}+40H_{0,-1,0,0}-44H_{0,0,0,0}+24H_{-1}\zeta_3-44H_0\zeta_3\Big)\\
&+\epsilon^5\Big(-\frac{248}{45}\pi^4H_{-1}+\frac{271}{24}\pi^4H_0-24\pi^2H_{-1,-1,-1}+\frac{88}{3}\pi^2H_{-1,-1,0}+64\pi^2H_{-1,0,-1}\\
&-\frac{140}{3}\pi^2H_{-1,0,0}+152\pi^2H_{0,-1,-1}-\frac{344}{3}\pi^2H_{0,-1,0}-160\pi^2H_{0,0,-1}+\frac{313}{3}\pi^2H_{0,0,0}\\
&-48H_{-1,-1,-1,0,0}+128H_{-1,-1,0,0,0}+128H_{-1,0,-1,0,0}-224H_{-1,0,0,0,0}+304H_{0,-1,-1,0,0}\\
&-384H_{0,-1,0,0,0}-320H_{0,0,-1,0,0}+420H_{0,0,0,0,0}-\frac{334\pi^2\zeta_3}{3}-48H_{-1,-1}\zeta_3-80H_{-1,0}\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+304H_{0,-1}\zeta_3-196H_{0,0}\zeta_3+\frac{6856\zeta_5}{15}\bigg)+\epsilon^6\bigg(-\frac{43585\pi^6}{27216}-\frac{197}{45}\pi^4H_{-1,-1}-\frac{341}{45}\pi^4H_{-1,0}\\
&+\frac{997}{45}\pi^4H_{0,-1}-\frac{3767}{360}\pi^4H_{0,0}-180\pi^2H_{-1,-1,-1,-1}+\frac{700}{3}\pi^2H_{-1,-1,-1,0}\\
&+316\pi^2H_{-1,-1,0,-1}-\frac{694}{3}\pi^2H_{-1,-1,0,0}+364\pi^2H_{-1,0,-1,-1}-332\pi^2H_{-1,0,-1,0}\\
&-316\pi^2H_{-1,0,0,-1}+\frac{416}{3}\pi^2H_{-1,0,0,0}+692\pi^2H_{0,-1,-1,-1}-\frac{1724}{3}\pi^2H_{0,-1,-1,0}\\
&-636\pi^2H_{0,-1,0,-1}+386\pi^2H_{0,-1,0,0}-748\pi^2H_{0,0,-1,-1}+\frac{1700}{3}\pi^2H_{0,0,-1,0}\\
&+540\pi^2H_{0,0,0,-1}-\frac{643}{3}\pi^2H_{0,0,0,0}-360H_{-1,-1,-1,-1,0,0}+1040H_{-1,-1,-1,0,0,0}\\
&+632H_{-1,-1,0,-1,0,0}-1320H_{-1,-1,0,0,0,0}+728H_{-1,0,-1,-1,0,0}-1264H_{-1,0,-1,0,0,0}\\
&-632H_{-1,0,0,-1,0,0}+1280H_{-1,0,0,0,0,0}+1384H_{0,-1,-1,-1,0,0}-2064H_{0,-1,-1,0,0,0}\\
&-1272H_{0,-1,0,-1,0,0}+1896H_{0,-1,0,0,0,0}-1496H_{0,0,-1,-1,0,0}+1904H_{0,0,-1,0,0,0}\\
&+1080H_{0,0,0,-1,0,0}-1532H_{0,0,0,0,0,0}-\frac{530}{3}\pi^2H_{-1}\zeta_3+\frac{721}{3}\pi^2H_0\zeta_3-360H_{-1,-1,-1}\zeta_3\\
&+136H_{-1,-1,0}\zeta_3+728H_{-1,0,-1}\zeta_3-584H_{-1,0,0}\zeta_3+1384H_{0,-1,-1}\zeta_3-904H_{0,-1,0}\zeta_3\\
&-1496H_{0,0,-1}\zeta_3+1076H_{0,0,0}\zeta_3-\frac{1364\zeta_3^2}{3}+984H_{-1}\zeta_5-\frac{3892}{5}H_0\zeta_5\bigg)+\mathcal{O}(\epsilon^7)\,. \qquad \text{(B.6)}
\end{aligned}
$$

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