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PRECISE HEAVY QUARK MASSES *

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Recent theoretical and experimental improvements in the determination of charmed- and bottom-quark masses are discussed. The final results, $m_c(3\,\mathrm{GeV}) = 986(13)\,\mathrm{MeV}$ and $m_b(m_b) = 4163(16)\,\mathrm{MeV}$, are among the most precise determinations of these two fundamental parameter. A critical analysis of the theoretical and experimental uncertainties is presented and possibilities for further improvements of the experimental input are discussed.

Keywords: Perturbative Quantum Chromodynamics, Charmed Quarks, Bottom Quarks

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1. Introduction

The most striking deficiency of the Standard Model (SM) of particle physics is its large number of presently uncalculable "fundamental" parameters. The gauge sector requires three constants only, which in the context of a Grand Unified Theory (GUT) might even be reduced to one universal gauge coupling. The quartic Higgs-boson self-coupling, together with the quadratic terms is sufficient for the description of spontaneous symmetry breaking and, as a consequence, the generation of gauge boson and fermion masses. A proliferation of parameters is observed in the fermion mass matrix, in other words in the Yukawa couplings which parametrize the strength of the interactions between fermions and the Higgs boson. The determination of this mass matrix, in turn, is equivalent to the determination of fermion masses and mixing angles. A large number of detailed experiments is devoted to the measurement of the Cabbibo-Kobayashi-Maskawa angles in the quark sector and the corresponding angles characteristic for the lepton mixing. The precise determination of the quark masses is of similar importance. In contrast to lepton masses the determination of quarks masses requires not only experimental but also considerable theoretical effort, as we will see below.

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To date no convincing framework exists which would allow to predict these parameters from first principles, and at best one might hope to find relations between quark and lepton masses within the third family. Despite this lack of predictive power there is strong interest in precise quark mass values which are decisive for numerous phenomenological issues. The u-, d- and s-quark masses are responsible for the explicit breaking of chiral symmetry and, as a consequence, for the non-vanishing masses of the pions, interpreted as pseudo-Goldstone bosons. Furthermore, they must be included in any analysis of pion and kaon scattering performed in the framework of chiral perturbation theory (χPT), (For a detailed discussion and further references see Ref. 1.) which leads to excellent predictions for a large variety of phenomena. For the strange quark with its mass of around 100 MeV the applicability of χPT is at the borderline. Mass determinations based on sum rules for various correlators [2,3,4] or on the analysis of strangeness violating tau-lepton decays[5,6,7] are based on perturbative QCD (pQCD), (despite the fact that they employ relatively low scales of order 1.5 to 2 GeV) and are in reasonable agreement with χPT and with lattice results[8,9].

The following discussion will be focused on charm and bottom quarks. With mass values around one and four GeV, respectively, these are sufficiently heavy to allow for a perturbative treatment, if suitable correlators are analyzed in well chosen momentum regions. Furthermore, for the observables considered in this paper, perturbation theory is known to sufficiently high order, such that the theory error is well under control.

The precise determination of m_c and m_b is motivated by phenomenological and conceptual considerations. Let us just recall that the charmed quark mass not only governs the properties of the charmonium states (See for example Ref. 10.), it is also a crucial ingredient in predictions[11,12] for the decay $K \to \pi\nu\bar{\nu}$. Both charm and bottom quark masses are required for predictions of $\Gamma(B \to X_b \ell\nu)$ and thus crucial for the determination of the CKM element V_{cb}. The bottom quark mass, finally, is required to predict rates, photon- and lepton-spectra in B-meson decays, it governs the masses of bottonium states (see e.g. Ref. 13) and determines the rate of the dominant decay mode of the Higgs boson, $\Gamma(H \to b\bar{b})$, and hence all experimentally observable branching ratios. Last not least, once the masses of top and bottom quark can be related e.g. in the framework of a Grand Unified Theory, the relative errors $\delta m_b/m_b$ and $\delta m_t/m_t$ (with both $\overline{MS}$ masses evaluated at the scale m_t) should be comparable. As we will see below, for $m_b(m_t) = 2.701 \pm 0.022\,\mathrm{GeV}$ and $m_t = 161.47 \pm 0.85\,\mathrm{GeV}$, (based on a pole mass of $M_t = 173.20 \pm 0.87\,\mathrm{GeV}$ as determined by a Tevatron[a] analysis[14] and $\alpha_s = 0.1189(20)$) this requirement is indeed fulfilled.

[a]This value is in very good agreement with the recent ATLAS and CMS results[15] of $173.2 \pm 0.6 \pm 0.8\,\mathrm{GeV}$ and $173.2 \pm 0.6 \pm 0.8\,\mathrm{GeV}$ respectively.

2. Quark Mass Definitions

Masses of stable elementary particles are obviously determined by kinematic measurements, and the same is true for unstable particles, as long as their decay rate is far smaller than the particle mass. This "pole mass" definition M_{pole} is directly related to the location of the pole of the corresponding propagator. In practical calculations (For an overview see e.g. Ref 16.) involving fundamental fermions (both quarks and leptons) it is often useful to use the $\overline{MS}$ mass $m(\mu)$ in intermediate steps. In the case of leptons, however, the pole mass remains the fundamental observable that can be measured with arbitrary precision, at least in principle. The situation is different for quarks which cannot be produced or studied in isolation. Correspondingly the pole mass cannot be determined through kinematic measurements with arbitrary precision and uncertainties of order Λ_{QCD} necessarily remain. In perturbative calculations this is reflected in relatively large higher order corrections, indicating the appearance of of a divergent series related to the presence of "renomalons"[17]. Nevertheless, in some cases (e.g. the determination of the top quark mass from its decay products) it is convenient or even unavoidable to use the pole mass as primary quantity.

The $\overline{MS}$-mass as second and for many cases most convenient choice is directly related to the mass parameter in the Lagrangian. In perturbation theory a regularisation and renormalisation prescription is required to arrive at an ultraviolet finite result, with dimensional regularisation and minimal subtraction as convenient conventions. To complete the prescription a choice for the renormalisation scale μ has to be adopted, and the μ-dependence of the $\overline{MS}$-mass $m(\mu)$ is governed by the mass anomalous dimension,

$$\mu^2 \frac{d}{d\mu^2}\overline{m}(\mu) = \overline{m}(\mu)\gamma_m(\alpha_s) \equiv -\overline{m}\sum_{i\geq 0}\gamma_m^i \left(\frac{\alpha_s}{\pi}\right)^{i+1} , \tag{1}$$

with γ^i known[18,19] to $i = 3$, corresponding to four-loop precision[b]. Two-, three- and partial four-loop results [21,22,23,24,25] are available for the conversion[c] between M_{pole} and $m(\mu)$. Various other mass definitions, particularly suited for specific applications, have been suggested in the literature. Examples are "kinetic mass"[28], employed e.g. in weak decays of B mesons, (Note that the kinetic mass has not yet been related to the $\overline{MS}$-mass in $\mathcal{O}(\alpha_s^3)$ accuracy.) "potential subtracted mass"[29] and "1S-mass"[30], used in connection with quarkonium physics. The following discussion will be restricted to the $\overline{MS}$-mass which, as stated above, is closely related to the Lagrangian and is used, at least in intermediate steps, in most higher order calculations. Last not least it is best suited for the mass determination based on moments of the R-ratio as discussed in the following.

[b] Recently[20] even the five-loop result has become available.
[c] These relations, together with the solution of Eq. 1 and issues of "matching" at flavour thresholds are conveniently encoded in the `Mathematica` and `C++` programs `RunDec`[26] and CRunDec[27].

3. Correlators, Moments and Quark Masses

The idea behind this approach relies on the observation that the correlator of the electromagnetic current j_μ

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi(q^2) \equiv i \int \mathrm{d}x e^{iqx} \langle T j_\mu(x) j_\nu(0)\rangle \tag{2}$$

can be evaluated perturbatively in the deep-Euklidean region $q^2 \ll 0$. As a consequence of the large quark mass the lowest pole of $\Pi(q^2)$ is located at $q^2 = M^2_{J/\psi}$, the branching cut even starts at $(2M_D)^2 \approx (3.739\,\mathrm{GeV})^2$. The perturbative evaluation is, therefore, possible even at $q^2 = 0$ which is already sufficiently far away from the physical, the threshold region. The same statements apply, a forteriori, to bottom quark production. However, some caveats must be considered. If perturbative QCD (pQCD) could be applied to $\Pi(q^2)$ at q^2 and all its derivatives, this would correspond to a perturbative description of $\Pi(q^2)$ for arbitrary q^2. It has therefore been argued that pQCD should be applicable only to the lowest terms of the Taylor series

$$\Pi_Q(q^2) \equiv Q_Q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n z^n \,, \tag{3}$$

with $z = q^2/4m_Q^2(\mu)$ and $m_Q(\mu)$ being the $\overline{MS}$ mass at scale μ. As stated before, the coefficients $\overline{C}_n$ can be calculated in pQCD and cast into the following generic form

$$\begin{aligned}\bar{C}_n = & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_Q}\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_Q} + \bar{C}_n^{(22)} l_{m_Q}^2\right) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_Q} + \bar{C}_n^{(32)} l_{m_Q}^2 \right. \\ & \left. \quad + \bar{C}_n^{(33)} l_{m_Q}^3\right) + \dots \,.\end{aligned} \tag{4}$$

with $\alpha_s \equiv \alpha_s(\mu)$, $\ell_{m_Q} \equiv \ln m_Q^2/\mu^2$ and $\bar{C}_n^{(ij)}$ being pure numbers. The coefficients $\bar{C}_n^{(ij)}$ with $j = 0$ are obtained from increasingly complex calculations, those with $n \geq 1$ can be reconstructed from the lower ones, employing the renormalisation group with the coefficients of the anomalous mass dimension γ_m and the β function to the appropriate order. The two-loop result $\overline{C}_n^{(10)}$ can be obtained directly from the Taylor expansion of the QED polarisation function, evaluated by Kallen and Sabry long time ago. The three-loop result has been calculated in Ref. 31, 32 up to $n = 4$, (later[33] up to $n = 8$) in terms of rational and transcendental numbers with the help of the FORM program MATAD[34], using a recursive algorithm. (More recently the results even up to $n = 30$ have become available[35,36].) The four-loop coefficients up to $n = 3$ for vector, axial vector, scalar and pseudoscalar correlators were obtained[37,38,39,40,41] using Integration-by-Parts identities[42] in combination with Laporta's algorithm[43,44] which leads to an algebraic reduction of C_n^{40}

to a small set of "master integrals". These have been evaluated first numerically with high precision[45,46], subsequently in a series of papers in terms of rational and transcendental numbers. (A complete list of references can be found in Ref. 40.) Numerical results for moments with higher n (with an estimated precision better than one percent for $n = 4$) are based on the approximate reconstruction of the full function $\Pi(q^2)$, using information at $q^2 = 0$, the high energy behaviour and the threshold region combined with interpolations based on Padè approximations[47]. (For an analysis along similar lines see Ref. 48.)

Exploiting the analyticity of $\Pi(q^2)$ around $q^2 = 0$ and using dispersion relations, the derivatives at $q^2 = 0$ can be expressed as weighted integrals over the imaginary part of $\Pi(q^2)$, which in turn is given by the cross section for electron-positron annihilation into hadrons. Let us denote the normalised cross section for heavy quark production as $R_Q(s) \equiv \sigma_Q(s)/\sigma_{\rm point}(s)$. The moments of R_Q, defined as

$$\mathcal{M}_n^{\rm exp} \equiv \int \frac{{\rm d}s}{s^{n+1}} R_Q(s)\,, \tag{5}$$

can be directly related to the perturbatively calculated Taylor coefficients. In total one thus obtains the $\overline{MS}$ quark mass in terms of experimentally weighted integrals of R_Q and the perturbatively calculable coefficients $\bar{C}_n$,

$$m_Q(\mu) = \frac{1}{2}\left(\frac{9Q_Q^2\bar{C}_n}{4\mathcal{M}_n^{\rm exp}}\right)^{1/(2n)}\,. \tag{6}$$

This strategy has been suggested originally in Ref. 49 and applied to a precise charm and bottom mass determination in Ref. 50 once the three-loop results had become available. A significantly improved reanalysis based on four-loop moments and with new data has been performed in Ref. 51, additional updates and improvements from new data and the precis evaluation of the higher moments can be found in Refs. 52, 53, which are the basis of the subsequent discussion. For the extraction of R_Q from the data the issue of singlet contributions and secondary radiation of heavy quarks has been discussed in some detail in Ref. 51. Furthermore, the potential influence of a non-vanishing gluon condensate $\langle \frac{\alpha_s}{\pi} GG\rangle = 0.006 \pm 0.012\,{\rm GeV}^4$ has been analysed in Refs. 51, 53, which, for moments with $n \leq 3$, leads to shifts of one to two MeV and an associated uncertainty of three MeV at most.

4. Results

Let us now present the experimental results for the moments, first for charm, later for bottom. For charm the integration region is split into one covering the narrow resonances J/ψ and ψ', the "threshold region" between $2m_D$ and 4.8 GeV and the perturbative continuum above. Note that we assume the validity of pQCD in this region with high precision, an assumption that is well consistent with present measurements (see Table 1) but for the moment remains an assumption, to be verified e.g. by future BESS experiments.

Table 1. Comparison of the theory predictions for $R(s)$ with the experimental results at a few selected values for $\sqrt{s}$.

$\sqrt{s}$(GeV)	2.00	3.65	3.732	4.80	9.00	10.52
$R^{\mathrm{th}}(s)$	2.209(91)	2.161(18)	2.160(17)	3.764(64)	3.564(17)	3.548(12)
$R^{\mathrm{exp}}(s)$	2.18(7)(18)	2.157(35)(86)	2.156(86)(86)	3.66(14)(19)	3.62(7)(14)	3.56(1)(7)
Experiment	BESS	BESS	BESS	BESS	MD-1	CLEO

Table 2. Experimental moments in $(\mathrm{GeV})^{-2n}$ as defined in Eq. (5), separated according to the contributions from the narrow resonances, the charm threshold region and the continuum region above $\sqrt{s} = 4.8$ GeV. In the last column the NLO contribution from the gluon condensate is shown.

n	$\mathcal{M}_n^{\mathrm{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\mathrm{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\mathrm{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\mathrm{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\mathrm{np}}(\mathrm{NLO})$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0002(5)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	−0.0005(10)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	−0.0008(16)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	−0.0013(25)

Table 3. Results for $m_c(3\,\mathrm{GeV})$ in GeV. The errors are from experiment, α_s, variation of μ and the gluon condensate.

n	$m_c(3\text{ GeV})$	exp	α_s	μ	$\mathrm{np}_{\mathrm{NLO}}$	total
1	0.986	0.009	0.009	0.002	0.001	0.013
2	0.975	0.006	0.014	0.005	0.002	0.016
3	0.975	0.005	0.015	0.007	0.003	0.017
4	0.999	0.003	0.009	0.031	0.003	0.032

The results for the moments from one to four and the error budget are listed in Table 2, those for the quark mass in Table 3. The moment with $n = 1$ is most robust from the theory side, as evident from the relatively smaller coefficient in the perturbative series. This argument can be made more quantitatively by rewriting eq. (6) in the form

$$
\begin{aligned}
m_c &= \frac{1}{2}\left(\frac{9Q_c^2}{4}\frac{\bar{C}_n^{\mathrm{Born}}}{\mathcal{M}_n^{\mathrm{exp}}}\right)^{\frac{1}{2n}}\left(1 + r_n^{(1)}\alpha_s + r_n^{(2)}\alpha_s^2 + r_n^{(3)}\alpha_s^3\right) \\
&\propto 1 - \begin{pmatrix}0.328\\0.524\\0.618\\0.662\end{pmatrix}\alpha_s - \begin{pmatrix}0.306\\0.409\\0.510\\0.575\end{pmatrix}\alpha_s^2 - \begin{pmatrix}0.262\\0.230\\0.299\\0.396\end{pmatrix}\alpha_s^3, \qquad (7)
\end{aligned}
$$

where the entries correspond to the moments with $n = 1$, 2, 3 and 4. Note, that the coefficients are decreasing with increasing order of α_s. Estimating the relative error through $r_n^{\mathrm{max}}(\alpha_s(3\text{ GeV}))^4$ leads to 1.4 / 2.3 / 2.7 / 2.9 per mille and thus to an estimate even smaller than the one based on the μ dependence listed as theory error in the fifth column of Table 3. Also the contribution from the gluon condensate is smallest for the lowest values of n. The advantage of $n = 2$ and even more so of $n = 3$ lies in the reduced contribution from the continuum region. Note that all four results are mutually consistent. In view of the smallest sensitivity to α_s and the choice of the renormalisation scale μ we adopt the value $m_c(3\,\mathrm{GeV}) = 986 \pm 13\,\mathrm{MeV}$ as derived from $n = 1$ as our final result. This is in very good agreement with the result $m_c(3\,\mathrm{GeV}) = 987 \pm 9\,\mathrm{MeV}$ obtained in Ref. 54, which is based on a different integration kernel.[d]

Tables 2 and 3 and also illustrate the path to a further reduction of the error. Let us restrict the discussion to $n = 1$,2 and 3. For $n = 1$ important contributions arise from all three regions. Improved determinations of $\Gamma_e(J/\psi)$ would reduce the errors of all three moments. Improved measurements of R_Q in the threshold region and at 4.8 GeV would have a strong impact on $n = 1$ and strengthen our confidence in the validity of pQCD close to, but above 4.8 GeV. Another interesting option would be a simultaneous fit to all three moments, taking the proper experimental correlations into account.

The theory error is presently deduced from a variation of μ between two and four GeV. Taking, as an alternative, the last calculated perturbative coefficient would lead to a comparable, somewhat smaller error. An important contribution to the error budget arises from the uncertainty in the strong coupling, where $\alpha_s(M_Z) = 0.1189 \pm 0.002$ has been taken from Ref. 55. Taking as an alternative $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ as obtained from a more recent compilation[57] would obviously lead to a significant reduction of δm. A comparison of selected m_c determinations is shown in Fig. 1. The excellent agreement between lattice[58,59] and perturbative QCD results is particularly encouraging.

Similar statements can be made for the determination of the bottom quark mass. A recent measurement of the cross section in the threshold region between 10.6 GeV and 11.2 GeV was employed in Ref. 52 and has lead to a significant reduction of the experimental error on m_b. Still, additional measurements in the region around and above 11 GeV would be welcome in order to confirm the validity of perturbative QCD relatively close to threshold. The result for the second moment has been adopted as our final answer

$$m_b(10\,\mathrm{GeV}) = 3610(16)\,\mathrm{MeV} \tag{8}$$

and corresponds to $m_b(m_b) = 3610(16)\,\mathrm{MeV}$. Our results are in excellent agreement with a completely independent lattice determination, as shown in Fig. 1.

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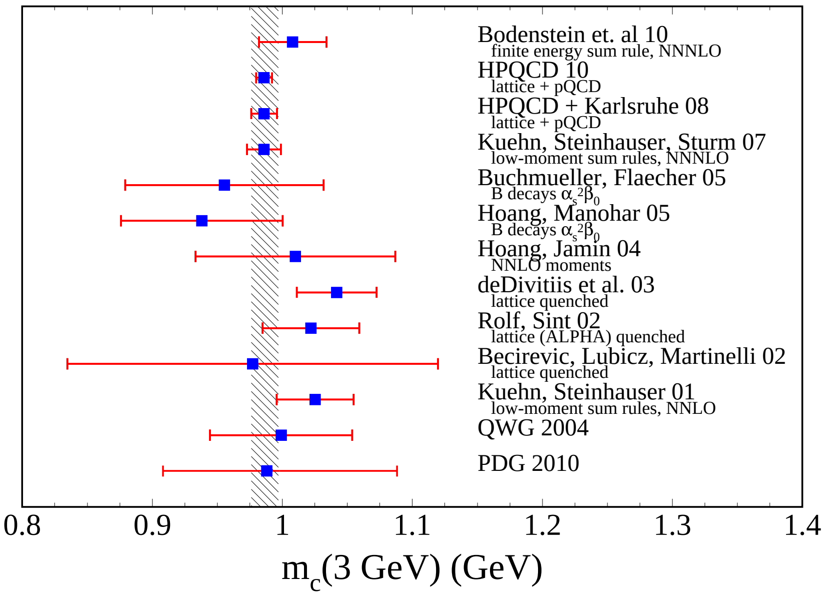
[d]For more details, references and a closely related discussion of m_b see Ref. 56.



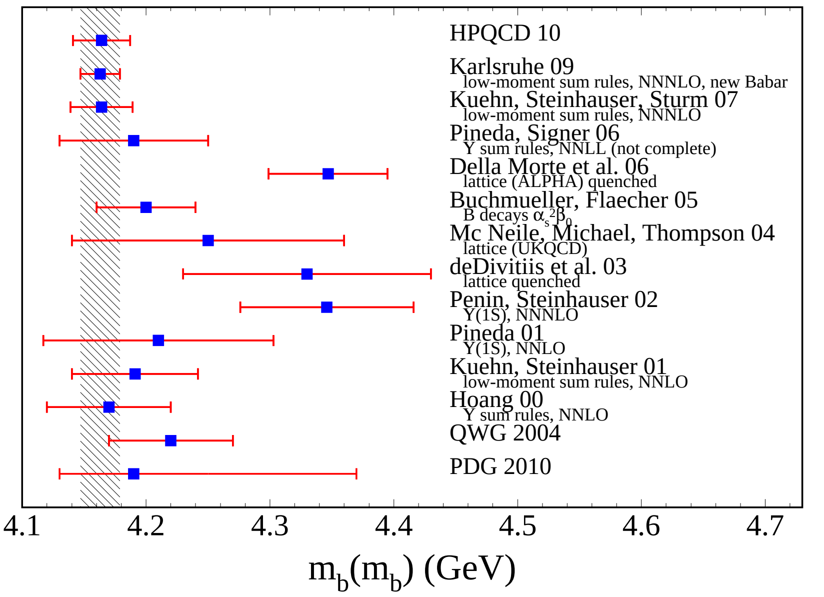


Fig. 1. Comparison of recent determinations of $m_c(3\text{ GeV})$ and $m_b(m_b)$.

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