

Resummation of non-global logarithms at finite N_c

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Abstract

In the context of inter-jet energy flow, we present the first quantitative result of the resummation of non-global logarithms at finite N_c . This is achieved by refining Weigert's approach in which the problem is reduced to the simulation of associated Langevin dynamics in the space of Wilson lines. We find that, in e^+e^- annihilation, the exact result is rather close to the result previously obtained in the large- N_c mean field approximation. However, we observe enormous event-by-event fluctuations in the Langevin process which may have significant consequences in hadron collisions.

1 Introduction

In certain search channels of the Higgs boson and new particles at the Large Hadron Collider (LHC), it is often desirable to be able to control QCD radiation from tagged jets in order to suppress large backgrounds. A prime example is the Higgs boson production in association with di-jets. The two competing production mechanisms, the gluon fusion and the vector boson fusion processes, have different patterns of soft gluon radiation due to the difference in their color structure. Discriminating these processes quantitatively using some measure of radiation is therefore a useful strategy to determine the Higgs couplings [1,2].

A related class of observables which are particularly sensitive to soft radiation is the cross section with a veto on unwanted jets in the full or partial region of the phase space. This generally requires the resummation of logarithms in p_T^{veto} , the threshold transverse momentum of vetoed jets. Steady progress in this direction has been made for *global* observables which involve all the particles and jets in the final state including those close to the beam axis. The state-of-the-art is that one can resum the leading logarithms (LL) $(\alpha_s \ln^2 p_T^{\text{veto}})^n$, the next-to-leading logarithms (NLL) $(\alpha_s \ln p_T^{\text{veto}})^n$ and even the next-to-next-to-leading logarithms (NNLL) $(\alpha_s^2 \ln p_T^{\text{veto}})^n$ [3–5].

However, in contrast to such progress, there exists a severe limitation in our ability to resum *non-global* logarithms [6,7] which arise when measurements are restricted to a part of the phase space excluding the beam and jet regions. In this case, double-logarithms $(\alpha_s \ln^2 p_T^{\text{veto}})^n$ are absent due to the lack of the collinear singularity. The leading contribution is then comprised of single-logarithms $(\alpha_s \ln p_T^{\text{veto}})^n$ which originate from the soft singularity. The problem is that these logarithms do not exponentiate, and because of this difficulty their resummation has been hitherto done only in the large- N_c limit [6–9]. In other words, even the leading logarithms cannot be fully satisfactorily resummed. This could be a potentially serious drawback in actual experiments considering the fact that, strictly speaking, any vetoed cross section at hadron colliders is inevitably non-global due to the finite acceptance of detectors.

In fact, there is a single work by Weigert [10] which did discuss the resummation of non-global logarithms at finite N_c in a simpler setup of e^+e^- annihilation where complications from the initial state radiation do not arise. His approach is based on an analogy with another, seemingly unrelated resummation in QCD, namely, that of the small- x (or ‘BFKL’) logarithms in Regge scattering. In this context, a very similar issue arises as to how one can generalize the equation which resums small- x logarithm in the large- N_c limit to one at finite N_c . It turns out that these equations bear a striking resemblance to the equation which resums non-global logarithms in the large- N_c limit. Since technologies to solve the former problem are well-developed, they may be suitably adapted to address the latter problem as well. Somewhat surprisingly, however, Weigert’s approach has not been pursued for a decade. Part of the reason of this may perhaps be that, as we shall point out, there is actually a flaw in his formulation which deters a straightforward numerical implementation. In this paper we overcome this difficulty and present the first quantitative results of the resummation of non-global logarithms at finite N_c .

In Section 2, we quickly review the nonlinear evolution equations which resum the small- x logarithms in high energy QCD. The subject may seem utterly unfamiliar to the readers whose primary interest is jet physics. However, the similarity (or even equivalence) to the resummation of non-global logarithms will soon become apparent in Section 3 where we introduce the relevant evolution equations. Using this similarity, we discuss how to solve the equation for non-global logs at finite N_c in Section 4, and present numerical results in Section 5. Finally, we examine the results and conclude in Section 6.

2 B–JIMWLK equation

Consider a ‘dipole’ consisting of a quark and an antiquark located at transverse coordinates \boldsymbol{x} and \boldsymbol{y} , respectively. The S -matrix of the dipole moving in the $x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$ direction and scattering off some target in the eikonal approximation is

$$\langle S_{\boldsymbol{x}\boldsymbol{y}} \rangle_\tau = \frac{1}{N_c} \langle \text{tr}(U_{\boldsymbol{x}} U_{\boldsymbol{y}}^\dagger) \rangle_\tau, \quad (1)$$

where $U_{\boldsymbol{x}}$ is the Wilson line in the fundamental representation

$$(U_{\mathbf{x}})_{ij} = P \exp \left(i \int_{-\infty}^{\infty} dx^- A_a^+(x^-, \mathbf{x}) t^a \right)_{ij}, \quad 1 \leq i, j \leq N_c, \quad (2)$$

and τ is the rapidity of the dipole. The expectation value $\langle \dots \rangle$ is taken in the target wavefunction. In deep inelastic scattering (DIS), τ is essentially the logarithm of the Bjorken- x variable

$$\tau \equiv \frac{\alpha_s}{\pi} \ln \frac{1}{x}. \quad (3)$$

The dipole S -matrix (1) (more precisely, $1 - \langle S \rangle$) then measures the total cross section of the subprocess $\gamma^* p \rightarrow q \bar{q} p \rightarrow X$ at the corresponding values of x and the photon virtuality $Q \sim 1/|\mathbf{x} - \mathbf{y}|$.

The B-JIMWLK¹ equation [11–13] resums the small- x logarithms $\tau^n \sim (\alpha_s \ln 1/x)^n$ which arise in the high energy evolution of $\langle S \rangle$. It reads

$$\partial_\tau \langle S_{\mathbf{x}\mathbf{y}} \rangle_\tau = N_c \int \frac{d^2 \mathbf{z}}{2\pi} \mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) \left(\langle S_{\mathbf{x}\mathbf{z}} S_{\mathbf{z}\mathbf{y}} \rangle_\tau - \langle S_{\mathbf{x}\mathbf{y}} \rangle_\tau \right), \quad (4)$$

where the dipole kernel is given by

$$\mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) = \frac{(\mathbf{x} - \mathbf{y})^2}{(\mathbf{x} - \mathbf{z})^2 (\mathbf{z} - \mathbf{y})^2}. \quad (5)$$

The equation (4) is not closed because the right-hand-side contains the double dipole S -matrix $\langle SS \rangle$. This non-linearity reflects the gluon saturation effect in the target. The consequence is that (4) is actually the first equation of an infinite hierarchy of coupled equations. However, it becomes a closed equation—the Balitsky–Kovchegov (BK) equation [11,14]—if one truncates the hierarchy and assumes factorization $\langle SS \rangle \rightarrow \langle S \rangle \langle S \rangle$

$$\partial_\tau \langle S_{\mathbf{x}\mathbf{y}} \rangle_\tau = N_c \int \frac{d^2 \mathbf{z}}{2\pi} \mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) \left(\langle S_{\mathbf{x}\mathbf{z}} \rangle_\tau \langle S_{\mathbf{z}\mathbf{y}} \rangle_\tau - \langle S_{\mathbf{x}\mathbf{y}} \rangle_\tau \right). \quad (6)$$

While the BK equation (6) can be solved numerically in a straightforward manner, solving the B-JIMWLK equation (4) had been difficult until the ingenious reformulation of the problem as random walk [15]. To explain this, it is essential to write the equation in the operator form

$$\partial_\tau \langle S_{\mathbf{x}\mathbf{y}} \rangle_\tau = -\langle \hat{H} S_{\mathbf{x}\mathbf{y}} \rangle_\tau. \quad (7)$$

The effective Hamiltonian \hat{H} takes the form

$$\begin{aligned} \hat{H} &= \int d^2 \mathbf{x} d^2 \mathbf{y} \frac{d^2 \mathbf{z}}{2\pi} \mathcal{K}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) \nabla_{\mathbf{x}}^a \left(1 + \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{y}} - \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{z}} - \tilde{U}_{\mathbf{z}}^\dagger \tilde{U}_{\mathbf{y}} \right)^{ab} \nabla_{\mathbf{y}}^b \\ &= \int d^2 \mathbf{x} d^2 \mathbf{y} \frac{d^2 \mathbf{z}}{2\pi} \mathcal{K}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) (1 - \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{z}})^{ac} (1 - \tilde{U}_{\mathbf{z}}^\dagger \tilde{U}_{\mathbf{y}})^{cb} \nabla_{\mathbf{x}}^a \nabla_{\mathbf{y}}^b - \int d^2 \mathbf{x} \sigma_{\mathbf{x}}^a \nabla_{\mathbf{x}}^a, \end{aligned} \quad (8)$$

¹ Acronym for Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner.

where $\tilde{U}^{ab} = (\tilde{U}^\dagger)^{ba}$ is the Wilson line in the adjoint representation and \mathcal{K} is the gluon emission kernel

$$\mathcal{K}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) = \frac{(\mathbf{x} - \mathbf{z}) \cdot (\mathbf{z} - \mathbf{y})}{(\mathbf{x} - \mathbf{z})^2 (\mathbf{z} - \mathbf{y})^2}. \quad (9)$$

In the last line of (8), we defined

$$\sigma_{\mathbf{x}}^a = -i \int \frac{d^2 \mathbf{z}}{2\pi} \frac{1}{(\mathbf{x} - \mathbf{z})^2} \text{tr}(T^a \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{z}}). \quad (10)$$

The derivative ∇ acts on the Wilson line in the fundamental representation as

$$\nabla_{\mathbf{x}}^a U_{\mathbf{y}} = i U_{\mathbf{x}} t^a \delta^{(2)}(\mathbf{x} - \mathbf{y}), \quad \nabla_{\mathbf{x}}^a U_{\mathbf{y}}^\dagger = -i t^a U_{\mathbf{x}}^\dagger \delta^{(2)}(\mathbf{x} - \mathbf{y}), \quad (11)$$

and also on the adjoint Wilson line with $(T^a)^{bc} = -i f^{abc}$

$$\nabla_{\mathbf{x}}^a \tilde{U}_{\mathbf{y}} = i \tilde{U}_{\mathbf{x}} T^a \delta^{(2)}(\mathbf{x} - \mathbf{y}), \quad \nabla_{\mathbf{x}}^a \tilde{U}_{\mathbf{y}}^\dagger = -i T^a U_{\mathbf{x}}^\dagger \delta^{(2)}(\mathbf{x} - \mathbf{y}). \quad (12)$$

The latter operation has been used in obtaining the σ -term in (8). An important observation is that (7) may be viewed as the Fokker–Planck equation treating Wilson lines U as dynamical variables. As is well-known, there is an associated Langevin equation which describes the random walk of these variables. The latter can be simulated numerically on a lattice, and the solution of the B–JIMWLK equation has thus been obtained in [16]. We shall later discuss this approach in detail.

Before leaving this section, it is useful to show another form of the Hamiltonian derived in [17]

$$\begin{aligned} \hat{H} &= \frac{1}{2} \int d^2 \mathbf{x} d^2 \mathbf{y} \frac{d^2 \mathbf{z}}{2\pi} \mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) \nabla_{\mathbf{x}}^a \left(1 + \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{y}} - \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{z}} - \tilde{U}_{\mathbf{z}}^\dagger \tilde{U}_{\mathbf{y}} \right)^{ab} \nabla_{\mathbf{y}}^b \\ &= \frac{1}{2} \int d^2 \mathbf{x} d^2 \mathbf{y} \frac{d^2 \mathbf{z}}{2\pi} \mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) \left(1 + \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{y}} - \tilde{U}_{\mathbf{x}}^\dagger \tilde{U}_{\mathbf{z}} - \tilde{U}_{\mathbf{z}}^\dagger \tilde{U}_{\mathbf{y}} \right)^{ab} \nabla_{\mathbf{x}}^a \nabla_{\mathbf{y}}^b. \end{aligned} \quad (13)$$

This Hamiltonian can be used only when it acts on ‘gauge invariant’ operators of the form

$$\text{tr}(U_{\mathbf{x}} U_{\mathbf{y}}^\dagger), \quad \text{tr}(U_{\mathbf{x}} U_{\mathbf{y}}^\dagger U_{\mathbf{z}} U_{\mathbf{w}}^\dagger \dots), \quad \text{tr}(U_{\mathbf{x}} U_{\mathbf{y}}^\dagger) \text{tr}(U_{\mathbf{z}} U_{\mathbf{w}}^\dagger), \quad \dots \quad (14)$$

and generates the same equations as those obtained from (8). Eq. (13) features the dipole kernel (5) instead of the gluon emission kernel (9). The following relation between the two kernels is worth noting

$$\mathcal{M}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) = 2\mathcal{K}_{\mathbf{x}\mathbf{y}}(\mathbf{z}) - \mathcal{K}_{\mathbf{x}\mathbf{x}}(\mathbf{z}) - \mathcal{K}_{\mathbf{y}\mathbf{y}}(\mathbf{z}). \quad (15)$$

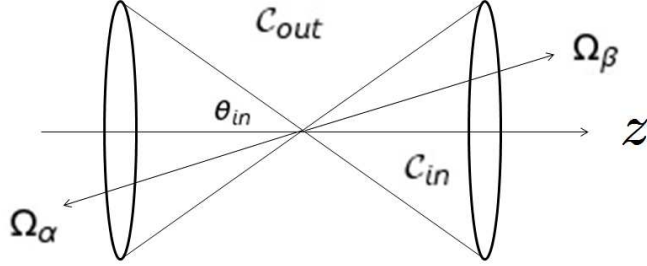


Fig. 1. Back-to-back jets.

3 Non-global logs at finite N_c

We now turn to the resummation of non-global logarithms which is our primary interest. Consider, as in the final state of e^+e^- annihilation, a pair of jets that is overall color singlet and pointing in the direction of the solid angle $\Omega_{\alpha,\beta} = (\theta_{\alpha,\beta}, \phi_{\alpha,\beta})$ measured with respect to the positive z -axis (see Fig. 1). Let \mathcal{C}_{in} be the region inside a pair of back-to-back cones with opening angle θ_{in} which include the jets, and let \mathcal{C}_{out} be its complementary region. We then ask what is the probability $P(\Omega_\alpha, \Omega_\beta)$ that the total flow of energy into \mathcal{C}_{out} is less than E_{out} . In the perturbative calculation of P in the regime $Q \gg E_{out} \gg \Lambda_{QCD}$ where Q is the hard scale, logarithms of the form $(\alpha_s \ln Q/E_{out})^n$ appear which have to be resummed. These logarithms are non-global because the measurement is done only in \mathcal{C}_{out} . To leading logarithmic accuracy, one may identify E_{out} with the jet veto scale p_T^{veto} mentioned in the introduction.

It has been shown by Banfi, Marchesini and Smye (BMS) that P satisfies the following evolution equation [8]

$$\partial_\tau P_{\alpha\beta} = N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \left(\Theta_{in}(\gamma) P_{\alpha\gamma} P_{\gamma\beta} - P_{\alpha\beta} \right), \quad (16)$$

where we abbreviated $P_{\alpha\beta} = P(\Omega_\alpha, \Omega_\beta)$ and the parameter τ is defined as

$$\tau = \begin{cases} \frac{\alpha_s}{\pi} \ln \frac{Q}{E_{out}}, & \text{(fixed coupling)} \\ \frac{6}{11N_c - 2n_f} \ln \left(\frac{\ln Q/\Lambda_{QCD}}{\ln E_{out}/\Lambda_{QCD}} \right), & \text{(running coupling)} \end{cases} \quad (17)$$

The equation (16) resums logarithms $\tau^n \sim (\alpha_s \ln E_{out})^n$ to all orders. The integral kernel

$$\begin{aligned}
\mathcal{M}_{\alpha\beta}(\gamma) &\equiv \frac{1 - \cos \theta_{\alpha\beta}}{(1 - \cos \theta_{\alpha\gamma})(1 - \cos \theta_{\gamma\beta})} \\
&= \frac{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\beta}{(1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma)(1 - \mathbf{n}_\gamma \cdot \mathbf{n}_\beta)} \\
&= \frac{n_\alpha \cdot n_\beta}{(n_\alpha \cdot n_\gamma)(n_\gamma \cdot n_\beta)},
\end{aligned} \tag{18}$$

is composed of null vectors $n^\mu = (1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (1, \mathbf{n})$ proportional to the four-vector of hard partons. The ‘step function’

$$\Theta_{in}(\gamma) \equiv \begin{cases} 1 & (\Omega_\gamma \in \mathcal{C}_{in}), \\ 0 & (\Omega_\gamma \in \mathcal{C}_{out}), \end{cases} \tag{19}$$

ensures that real gluons are emitted only in \mathcal{C}_{in} .² A trivial rewriting of (16)

$$\partial_\tau P_{\alpha\beta} = -N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{out}(\gamma) P_{\alpha\beta} + N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{in}(\gamma) (P_{\alpha\gamma} P_{\gamma\beta} - P_{\alpha\beta}), \tag{20}$$

where $\Theta_{out}(\gamma) \equiv 1 - \Theta_{in}(\gamma)$ illuminates the physical meaning of the right-hand-side. The first term represents the familiar Sudakov suppression, whereas non-global logarithms are resummed by the second term describing the emission of an arbitrary number of gluons into \mathcal{C}_{in} which then coherently emit the softest gluons into \mathcal{C}_{out} .

For the purpose of the present work, it is indispensable to note that the equation (16) has been derived in the large- N_c limit. Its finite- N_c generalization was discussed by Weigert [10] and the result reads

$$\begin{aligned}
\partial_\tau \langle P_{\alpha\beta} \rangle_\tau &= N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \left\langle \Theta_{in}(\gamma) \left(P_{\alpha\gamma} P_{\gamma\beta} - \frac{P_{\alpha\beta}}{N_c^2} \right) - \frac{2C_F}{N_c} P_{\alpha\beta} \right\rangle_\tau \\
&= -2C_F \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{out}(\gamma) \langle P_{\alpha\beta} \rangle_\tau \\
&\quad + N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{in}(\gamma) \langle P_{\alpha\gamma} P_{\gamma\beta} - P_{\alpha\beta} \rangle_\tau,
\end{aligned} \tag{21}$$

where $C_F = \frac{N_c^2 - 1}{2N_c}$. However, (21) itself is highly formal in that the meaning of the averaging $\langle \dots \rangle$ can be specified only indirectly, as a proxy of certain complicated functional integrals [10]. Nevertheless, putting this qualification aside, the striking similarity between Eqs. (16), (21) and Eqs. (4), (6) is unmistakable. In fact, it is possible to establish a rigorous mathematical equivalence between the two problems. As shown in [18,19], a conformal transformation known as the stereographic projection

² In principle, real gluons can be directly emitted from hard partons into \mathcal{C}_{out} provided their energy is less than E_{out} . However, to leading logarithmic accuracy such contributions may be omitted [8].

$$\mathbf{x} = (x^1, x^2) = \left(\frac{\sin \theta \cos \phi}{1 + \cos \theta}, \frac{\sin \theta \sin \phi}{1 + \cos \theta} \right), \quad (22)$$

exactly maps the respective kernels onto each other

$$\frac{d^2 \mathbf{z}}{2\pi} \frac{(\mathbf{x} - \mathbf{y})^2}{(\mathbf{x} - \mathbf{z})^2 (\mathbf{z} - \mathbf{y})^2} = \frac{d\Omega_\gamma}{4\pi} \frac{1 - \cos \theta_{\alpha\beta}}{(1 - \cos \theta_{\alpha\gamma})(1 - \cos \theta_{\gamma\beta})}. \quad (23)$$

Aside from the kinematical constraint factor Θ_{in} , the map (23) dispels any structural difference between the two sets of equations.³ This equivalence probably has a deep geometrical origin which goes beyond the perturbative framework. Indeed, such a correspondence persists even in the strong coupling limit of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [18].

Fortunately, a powerful machinery to solve the JIMWLK problem (4) is available, and this brings hope that the jet problem (21) can be solved in a similar manner. For this purpose, one seeks the operator form of (21) with a formal identification

$$P_{\alpha\beta} \leftrightarrow \frac{1}{N_c} \text{tr}(U_\alpha U_\beta^\dagger). \quad (24)$$

Here the Wilson lines $U_{\alpha,\beta}$ represent eikonal (jet) lines starting from the space-time origin and extending to infinity in the direction of $\Omega_{\alpha,\beta}$.⁴ Eq. (21) can then be written as a Fokker–Planck equation in group space

$$\partial_\tau \langle P_{\alpha\beta} \rangle_\tau = -\langle \hat{H} P_{\alpha\beta} \rangle, \quad (25)$$

where [10]

$$\begin{aligned} \hat{H} &= \frac{1}{2} \int d\Omega_\alpha d\Omega_\beta \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \nabla_\alpha^a \left(1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \right)^{ab} \nabla_\beta^b \\ &= \frac{1}{2} \int d\Omega_\alpha d\Omega_\beta \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \left(1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \right)^{ab} \nabla_\alpha^a \nabla_\beta^b. \end{aligned} \quad (26)$$

In (26), the derivative ∇ is defined by $(\delta(\Omega - \Omega') \equiv \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'))$

$$\nabla_\alpha^a U_\beta = i U_\alpha t^a \delta(\Omega_\alpha - \Omega_\beta), \quad \nabla_\alpha^a U_\beta^\dagger = -i t^a U_\alpha^\dagger \delta(\Omega_\alpha - \Omega_\beta), \quad (27)$$

and similarly to (12) in the adjoint case ($t^a \rightarrow T^a$).

³ Note that if one sets $\Theta_{in} \rightarrow 1$, (21) becomes identical to (4) under the map (23).

⁴ Given the probabilistic nature of P , one should more properly consider U as the product of a Wilson line in the amplitude and that in the complex–conjugate amplitude [10]. In practice, however, this does not matter in the equivalent Langevin approach.

4 Equivalent Langevin dynamics

The Fokker–Planck equations (7) and (25) can be solved by making use of an equivalent Langevin formulation. To illustrate the idea, consider the following Fokker–Planck equation for some probability distribution $P_\tau(x)$ of dynamical variables $\{x^a\}$

$$\frac{dP_\tau(x)}{d\tau} = \frac{1}{2} \partial_a \left(\chi^{ab}(x) \partial_b P_\tau(x) \right) \quad (28)$$

$$= \left(\frac{1}{2} \chi^{ab}(x) \partial_a \partial_b + \sigma^a \partial_a \right) P_\tau(x) \quad (29)$$

$$= \partial_a \left[\frac{1}{2} \partial_b \left(\chi^{ab}(x) P_\tau(x) \right) - \sigma^a P_\tau(x) \right], \quad (30)$$

where $\chi^{ab} = \chi^{ba}$ and $\sigma^a \equiv \frac{1}{2} \partial_b \chi^{ba}$. We assume that χ is factorized in the form $\chi^{ab} = \mathcal{E}^{ac} \mathcal{E}^{cb}$. The equivalent Langevin equation is then

$$\frac{d}{d\tau} x^a(\tau) = \sigma^a(x) + \mathcal{E}^{ac}(x) \xi_c(\tau), \quad (31)$$

where ξ is the Gaussian white noise characterized by the correlator

$$\langle \xi_a(\tau) \xi_b(\tau') \rangle = \delta_{ab} \delta(\tau - \tau'). \quad (32)$$

The distribution can then be obtained by averaging over an ensemble $\{x^a(\tau)\}$ of random walk trajectories

$$P_\tau(x) = \langle \delta(x - x(\tau)) \rangle. \quad (33)$$

To be more precise, the equation (31) makes sense only in a τ -discretized form. There is a well-known ambiguity in how we discretize the equation, the so-called Itô–Stratonovich dilemma. The appropriate choice corresponding to (30) is the Itô scheme

$$x^a(\tau + \varepsilon) = x^a(\tau) + \varepsilon \sigma^a(x) + \sqrt{\varepsilon} \mathcal{E}^{ac}(x(\tau)) \xi_c(\tau), \quad (34)$$

where ε is the time step and the argument of \mathcal{E} is evaluated at the previous time τ , respecting causality [20]. In (34), we have rescaled the noise as $\sqrt{\varepsilon} \xi \rightarrow \xi$ in order to make explicit the fact that the typical variation Δx is $\mathcal{O}(\sqrt{\varepsilon})$ in a random walk. With this normalization, the noise correlator in discrete time reads

$$\langle \xi_a(\tau) \xi_b(\tau') \rangle = \delta_{ab} \delta_{\tau\tau'}. \quad (35)$$

The operator form of the B–JIMWLK equation (7) has precisely the structure (30) with a factorized kernel (8). It can thus be described by the Langevin dynamics (34), with the SU(3) matrices $U_{x,y}$ playing the role of $\{x^a\}$ [15]. One can simulate the random walk in group space

on a transverse lattice, and this is how the solution to B–JIMWLK equation has been originally obtained [16].

In [10], Weigert suggested to follow the same strategy in solving (25). A comparison of (26) and (30) implies that $\sigma = 0$. The kernel of (26)

$$\begin{aligned} 1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \\ = \Theta_{out}(\gamma) + \Theta_{out}(\gamma) \tilde{U}_\alpha^\dagger \tilde{U}_\beta + \Theta_{in}(\gamma) (1 - \tilde{U}_\alpha^\dagger \tilde{U}_\gamma) (1 - \tilde{U}_\gamma^\dagger \tilde{U}_\beta), \end{aligned} \quad (36)$$

can be written as a sum of three factorized terms. [Note that $(\Theta_{in})^2 = \Theta_{in}$.] Exploiting the factorized form of \mathcal{M} as seen in the last line of (18), Weigert deduced an analog of the stochastic term $\mathcal{E}^{ac} \xi_c$ in (31) by introducing three independent noises $\xi^{(I)}$, ($I = 1, 2, 3$)

$$\mathcal{E}^{ac} \xi_c \sim \int d\Omega_\gamma \frac{n_\alpha^\mu}{n_\alpha \cdot n_\gamma} \left\{ \Theta_{out}(\gamma) \left(\delta^{ac} \xi_{\gamma c \mu}^{(1)} + (\tilde{U}_\alpha^\dagger)^{ac} \xi_{\gamma c \mu}^{(2)} \right) + \Theta_{in}(\gamma) (1 - \tilde{U}_\alpha^\dagger \tilde{U}_\gamma)^{ac} \xi_{\gamma c \mu}^{(3)} \right\}, \quad (37)$$

which generates a random walk $U \rightarrow U e^{it^a \mathcal{E}^{ac} \xi_c}$ in group space. However, the fact that \mathcal{M} is factorized in four–vector space means that the noises must have the correlator

$$\langle \xi_a^{(I)\mu} \xi_b^{(J)\nu} \rangle \sim \delta_{ab} \delta^{IJ} g^{\mu\nu}, \quad (38)$$

which is negative for the spatial components $\mu, \nu = 1, 2, 3$, hence they cannot be simulated in practice.

The resolution of this problem again comes from the correspondence with the B–JIMWLK evolution. Firstly, the identity (23) clearly shows the two–dimensional nature of the dipole kernel, so introducing four–vectors is an excess. We then notice the close similarity between (26) and (13), the latter being equivalent to (8). This suggests that we can rewrite the effective Hamiltonian in a form analogous to (8) also for the jet problem. We thus look for a kernel \mathcal{K} satisfying (cf. (15))

$$\mathcal{M}_{\alpha\beta}(\gamma) = 2\mathcal{K}_{\alpha\beta}(\gamma) - \mathcal{K}_{\alpha\alpha}(\gamma) - \mathcal{K}_{\beta\beta}(\gamma). \quad (39)$$

The solution is found to be (cf. (9))

$$\mathcal{K}_{\alpha\beta}(\gamma) = \frac{\cos \theta_{\alpha\gamma} + \cos \theta_{\gamma\beta} - \cos \theta_{\alpha\beta} - 1}{2(1 - \cos \theta_{\alpha\gamma})(1 - \cos \theta_{\gamma\beta})} = \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma) \cdot (\mathbf{n}_\gamma - \mathbf{n}_\beta)}{2(1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma)(1 - \mathbf{n}_\gamma \cdot \mathbf{n}_\beta)}, \quad (40)$$

which has a factorized structure on the unit sphere ($|\mathbf{n}| = 1$) embedded in three spatial dimensions. Reversing the argument in [17] which led (8) to (13), we arrive at an effective Hamiltonian equivalent to (26)

$$\begin{aligned}
\hat{H} &= \int d\Omega_\alpha d\Omega_\beta \frac{d\Omega_\gamma}{4\pi} \mathcal{K}_{\alpha\beta}(\gamma) \nabla_\alpha^a \left(1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \right)^{ab} \nabla_\beta^b \\
&= \int d\Omega_\alpha d\Omega_\beta \frac{d\Omega_\gamma}{4\pi} \mathcal{K}_{\alpha\beta}(\gamma) \left(1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \right)^{ab} \nabla_\alpha^a \nabla_\beta^b \\
&\quad - \int d\Omega_\alpha \sigma_\alpha^a \nabla_\alpha^a,
\end{aligned} \tag{41}$$

where

$$\sigma_\alpha^a = -i \int \frac{d\Omega_\gamma}{4\pi} \frac{\Theta_{in}(\gamma)}{(1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma)} \text{tr}(T^a \tilde{U}_\alpha^\dagger \tilde{U}_\gamma). \tag{42}$$

Moreover, differently from (36), we write the expression in the brackets as a sum of two factorized terms

$$\begin{aligned}
&1 + \tilde{U}_\alpha^\dagger \tilde{U}_\beta - \Theta_{in}(\gamma) (\tilde{U}_\alpha^\dagger \tilde{U}_\gamma + \tilde{U}_\gamma^\dagger \tilde{U}_\beta) \\
&= (1 - \Theta_{in}(\gamma) \tilde{U}_\alpha^\dagger \tilde{U}_\gamma) (1 - \Theta_{in} \tilde{U}_\gamma^\dagger \tilde{U}_\beta) + \Theta_{out}(\gamma) \tilde{U}_\alpha^\dagger \tilde{U}_\beta.
\end{aligned} \tag{43}$$

This reduces the number of independent noises from three to two $\xi^{(I)}$ ($I = 1, 2$). They are characterized by the following correlator (in discretized ‘time’ τ , cf. (35))

$$\langle \xi_{\alpha a}^{(I)k}(\tau) \xi_{\beta b}^{(J)l}(\tau') \rangle = \delta_{\tau\tau'} \delta(\cos \theta_\alpha - \cos \theta_\beta) \delta(\phi_\alpha - \phi_\beta) \delta^{IJ} \delta_{ab} \delta^{kl}, \tag{44}$$

where $k, l = 1, 2, 3$ are the spatial indices. The problem of negative metric has been circumvented. We can now write down the associated Langevin evolution for U_α with $\Omega_\alpha \in \mathcal{C}_{in}$ (cf. (34))

$$U_\alpha(\tau + \varepsilon) = U_\alpha(\tau) \exp \left\{ it^a \left(\sqrt{\varepsilon} \int d\Omega_\gamma \left(\boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(1)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(1)} + \boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(2)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(2)} \right) + \varepsilon \sigma_\alpha^a \right) \right\}, \tag{45}$$

where

$$(\boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(1)ac})^k = \frac{1}{\sqrt{4\pi}} \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma)^k}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} [1 - \Theta_{in}(\gamma) \tilde{U}_\alpha^\dagger \tilde{U}_\gamma]^{ac}, \tag{46}$$

$$(\boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(2)ac})^k = \frac{1}{\sqrt{4\pi}} \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma)^k}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} \Theta_{out}(\gamma) (\tilde{U}_\alpha^\dagger)^{ac}. \tag{47}$$

All the matrices on the right-hand-side of (45) are evaluated at τ according to the Itô scheme. Expanding the exponential up to $\mathcal{O}(\varepsilon)$, we get

$$\begin{aligned}
U_\alpha(\tau + \varepsilon) &\approx U_\alpha(\tau) + i\sqrt{\varepsilon} U_\alpha(\tau) t^a \int d\Omega_\gamma \left(\boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(1)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(1)} + \boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(2)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(2)} \right) \\
&\quad + \varepsilon U_\alpha(\tau) \left\{ it^a \sigma_\alpha^a - \frac{1}{2} \left(t^a \int d\Omega_\gamma \left(\boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(1)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(1)} + \boldsymbol{\mathcal{E}}_{\alpha\gamma}^{(2)ac} \cdot \boldsymbol{\xi}_{\gamma c}^{(2)} \right) \right)^2 \right\}.
\end{aligned} \tag{48}$$

The second line of (48) can be simplified as follows. In the σ -term, we use the identities $\tilde{U}^{ab}t^b = U^\dagger t^a U$ and $t^a U t^a = \frac{1}{2}(\text{tr}U) - \frac{1}{2N_c}U$ to obtain

$$\begin{aligned} U_\alpha t^a \text{tr}(T^a \tilde{U}_\alpha^\dagger \tilde{U}_\gamma) &= -U_\alpha [t^b, t^c] (\tilde{U}_\alpha^\dagger)^{cd} (\tilde{U}_\gamma)^{db} \\ &= \frac{1}{2} \text{tr}(U_\alpha U_\gamma^\dagger) U_\gamma - \frac{1}{2} \text{tr}(U_\gamma U_\alpha^\dagger) U_\alpha U_\gamma^\dagger U_\alpha. \end{aligned} \quad (49)$$

To the accuracy of $\mathcal{O}(\varepsilon)$, the terms quadratic in noise $\xi\xi$ may be replaced by their expectation values using (44). After these manipulations, (48) takes the form

$$\begin{aligned} U_\alpha(\tau + \varepsilon) &= U_\alpha(\tau) \\ &+ i\sqrt{\frac{\varepsilon}{4\pi}} \int d\Omega_\gamma \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma)^k}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} \left(U_\alpha t^a \xi_{\gamma a}^{(1)k} - \Theta_{in}(\gamma) U_\gamma t^a U_\gamma^\dagger U_\alpha \xi_{\gamma a}^{(1)k} + \Theta_{out}(\gamma) t^a U_\alpha \xi_{\gamma a}^{(2)k} \right) \\ &+ \varepsilon \int \frac{d\Omega_\gamma}{4\pi} \frac{1}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} \left(-2C_F U_\alpha + \Theta_{in}(\gamma) \left(\text{tr}(U_\alpha U_\gamma^\dagger) U_\gamma - \frac{1}{N_c} U_\alpha \right) \right). \end{aligned} \quad (50)$$

Note that there is no singularity at $\Omega_\gamma = \Omega_\alpha \in \mathcal{C}_{in}$. By computing the difference

$$\frac{1}{N_c} \text{tr}(U_\alpha(\tau + \varepsilon) U_\beta^\dagger(\tau + \varepsilon)) - \frac{1}{N_c} \text{tr}(U_\alpha(\tau) U_\beta^\dagger(\tau)), \quad (51)$$

to $\mathcal{O}(\varepsilon)$ and using (44), (39) and the relation $\mathcal{K}_{\alpha\alpha}(\gamma) = -1/(1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma)$, one can recover (21) after the identification (24).

For the sake of numerical simulations, it is more economical to express the evolution (50) in a left-right symmetric form⁵

$$U_\alpha(\tau + \varepsilon) = e^{iA_\alpha^L} U_\alpha(\tau) e^{iA_\alpha^R}, \quad (52)$$

where

$$A_\alpha^L = \sqrt{\frac{\varepsilon}{4\pi}} \int d\Omega_\gamma \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma)^k}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} \left(-\Theta_{in}(\gamma) U_\gamma t^a U_\gamma^\dagger \xi_{\gamma a}^{(1)k} + \Theta_{out}(\gamma) t^a \xi_{\gamma a}^{(2)k} \right), \quad (53)$$

$$A_\alpha^R = \sqrt{\frac{\varepsilon}{4\pi}} \int d\Omega_\gamma \frac{(\mathbf{n}_\alpha - \mathbf{n}_\gamma)^k}{1 - \mathbf{n}_\alpha \cdot \mathbf{n}_\gamma} t^a \xi_{\gamma a}^{(1)k}. \quad (54)$$

In this representation only the terms proportional to noise are kept in the exponential. It is easy to check that (52) and (50) are equivalent to $\mathcal{O}(\varepsilon)$ under the identification $\xi\xi \approx \langle \xi\xi \rangle$. Eqs. (44) and (52) will serve as the starting point of our numerical simulation.

⁵ See Ref. [21] for a similar rewriting of the JIMWLK evolution. As noted in this paper, the right-multiplication rule such as (45) is related to the choice of the Hamiltonian (8) or (26) being expressed in terms of ‘right-derivatives’ $\nabla_R^a U \sim U t^a$. The σ -term can be eliminated in the process of converting one of the right-derivatives into a left-derivative $\nabla_L^a U \sim t^a U$.

5 Numerical simulation

We simulate the random walk (52) of Wilson lines living on the unit sphere by discretizing the coordinates $1 \geq \cos \theta \geq -1$ and $2\pi > \phi \geq 0$ with lattice spacings a_c and a_ϕ , respectively. The $SU(3)$ matrices U_α are defined only at the grid points belonging to \mathcal{C}_{in} , whereas the noises $\xi_\alpha^{(I)}$ are defined at all grid points.⁶ The initial condition at $\tau = 0$ is simply given by

$$U_\alpha = 1, \quad (\text{unit matrix}) \quad (55)$$

for all $\Omega_\alpha \in \mathcal{C}_{in}$, or equivalently, $P_{\alpha\beta} = 1$ for all pairs $(\Omega_\alpha, \Omega_\beta)$ corresponding to no radiation before evolution.⁷ We then update $\{U_\alpha\}$ after each time step ε according to the formula (52) with noises $\xi^{(I)}$ ($I = 1, 2$) randomly generated from the Gaussian distribution

$$\prod_{\gamma, a, k} \sqrt{\frac{a_c a_\phi}{2\pi}} \exp\left(-\frac{a_c a_\phi}{2} \xi_{\gamma a}^{(I)k} \xi_{\gamma a}^{(I)k}\right), \quad \langle \xi_{\alpha a}^{(I)k} \xi_{\beta b}^{(J)l} \rangle = \frac{1}{a_c a_\phi} \delta^{IJ} \delta_{ab} \delta^{kl} \delta_{\alpha\beta}. \quad (56)$$

In order to ensure that U 's remain unitary during the evolution, we need to evaluate the exponential of matrices $e^{iA_{L/R}}$ accurately (although the equation (52) makes sense only to $\mathcal{O}(\varepsilon)$). In practice, we use an approximation $e^{iA} = (e^{iA/2^n})^{2^n} \approx \left(1 + \frac{iA}{2^n} + \dots + \frac{1}{m!} \left(\frac{iA}{2^n}\right)^m\right)^{2^n}$ with m, n large enough. On top of this, we perform the ‘reunitarization’ of U 's using polar decomposition method after every 100 steps of evolution. The effect of this latter operation is actually very small due to our accurate evaluation of $e^{iA_{L/R}}$.

The above procedure is repeated a desired number of times $N = \tau/\varepsilon$, and at the end of this random walk trajectory we compute the trace

$$\frac{1}{N_c} \text{tr}(U_\alpha(\tau) U_\beta^\dagger(\tau)). \quad (57)$$

We then average it over many such trajectories and identify the result with $\langle P_{\alpha\beta} \rangle_\tau$. In practice, we choose $\varepsilon = 10^{-5}$ and average over 500 independent trajectories.

Fig. 2 shows the evolution of $P_\tau(\Omega_\alpha, \Omega_\beta)$ for $\cos \theta_\alpha = 1$ and $\cos \theta_\beta = -1$, corresponding to back-to-back jets in the beam direction.⁸ The opening angle of the cones (see Fig. 1) is fixed to $\cos \theta_{in} = \frac{1}{2}$. The simulation was done on a lattice with 80 grid points in the $\cos \theta$ direction, and 40 grid points in the ϕ direction. The exact $N_c = 3$ solution to (21) (solid red line) is compared

⁶ At $\cos \theta_\alpha = \pm 1$, U_α and ξ_α are independent of ϕ , as they should. In the case of U_α , this is guaranteed by our initial condition (55) and the structure of the evolution (50) which preserves this property.

⁷ In the JIMWLK case, the initial condition is the value of the S -matrix (1) at small, but not too small value of x . This is model dependent and its initial sampling is non-trivial [16].

⁸ We actually plot the real part of the average $\langle \text{tr} U_\alpha U_\beta^\dagger \rangle$. $\text{tr} U_\alpha U_\beta^\dagger$ is in general complex-valued for each trajectory, but we have checked that the imaginary part of the average is consistent with zero within errors.

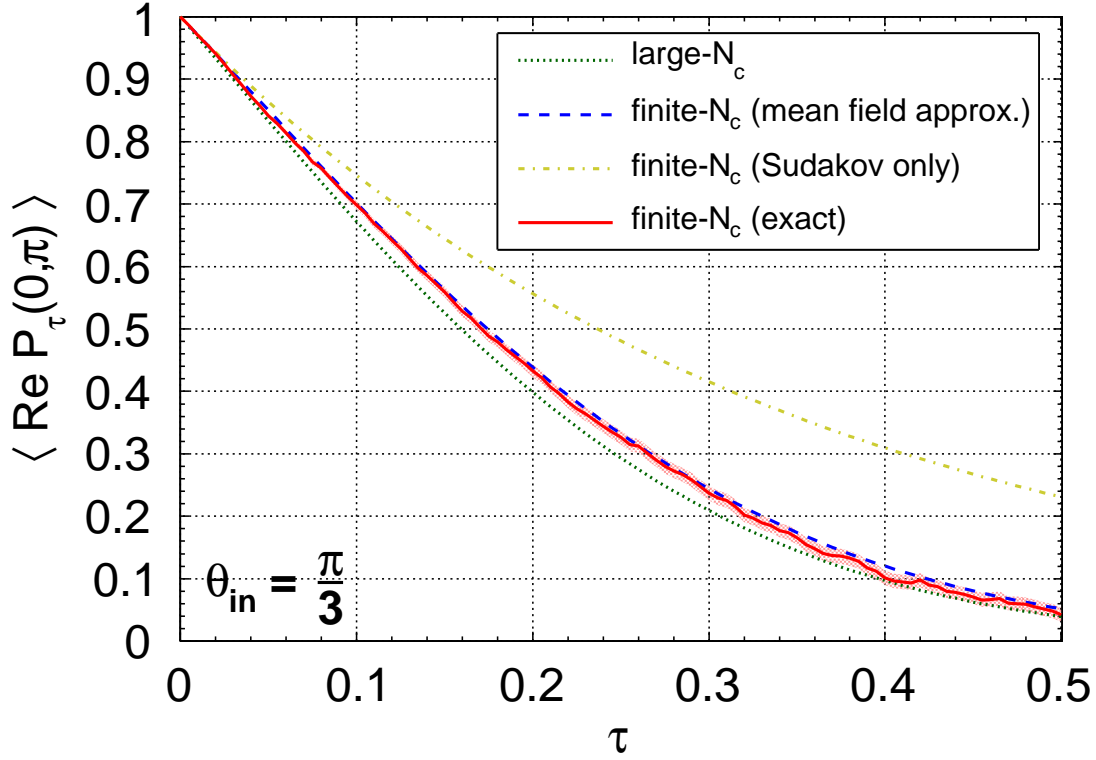


Fig. 2. Solid line (red): exact $N_c = 3$ solution to (21). The band indicates the standard error. Dashed line (blue): $N_c = 3$, mean-field solution to (58). Dotted line (green): solution to the BMS equation (16) from [22]. Dash-dotted line (yellow): result with only the Sudakov term.

with the solution of the large- N_c BMS equation (16) (dotted green line) previously obtained in [22],⁹ and also with the solution of the ‘mean field approximation’ to (21) (dashed blue line)

$$\begin{aligned} \partial_\tau \langle P_{\alpha\beta} \rangle_\tau = & -2C_F \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{out}(\gamma) \langle P_{\alpha\beta} \rangle_\tau \\ & + N_c \int \frac{d\Omega_\gamma}{4\pi} \mathcal{M}_{\alpha\beta}(\gamma) \Theta_{in}(\gamma) \left(\langle P_{\alpha\gamma} \rangle_\tau \langle P_{\gamma\beta} \rangle_\tau - \langle P_{\alpha\beta} \rangle_\tau \right), \end{aligned} \quad (58)$$

which differs from the BMS equation only by the coefficient of the Sudakov term $N_c = 3 \leftrightarrow 2C_F = 8/3$. The latter serves as an indicator of the quality of the mean field approximation $\langle PP \rangle \rightarrow \langle P \rangle \langle P \rangle$. For the sake of reference, we also plot the solution obtained by keeping only the Sudakov term (first term on the right-hand-side) in (58) (dash-dotted yellow line).

⁹ Note that the definition of τ in [22] differs from (17) by a factor of N_c .

6 Discussion

A comparison of the solid (red) and dashed (blue) lines in Fig. 2 shows that the exact solution is rather close to the mean field solution with the finite- N_c corrected Sudakov term for all values of τ explored in this work. This may come as no surprise to those who are acquainted with the solution of the JIMWLK equation which agrees well with the BK solution [16,23]. Nevertheless, we find the present result quite intriguing because we actually observed enormous trajectory-by-trajectory fluctuations. In Fig. 3 we show 50 (out of 500) individual random walk trajectories used in the computation of the average. It turns out that the fluctuations are so large that the standard deviation $\delta P \equiv \sqrt{\langle PP \rangle - \langle P \rangle \langle P \rangle}$ is of the order of $\langle P \rangle$ itself for not-so-small values of τ . Actually, this is the reason why we needed to run $\mathcal{O}(100)$ trajectories to obtain a reasonably stable result. Such large fluctuations have not been seen in the previous simulation of the JIMWLK equation where ‘already one trajectory gives a good estimate of the final result’ [16].

In our opinion, the crucial difference between the two problems which has resulted in such different behaviors of fluctuations is the initial condition. In the jet problem, the initial condition $\langle P \rangle_{\tau=0} = 1$ means that there are no partons besides the $q\bar{q}$ pair. In the parlance of saturation physics, the system is initially very ‘dilute’. The evolution then produces soft gluons which multiply exponentially in τ according to the BFKL formula [9,18]. It has been demonstrated that these gluons have very strong number fluctuations [24] and spatial correlations [25,26,19]. We thus find it natural to attribute the observed large values of δP to such fluctuations and correlations. On the other hand, when solving the JIMWLK equation, one often uses ‘dense’ or ‘classically saturated’ initial conditions; $\langle S_{xy} \rangle_{\tau=0} \ll 1$ for $|\mathbf{x} - \mathbf{y}|$ larger than some value. Since there are many uncorrelated gluons in the system from the beginning, there is little room left for pure-BFKL evolution, i.e., it is suppressed by the saturation effect. Accordingly, fluctuations and correlations can develop only weakly, and this is consistent with what has been found in the previous simulations.

In conclusion, we have demonstrated for the first time the resummation of non-global logarithms at finite N_c to leading logarithmic accuracy. Our study shows that, at least in e^+e^- annihilation and for phenomenologically interesting values of $\tau \lesssim 0.2 \sim 0.3$, event-averaged non-global observables may be reliably computed in the mean field approximation by solving the (modified) BMS equation (58), or equivalently, by Monte Carlo simulations [6]. However, the observed large fluctuations imply that the situation may be drastically different for hadron-hadron collisions (cf. [27]). Since four partons are involved in hard scattering, one has to deal with multiple products of Wilson lines such as $\text{tr}(UU^\dagger)\text{tr}(UU^\dagger)$ and $\text{tr}(UU^\dagger UU^\dagger)$ (cf. [28]). Moreover, if there are gluons in the initial and final states, each of them picks up an adjoint Wilson line \tilde{U} which further increases the number of (fundamental) Wilson lines. [Roughly, \tilde{U} acts like the square of U .] The proper treatment of fluctuations laid out in this paper is potentially very important for such observables. We leave this problem for future work.

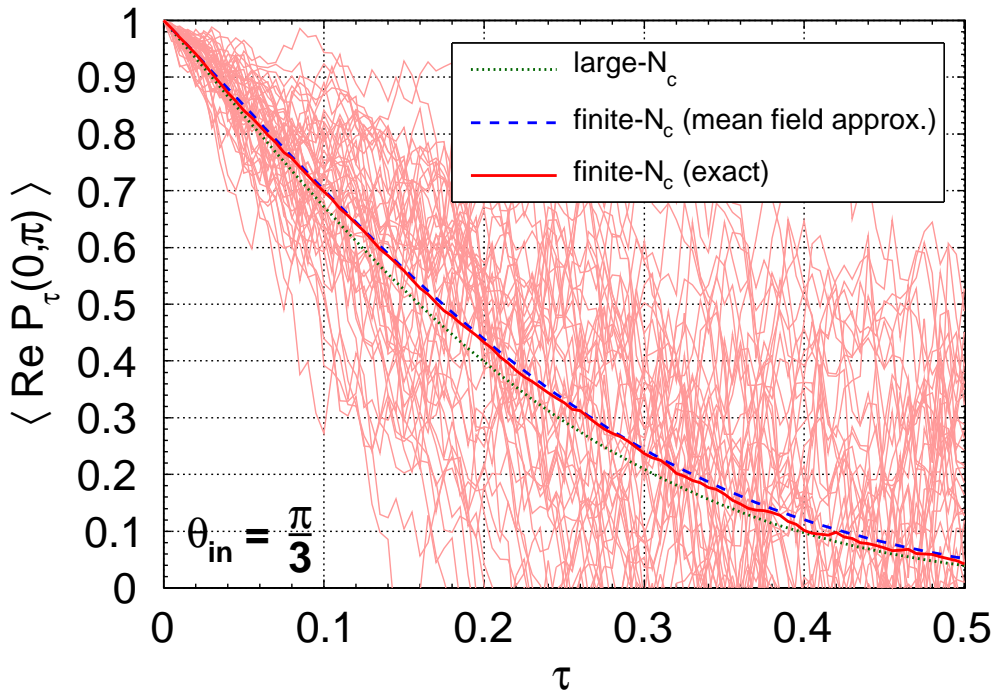


Fig. 3. Thin lines (pink): 50 random walk trajectories used in the computation of the average (thick red line).

Acknowledgments

We thank Tuomas Lappi for useful correspondence. The work of T. U. was supported by the DFG through SFB/TR 9 “Computational Particle Physics”. The calculations were partially performed on the HP XC3000 at Steinbuch Centre for Computing of KIT.

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