Prepared for submission to JHEP

TTP13-008
SFB/CPP-13-19

β-function for the Higgs self-interaction in the Standard Model at three-loop level

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Abstract: We analytically compute the QCD, electroweak, Higgs and third generation Yukawa contributions to the β-function for the Higgs self-coupling as well as for the Higgs mass parameter in the unbroken phase of the Standard Model at three-loop level.

Keywords: Renormalization Group, Higgs Physics, Standard Model

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1 Introduction

Renormalization Group functions, i.e. β-functions and anomalous dimensions, are of great importance in quantum field theory. Recently most of these functions have been computed at three-loop accuracy in the Standard Model (SM). The results for the gauge couplings have first been derived in [1, 2] and have been confirmed independently in [3]. For the top-Yukawa coupling and the Higgs self-interaction λ the QCD, top-Yukawa and Higgs contributions have been derived in [4]. The result for the top-Yukawa β-function has been extended to include the electroweak and all third generation Yukawa couplings in [5] where also the β-functions for bottom and τ-Yukawa have been presented. The one-loop and two-loop results for all SM couplings have been known for a long time [6–20] as have been partial three-loop results [21–26]. Four-loop β-functions are available for QCD [27, 28] and the purely scalar part of the SM [29–31]. In this paper we present the extension of our result for the Higgs self-interaction and the anomalous dimension of the Higgs mass parameter to include the electroweak and all third generation Yukawa couplings as well. Especially the β-function for the Higgs self-coupling is interesting because of its close connection to the question of vacuum stability in the Standard Model. It has been shown that the stability of the SM vacuum up to some energy scale Λ is approximately equivalent to the requirement that the running coupling $\lambda(\mu) > 0$ for $\mu \leq \Lambda$ [32–34]. Many analyses of this question have been performed [4, 35–41] during the last years. The main uncertainty stems from the experimental error on the top mass followed by the uncertainty in α_s. But future linear colliders could greatly reduce these uncertainties and a possible Higgs mass of about 126 GeV [42, 43] allows for both scenarios, a stable and an unstable (or to be more precise metastable) SM vacuum, within the present experimental and theoretical errors. For this reason we think that the present work will reduce the theoretical uncertainty connected to the running of λ even further than our previous calculation [4].

In the following section the setup and the technical details of the calculation are discussed. After that we present our results and some numerics in order to determine the significance of the new terms.

2 Calculation

The gauge group of the SM is an $\mathrm{SU}_C(3) \times \mathrm{SU}(2) \times \mathrm{U}_Y(1)$ which is spontaneously broken to an $\mathrm{SU}_C(3) \times \mathrm{U}_Q(1)$ at the Fermi scale. Our calculation is performed in the unbroken phase of the SM which is justified by the fact that the UV behaviour and therefore the renormalization constants for fields and vertices do not depend on masses in the $\overline{\mathrm{MS}}$-scheme [44].

The Lagrangian of the SM can be decomposed into the following pieces:

$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{EW} + \mathcal{L}_{Yukawa} + \mathcal{L}_{\Phi}. \tag{2.1}$$

The QCD and electroweak (EW) part are implemented in the usual way with the gauge fields A^a_μ ($\mathrm{SU}_C(3)$, $a = 1, \ldots, 8$), W^a_μ ($\mathrm{SU}(2)$, $a = 1, 2, 3$) and B_μ ($\mathrm{U}_Y(1)$). These appear in the covariant derivative

$$D^\mu = \partial^\mu - i g_1 Y_f B^\mu - i \frac{g_2}{2} \sigma^a W^{a\,\mu} - i g_s T^a A^{a\,\mu} \tag{2.2}$$

with the Pauli matrices σ^a and the hypercharge Y_f of the field f on which the covariant derivative acts. In the Yukawa part we neglect the first two generations and the mixing of generations. The W-fermion-vertices are taken to be diagonal in the generations as well, i.e. we set the CKM matrix to the unit matrix. Light quarks and leptons are present in the QCD and electroweak sector however. This leads to the Lagrangian

$$\begin{aligned} \mathcal{L}_{Yukawa} = &- y_t \left\{ \bar{t}_R \Phi^{\dagger\, c} Q_L + \bar{Q}_L \Phi^c t_R \right\} - y_b \left\{ \bar{b}_R \Phi^\dagger Q_L + \bar{Q}_L \Phi t_R \right\} \\ &- y_\tau \left\{ \bar{\tau}_R \Phi^\dagger L_L + \bar{L}_L \Phi t_R \right\} \end{aligned} \tag{2.3}$$

for the Yukawa sector. The complex scalar field Φ and the left-handed quarks and leptons are doublets under SU(2):

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \qquad \Phi^c = i\sigma^2 \Phi^*, \qquad Q_L = \begin{pmatrix} t \\ b \end{pmatrix}_L, \qquad L_L = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L. \tag{2.4}$$

The indices L and R indicate the left- and right-handed part of the fields as obtained by the projectors

$$P_L = \frac{1}{2}(1 - \gamma_5) \qquad P_R = \frac{1}{2}(1 + \gamma_5). \tag{2.5}$$

Finally, we have the Higgs sector with

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi \right)^2. \tag{2.6}$$

For every field and vertex a counterterm is introduced and the corresponding renormalization constant is calculated order by order in perturbation theory. The left-handed and right-handed parts of fermion fields, quark-gluon-vertices and fermion-B-vertices are renormalized with different counterterms. The renormalization constant for the gauge, Yukawa and Higgs couplings can be obtained in different ways, e.g.

$$Z_{g_1} = \frac{Z_{1,L}^{(\tau\tau B)}}{Z_{2,L}^{(2\tau)} \sqrt{Z_3^{(2B)}}} = \frac{Z_{1,R}^{(ttB)}}{Z_{2,R}^{(2t)} \sqrt{Z_3^{(2B)}}} = \ldots \tag{2.7}$$

where $Z_1^{f_1\dots f_n}$ is the renormalization constant for the vertex of the (renormalized) fields $f_1,\dots,f_n$, $Z_{2,L/R}^{(2f)}$ the field strength renormalization constant for the left-handed (L)/ right-handed (R) part of the fermion field f and $Z_3^{(2g)}$ the field strength renormalization constant for the gauge field g.

Likewise, the renormalization constant for Yukawa couplings can be computed from the renormalization constant for any vertex proportional to this coupling and the renormalization constants for the external legs of this vertex, e.g.

$$Z_{y_t} = \frac{Z_1^{(tt\Phi)}}{\sqrt{Z_{2,L}^{(2t)} Z_{2,R}^{(2t)} Z_2^{(2\Phi)}}}, \tag{2.8}$$

where $Z_2^{(2\Phi)}$ is the field strength renormalization constant for the scalar doublet. All renormalization constants are defined in a minimal way as

$$Z = 1 + \delta Z, \tag{2.9}$$

with δZ containing only poles in the regulating parameter $\varepsilon = (4-D)/2$ where D is the engineering space-time dimension. The Higgs self-coupling λ is renormalized with a counterterm δZ_λ that is not proportional to λ but also has terms proportional to four Yukawa couplings. Consequently, this is a feature of the corresponding β-function as well. We find

$$\begin{aligned} -\lambda_B \left(\Phi_B^\dagger \Phi_B\right)^2 =& (-\lambda + \delta Z_1^{(4\Phi)}) \left(\Phi^\dagger \Phi\right)^2 \\ \Rightarrow -(\lambda + \delta Z_\lambda) \left(Z_2^{(2\Phi)}\right)^2 =& (-\lambda + \delta Z_1^{(4\Phi)}) \\ \Rightarrow \delta Z_\lambda =& \left(Z_2^{(2\Phi)}\right)^{-2} (\lambda - \delta Z_1^{(4\Phi)}) - \lambda\,, \end{aligned} \tag{2.10}$$

with the counterterm $\delta Z_1^{(4\Phi)}$ for the four-Φ-vertices and the index B marking bare quantities. The β-function for any coupling X is defined as

$$\beta_X = \mu^2 \frac{dX}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n}\, \beta_X^{(n)} \tag{2.11}$$

and is given as a power series in all considered couplings of the SM, i.e. $g_s, g_2, g_1, y_t, y_b, y_\tau$ and λ. The mass parameter m^2 of the scalar field is neglected in the calculation as it has no influence on the UV behaviour of the couplings in the $\overline{\text{MS}}$-scheme.

The β-function (or anomalous dimension) describing the running of this mass parameter m^2 in eq. (2.6) can be computed from the renormalization constant of the local operator $O_{2\Phi} := \Phi^\dagger\Phi$. An insertion of $O_{2\Phi}$ into a Green's function, e.g. with two external Φ-fields, is renormalized as $[O_{2\Phi}] = Z_{\Phi^2} O_{2\Phi}$ where $[O_{2\Phi}]$ is the corresponding finite operator. From $[O_{2\Phi}] = Z_{m^2} O_{2\Phi}^{\text{bare}}$ and $O_{2\Phi}^{\text{bare}} = Z_2^{(2\Phi)} O_{2\Phi}$ it follows that

$$Z_{m^2} = \left(Z_2^{(2\Phi)}\right)^{-1} Z_{\Phi^2}. \tag{2.12}$$

For this project we need the renormalization constants $\delta Z_1^{(4\Phi)}$, Z_{Φ^2} and $Z_2^{(2\Phi)}$ at three-loop level and all other counterterms at two-loop accuracy or less. We perform our calculation in a general R_ξ-gauge with different gauge parameters ξ_1, ξ_2 and ξ for the gauge fields B, W and A. The β-functions for all couplings are independent of the gauge parameters which serves as an important check for the result. In order to compute all counterterms up to two-loop level and the one and two-loop diagrams with counterterm insertions contributing to the three-loop result it has been convenient to use a setup where different isospin configurations of a field and different fermion generations (in our case the distinction between third generation and light is enough) are implemented as separate fields as many counterterms depend on those. Additionally, the $\mathrm{U}_Y(1)$ hypercharge depends on the isospin and differs for the left-handed and right-handed part of the field. So we use the set of fields

$$\underbrace{t, b, u, d}_{\text{quarks}}, \underbrace{e^-, \nu_e, \tau, \nu_\tau}_{\text{leptons}}, \underbrace{A^{a\mu}, B^\mu, W^{1\mu}, W^{2\mu}, W^{3\mu}}_{\text{gauge bosons}}, \underbrace{c^a, c_W^1, c_W^2, c_W^3}_{\text{ghosts}}, \Phi_1, \Phi_2$$

and their anti-fields. The fermion fields have to be split up in a left-handed and a right-handed part during the calculation. The price we pay, however, is that many diagrams are produced which look the same in momentum space if no counterterms. are inserted. For the 1PI process with four external Φ-legs $\sim 2.3 \times 10^6$ diagrams are generated at three-loop level. In order to reduce this number and because we do not need three-loop diagrams with counterterm insertions anyway we have chosen a second smaller set of fields for this part of the calculation

$$\underbrace{q_i}_{\text{quarks}}, \underbrace{l_i}_{\text{leptons}}, \underbrace{A^{a\mu}, B^\mu, W^{a\mu}}_{\text{gauge bosons}}, \underbrace{c^a, c_W^a}_{\text{ghosts}}, \Phi_i.$$

Here the index i marks the isospin of Φ and the left-handed fermions. For the right-handed fermions $i = 1, 2$ is just a label to mark the flavour, e.g. $q_{1,R} = t_R, q_{2,R} = b_R$. Fermion loops without Yukawa interactions are multiplied by the number of generations N_g in order to include the light fermions. The indices of external particles can be explicitly chosen. Using this setup only 573692 diagrams are produced for the four-Φ_1-process at three loops. The computation of the $\mathrm{SU}(2) \times \mathrm{U}_Y(1)$ group factors has been implemented with Mathematica using labels for the left-handed and right-handed part at each quark-Yukawa-vertex and B-fermion-vertex as well as for the three different structures in the four-W-vertex. With the help of these labels we can completely factorize the $\mathrm{SU}(2) \times \mathrm{U}_Y(1)$ part from the momentum space diagram. The QCD colour factors have been calculated with the FORM package COLOR [45]. All Feynman diagrams have been automatically generated with QGRAF [46].

In order to check our setup we have computed all Yukawa and gauge coupling renormalization constants at two-loop level from at least two different vertices with the first set of fields and compared the result to the literature. The same has been done for the renormalization constants of the gauge, ghost and scalar fields. We also explicitly checked that we get the same renormalization constants for Φ_1 and Φ_2 as well as for the left-handed fermion flavours of the same generation at two loops. Another check has been the finiteness of the three-B-vertex up to two loops.

The renormalization constant for Φ has been computed with both sets of fields at three-loop level which yields the same result as in [2]. The renormalization constants for the W, c_W, Φ_1 and Φ_2 fields as well as for the W-$\bar{c}_W$-c_W-vertex, the $O_{2\Phi}$-Φ_1-Φ_1-vertex, the $O_{2\Phi}$-Φ_1-Φ_1-vertex and the 4-Φ_1-vertex have been computed with both sets of fields up to two loops with the same result.

As explained in detail in [4] some diagrams with four external Φ-fields where two external momenta are set to zero suffer from IR divergences which mix with the UV ones in dimensional regularization. We therefore use the same method as in [4] and introduce the same auxiliary mass parameter M^2 in every propagator denominator. Subdivergences $\propto M^2$ are canceled by counterterms

$$\frac{M^2}{2}\delta Z^{(2g)}_{M^2}\, A^a_\mu A^{a\,\mu},\ \frac{M^2}{2}\delta Z^{(2W)}_{M^2}\, W^a_\mu W^{a\,\mu},\ \frac{M^2}{2}\delta Z^{(2B)}_{M^2}\, B_\mu B^\mu \text{ and } \frac{M^2}{2}\delta Z^{(2\Phi)}_{M^2}\, \Phi^\dagger\Phi. \tag{2.13}$$

Counterterms $\propto M$ that would arise for fermions cannot appear because there are no M in the numerators of propagators. Ghost mass terms $\frac{M^2}{2}\delta Z^{(2c)}_{M^2}\,\bar{c}^a c^a$ for the SU(3) and SU(2) ghosts do not appear because of the momentum dependence of the ghost-gauge boson-vertex. The remaining divergences are the mass-independent UV ones we are looking for. This method has been suggested in [47] and has been elaborated on in the context of three-loop calculations in [48]. The resulting massive tadpole integrals can be computed with the FORM-based program MATAD [49].

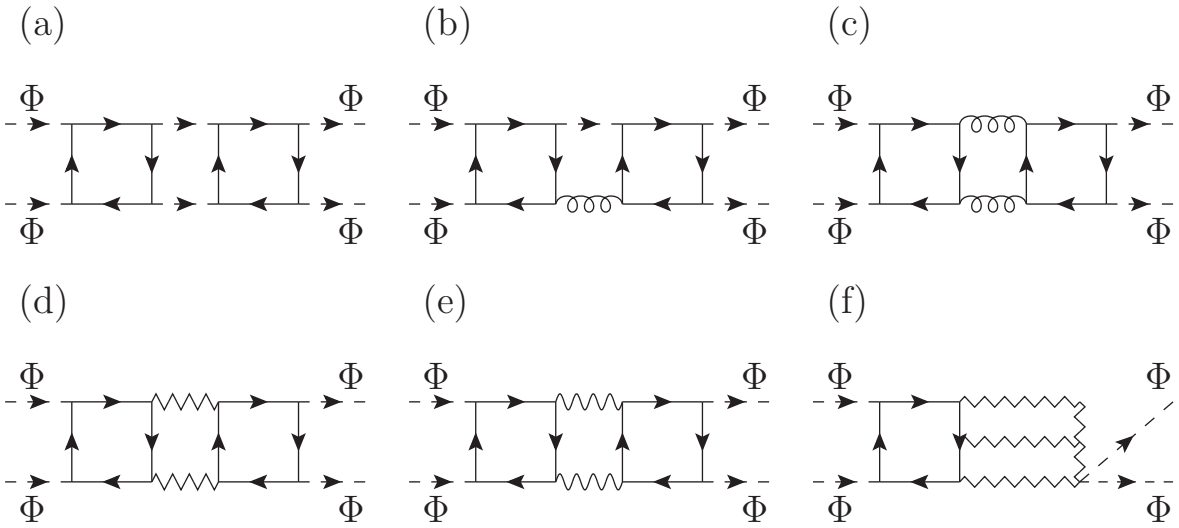

Figure 1: Some diagrams contributing to the renormalization of the 4-Φ-vertices

As opposed to the case of the Yukawa coupling β-functions (see [4, 5]) a completely naive treatment of γ_5 in dimensional regularization is possible for β_λ and β_{m^2}. In four dimensions we define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \text{ with } \varepsilon_{0123} = 1 = -\varepsilon^{0123}. \tag{2.14}$$

In order to have a non-naive contribution from a fermion loop with a γ_5 matrix in it at least four free Lorentz indices or momenta on the external lines of the minimal subgraph

containing this fermion loop are required. These can be indices from the gauge boson vertices or the internal momenta from other loops which act as external momenta to the minimal subgraph containing the fermion loop in question. External momenta of the whole diagram can be set to zero as the renormalization constants in the $\overline{\text{MS}}$-scheme do not depend on those. In four dimensions the trace of such a fermion line will produce a result $\propto \varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ where $\mu_1, \ldots, \mu_4$ are the aforementioned free Lorentz indices. If there is a second fermion line the trace of which also yields an ε-tensor these two ε-tensors can be contracted and we can get a non-naive contribution from γ_5.

Let us consider two examples. The diagram in Fig.1 (c) has two fermion loops. If we take one of the fermion loops with the momenta on the two external Φ-legs set to zero we have two indices from the gluon lines attached to the fermion loop and one loop momentum going through the two gluons and acting as an external momentum to the subgraph containing only this fermion loop. This is not enough to have a non-naive γ_5 contribution from this graph.

The fermion loop in Fig.1 (f) has three Lorentz indices and two external loop momenta but there is no second fermion line to produce a second ε-tensor. Furthermore - as we set all external momenta to zero - there are no free Lorentz indices or momenta in the final result to support an ε-tensor there. Therefore any contribution with an ε-tensor from this fermion loop (which is the only antisymmetric Lorentz structure) must vanish after contraction with the Lorentz structures from the W-bosons.

3 Results

In this section we give the results for the three-loop β-functions for the couplings λ and the mass parameter m^2 setting all gauge group factors to their SM values. All results of this work can found at
http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-008/.[1]
We denote the number of generations by N_g.

[1] There we also present the results in a form where the QCD colour factors are not set to numbers but generically expressed through the quadratic Casimir operators C_F and C_A of the quark and the adjoint representation of the corresponding Lie algebra, the dimension of the quark representation d_R and the trace T_F defined through $T_F\delta^{ab} = \mathbf{Tr}\left(T^aT^b\right)$ with the group generators T^a of the quark representation.

$$
\begin{aligned}
\beta_\lambda^{(1)} =& -y_\tau^4 - y_b^4 3 + g_2^4\frac{9}{16} + g_1^2 g_2^2\frac{3}{8} + g_1^4\frac{3}{16} + \lambda y_\tau^2 2 + \lambda y_b^2 6 - \lambda g_2^2\frac{9}{2} - \lambda g_1^2\frac{3}{2} \\
&+ \lambda^2 12 + y_t^2\lambda 6 - y_t^4 3\,, \\
\beta_\lambda^{(2)} =& y_\tau^6 5 + y_b^6 15 - g_2^4 y_\tau^2\frac{3}{8} - g_2^4 y_b^2\frac{9}{8} + g_2^6\left(\frac{497}{32} - 2N_g\right) - g_1^2 y_\tau^4 2 + g_1^2 y_b^4\frac{2}{3} \\
&+ g_1^2 g_2^2 y_\tau^2\frac{11}{4} + g_1^2 g_2^2 y_b^2\frac{9}{4} - g_1^2 g_2^4\left(\frac{97}{96} + \frac{2}{3}N_g\right) - g_1^4 y_\tau^2\frac{25}{8} + g_1^4 y_b^2\frac{5}{8} \\
&- g_1^4 g_2^2\left(\frac{239}{96} + \frac{10}{9}N_g\right) - g_1^6\left(\frac{59}{96} + \frac{10}{9}N_g\right) - \lambda y_\tau^4\frac{1}{2} - \lambda y_b^4\frac{3}{2} \\
&+ \lambda g_2^2 y_\tau^2\frac{15}{4} + \lambda g_2^2 y_b^2\frac{45}{4} + \lambda g_2^4\left(-\frac{313}{16} + 5N_g\right) + \lambda g_1^2 y_\tau^2\frac{25}{4} + \lambda g_1^2 y_b^2\frac{25}{12} \\
&+ \lambda g_1^2 g_2^2\frac{39}{8} + \lambda g_1^4\left(\frac{229}{48} + \frac{25}{9}N_g\right) - \lambda^2 y_\tau^2 24 - \lambda^2 y_b^2 72 + \lambda^2 g_2^2 54 \\
&+ \lambda^2 g_1^2 18 - \lambda^3 156 - y_t^2 y_b^4 3 - y_t^2 g_2^4\frac{9}{8} + y_t^2 g_1^2 g_2^2\frac{21}{4} - y_t^2 g_1^4\frac{19}{8} \\
&- y_t^2\lambda y_b^2 21 + y_t^2\lambda g_2^2\frac{45}{4} + y_t^2\lambda g_1^2\frac{85}{12} - y_t^2\lambda^2 72 - y_t^4 y_b^2 3 - y_t^4 g_1^2\frac{4}{3} \\
&- y_t^4\lambda\frac{3}{2} + y_t^6 15 - g_s^2 y_b^4 16 + g_s^2\lambda y_b^2 40 + g_s^2 y_t^2\lambda 40 - g_s^2 y_t^4 16.
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\beta_\lambda^{(3)} =& y_\tau^8\left(-\frac{143}{8} - 12\zeta_3\right) - y_b^2 y_\tau^6\frac{297}{8} - y_b^4 y_\tau^4 72 - y_b^6 y_\tau^2\frac{297}{8} + y_b^8\left(-\frac{1599}{8} - 36\zeta_3\right) \\
&+ g_2^2 y_\tau^6\left(\frac{1137}{32} - 9\zeta_3\right) + g_2^2 y_b^6\left(\frac{3411}{32} - 27\zeta_3\right) + g_2^4 y_\tau^4\left(\frac{4503}{128} - \frac{273}{16}\zeta_3 - \frac{13}{4}N_g\right) \\
&+ g_2^4 y_b^2 y_\tau^2\frac{9}{8} + g_2^4 y_b^4\left(\frac{13653}{128} - \frac{819}{16}\zeta_3 - \frac{39}{4}N_g\right) + g_2^6 y_\tau^2\left(-\frac{5739}{256} + \frac{99}{4}\zeta_3 + \frac{9}{2}N_g\right) \\
&+ g_2^6 y_b^2\left(-\frac{17217}{256} + \frac{297}{4}\zeta_3 + \frac{27}{2}N_g\right) + g_2^8\left(\frac{982291}{3072} - \frac{2781}{128}\zeta_3 - \frac{14749}{192}N_g\right. \\
&\left.-45N_g\zeta_3 - \frac{5}{3}N_g^2\right) + g_1^2 y_\tau^6\left(\frac{135}{32} + 33\zeta_3\right) + g_1^2 y_b^6\left(\frac{5111}{96} - 25\zeta_3\right) \\
&+ g_1^2 g_2^2 y_\tau^4\left(-\frac{15}{64} - \frac{381}{8}\zeta_3\right) - g_1^2 g_2^2 y_b^2 y_\tau^2\frac{5}{4} + g_1^2 g_2^2 y_b^4\left(-\frac{3239}{192} - \frac{311}{8}\zeta_3\right) \\
&+ g_1^2 g_2^4 y_\tau^2\left(\frac{1833}{256} - \frac{3}{2}\zeta_3 - \frac{1}{2}N_g\right) + g_1^2 g_2^4 y_b^2\left(\frac{4179}{256} + 9\zeta_3 + \frac{5}{2}N_g\right) \\
&+ g_1^2 g_2^6\left(-\frac{54053}{3456} - \frac{405}{32}\zeta_3 - \frac{8341}{864}N_g - \frac{10}{27}N_g^2\right) + g_1^4 y_\tau^4\left(\frac{5697}{128} + \frac{375}{16}\zeta_3 + \frac{65}{12}N_g\right) \\
&+ g_1^4 y_b^2 y_\tau^2\frac{41}{24} + g_1^4 y_b^4\left(\frac{15137}{3456} - \frac{2035}{144}\zeta_3 - \frac{415}{36}N_g\right) + g_1^4 g_2^2 y_\tau^2\left(\frac{6657}{256} - \frac{15}{2}\zeta_3 - \frac{5}{6}N_g\right) \\
&+ g_1^4 g_2^2 y_b^2\left(\frac{4403}{256} + \frac{9}{2}\zeta_3 + \frac{25}{6}N_g\right) + g_1^4 g_2^4\left(-\frac{64693}{3456} + \frac{873}{64}\zeta_3 + \frac{149}{648}N_g + 7N_g\zeta_3\right. \\
&\left.-\frac{50}{81}N_g^2\right) + g_1^6 y_\tau^2\left(\frac{3929}{256} - \frac{15}{4}\zeta_3 + \frac{55}{6}N_g\right) + g_1^6 y_b^2\left(\frac{12043}{2304} + \frac{5}{4}\zeta_3 + \frac{95}{18}N_g\right) \\
&+ g_1^6 g_2^2\left(-\frac{29779}{6912} + \frac{75}{32}\zeta_3 - \frac{18001}{2592}N_g + \frac{61}{9}N_g\zeta_3 - \frac{250}{243}N_g^2\right)
\end{aligned}
$$

$$+ g_1^8\left(-\frac{6845}{9216}+\frac{99}{128}\zeta_3-\frac{20735}{1728}N_g+\frac{95}{9}N_g\zeta_3-\frac{125}{81}N_g^2\right)+\lambda y_\tau^6\left(-\frac{1241}{8}-66\zeta_3\right)$$
$$+\lambda y_b^2y_\tau^4240+\lambda y_b^4y_\tau^2240+\lambda y_b^6\left(\frac{117}{8}-198\zeta_3\right)+\lambda g_2^2y_\tau^4\left(-\frac{1587}{8}+171\zeta_3\right)$$
$$-\lambda g_2^2y_b^2y_\tau^227+\lambda g_2^2y_b^4\left(-\frac{4977}{8}+513\zeta_3\right)+\lambda g_2^4y_\tau^2\left(-\frac{1311}{64}-\frac{117}{2}\zeta_3-\frac{21}{4}N_g\right)$$
$$+\lambda g_2^4y_b^2\left(-\frac{3933}{64}-\frac{351}{2}\zeta_3-\frac{63}{4}N_g\right)+\lambda g_2^6\left(-\frac{46489}{288}+\frac{2259}{8}\zeta_3+\frac{3515}{36}N_g\right.$$
$$\left.+90N_g\zeta_3+\frac{70}{9}N_g^2\right)+\lambda g_1^2y_\tau^4\left(\frac{507}{8}-117\zeta_3\right)-\lambda g_1^2y_b^2y_\tau^29+\lambda g_1^2y_b^4\left(-\frac{5737}{24}\right.$$
$$\left.+249\zeta_3\right)+\lambda g_1^2g_2^2y_\tau^2\left(-\frac{3771}{32}+126\zeta_3\right)+\lambda g_1^2g_2^2y_b^2\left(-\frac{3009}{32}+12\zeta_3\right)$$
$$+\lambda g_1^2g_2^4\left(\frac{4553}{32}-\frac{249}{8}\zeta_3+\frac{33}{2}N_g-6N_g\zeta_3\right)+\lambda g_1^4y_\tau^2\left(-\frac{1783}{64}-\frac{123}{2}\zeta_3-\frac{65}{4}N_g\right)$$
$$+\lambda g_1^4y_b^2\left(-\frac{127303}{1728}-\frac{47}{6}\zeta_3-\frac{155}{36}N_g\right)+\lambda g_1^4g_2^2\left(\frac{979}{8}-\frac{3}{8}\zeta_3+\frac{95}{4}N_g-6N_g\zeta_3\right)$$
$$+\lambda g_1^6\left(\frac{12679}{432}+\frac{9}{8}\zeta_3+\frac{5995}{162}N_g-\frac{190}{9}N_g\zeta_3+\frac{1750}{243}N_g^2\right)+\lambda^2y_\tau^4\left(\frac{717}{2}+252\zeta_3\right)$$
$$-\lambda^2y_b^2y_\tau^2216+\lambda^2y_b^4\left(\frac{1719}{2}+756\zeta_3\right)+\lambda^2g_2^2y_\tau^2\left(\frac{213}{4}-144\zeta_3\right)+\lambda^2g_2^2y_b^2\left(\frac{639}{4}\right.$$
$$\left.-432\zeta_3\right)+\lambda^2g_2^4\left(\frac{1995}{8}-513\zeta_3-141N_g\right)+\lambda^2g_1^2y_\tau^2\left(-\frac{541}{4}+96\zeta_3\right)$$
$$+\lambda^2g_1^2y_b^2\left(\frac{417}{4}-192\zeta_3\right)+\lambda^2g_1^2g_2^2\left(-333-162\zeta_3\right)+\lambda^2g_1^4\left(-183-81\zeta_3\right.$$
$$\left.-\frac{235}{3}N_g\right)+\lambda^3y_\tau^2291+\lambda^3y_b^2873+\lambda^3g_2^2\left(-474+72\zeta_3\right)+\lambda^3g_1^2\left(-158+24\zeta_3\right)$$
$$+\lambda^4\left(3588+2016\zeta_3\right)+y_t^2y_\tau^6\left(-\frac{297}{8}\right)+y_t^2y_b^2y_\tau^412+y_t^2y_b^4y_\tau^2\frac{45}{8}+y_t^2y_b^6\left(-\frac{717}{8}\right.$$
$$\left.-36\zeta_3\right)+y_t^2g_2^2y_b^4\frac{477}{32}+y_t^2g_2^4y_\tau^2\frac{9}{8}+y_t^2g_2^4y_b^2\left(-\frac{351}{64}+\frac{117}{2}\zeta_3-12N_g\right)$$
$$+y_t^2g_2^6\left(-\frac{17217}{256}+\frac{297}{4}\zeta_3+\frac{27}{2}N_g\right)+y_t^2g_1^2y_b^4\left(-\frac{2299}{96}+26\zeta_3\right)$$
$$+y_t^2g_1^2g_2^2y_\tau^2\frac{29}{4}+y_t^2g_1^2g_2^2y_b^2\left(\frac{1001}{96}+\frac{31}{2}\zeta_3\right)+y_t^2g_1^2g_2^4\left(\frac{3103}{256}+\frac{27}{4}\zeta_3+\frac{1}{2}N_g\right)$$
$$+y_t^2g_1^4y_\tau^2\frac{701}{24}+y_t^2g_1^4y_b^2\left(-\frac{709}{64}-\zeta_3\right)+y_t^2g_1^4g_2^2\left(\frac{23521}{768}-3\zeta_3+\frac{5}{6}N_g\right)$$
$$+y_t^2g_1^6\left(\frac{42943}{2304}-\frac{5}{2}\zeta_3+\frac{215}{18}N_g\right)+y_t^2\lambda y_\tau^4240+y_t^2\lambda y_b^2y_\tau^221$$
$$+y_t^2\lambda y_b^4\left(\frac{6399}{8}+144\zeta_3\right)-y_t^2\lambda g_2^2y_\tau^227+y_t^2\lambda g_2^2y_b^2\left(-\frac{531}{4}+54\zeta_3\right)$$
$$+y_t^2\lambda g_2^4\left(-\frac{3933}{64}-\frac{351}{2}\zeta_3-\frac{63}{4}N_g\right)-y_t^2\lambda g_1^2y_\tau^29+y_t^2\lambda g_1^2y_b^2\left(-\frac{929}{12}-2\zeta_3\right)$$
$$+y_t^2\lambda g_1^2g_2^2\left(-\frac{6509}{32}+177\zeta_3\right)+y_t^2\lambda g_1^4\left(-\frac{112447}{1728}-\frac{449}{6}\zeta_3-\frac{635}{36}N_g\right)$$
$$-y_t^2\lambda^2y_\tau^2216+y_t^2\lambda^2y_b^2\left(117-864\zeta_3\right)+y_t^2\lambda^2g_2^2\left(\frac{639}{4}-432\zeta_3\right)$$

$$
\begin{aligned}
&+ y_t^2\lambda^2 g_1^2\left(-\frac{195}{4} - 48\zeta_3\right) + y_t^2\lambda^3 873 - y_t^4 y_\tau^4 72 + y_t^4 y_b^2 y_\tau^2 \frac{45}{8} + y_t^4 y_b^4 72\zeta_3 \\
&+ y_t^4 g_2^2 y_b^2 \frac{477}{32} + y_t^4 g_2^4\left(\frac{13653}{128} - \frac{819}{16}\zeta_3 - \frac{39}{4}N_g\right) + y_t^4 g_1^2 y_b^2\left(\frac{1337}{96} - 28\zeta_3\right) \\
&+ y_t^4 g_1^2 g_2^2\left(-\frac{1079}{192} - \frac{743}{8}\zeta_3\right) + y_t^4 g_1^4\left(\frac{100913}{3456} + \frac{2957}{144}\zeta_3 - \frac{115}{36}N_g\right) \\
&+ y_t^4\lambda y_\tau^2 240 + y_t^4\lambda y_b^2\left(\frac{6399}{8} + 144\zeta_3\right) + y_t^4\lambda g_2^2\left(-\frac{4977}{8} + 513\zeta_3\right) \\
&+ y_t^4\lambda g_1^2\left(-\frac{2485}{24} + 57\zeta_3\right) + y_t^4\lambda^2\left(\frac{1719}{2} + 756\zeta_3\right) - y_t^6 y_\tau^2 \frac{297}{8} \\
&+ y_t^6 y_b^2\left(-\frac{717}{8} - 36\zeta_3\right) + y_t^6 g_2^2\left(\frac{3411}{32} - 27\zeta_3\right) + y_t^6 g_1^2\left(\frac{3467}{96} + 17\zeta_3\right) \\
&+ y_t^6\lambda\left(\frac{117}{8} - 198\zeta_3\right) + y_t^8\left(-\frac{1599}{8} - 36\zeta_3\right) + g_s^2 y_b^6\left(-38 + 240\zeta_3\right) \\
&+ g_s^2 g_2^2 y_b^4\left(-\frac{31}{2} + 24\zeta_3\right) + g_s^2 g_2^4 y_b^2\left(\frac{651}{8} - 54\zeta_3\right) + g_s^2 g_2^6\left(-\frac{153}{8}N_g + 18N_g\zeta_3\right) \\
&+ g_s^2 g_1^2 y_b^4\left(-\frac{641}{18} + \frac{136}{3}\zeta_3\right) + g_s^2 g_1^2 g_2^2 y_b^2\left(\frac{233}{4} - 36\zeta_3\right) + g_s^2 g_1^2 g_2^4\left(-\frac{51}{8}N_g + 6N_g\zeta_3\right) \\
&+ g_s^2 g_1^4 y_b^2\left(\frac{683}{24} - 18\zeta_3\right) + g_s^2 g_1^4 g_2^2\left(-\frac{187}{24}N_g + \frac{22}{3}N_g\zeta_3\right) + g_s^2 g_1^6\left(-\frac{187}{24}N_g\right. \\
&\left.+\frac{22}{3}N_g\zeta_3\right) + g_s^2\lambda y_b^4\left(895 - 1296\zeta_3\right) + g_s^2\lambda g_2^2 y_b^2\left(-\frac{489}{2} + 216\zeta_3\right) \\
&+ g_s^2\lambda g_2^4\left(\frac{135}{2}N_g - 72N_g\zeta_3\right) + g_s^2\lambda g_1^2 y_b^2\left(-\frac{991}{18} + 40\zeta_3\right) + g_s^2\lambda g_1^4\left(\frac{55}{2}N_g\right. \\
&\left.-\frac{88}{3}N_g\zeta_3\right) + g_s^2\lambda^2 y_b^2\left(-1224 + 1152\zeta_3\right) + g_s^2 y_t^2 y_b^4\left(-2 - 48\zeta_3\right) \\
&+ g_s^2 y_t^2 g_2^2 y_b^2\left(-8 + 96\zeta_3\right) + g_s^2 y_t^2 g_2^4\left(\frac{651}{8} - 54\zeta_3\right) + g_s^2 y_t^2 g_1^2 g_2^2\left(\frac{249}{4} - 36\zeta_3\right) \\
&+ g_s^2 y_t^2 g_1^4\left(\frac{587}{24} - 18\zeta_3\right) + g_s^2 y_t^2\lambda y_b^2\left(82 - 96\zeta_3\right) + g_s^2 y_t^2\lambda g_2^2\left(-\frac{489}{2} + 216\zeta_3\right) \\
&+ g_s^2 y_t^2\lambda g_1^2\left(-\frac{2419}{18} + 136\zeta_3\right) + g_s^2 y_t^2\lambda^2\left(-1224 + 1152\zeta_3\right) + g_s^2 y_t^4 y_b^2\left(-2 - 48\zeta_3\right) \\
&+ g_s^2 y_t^4 g_2^2\left(-\frac{31}{2} + 24\zeta_3\right) + g_s^2 y_t^4 g_1^2\left(\frac{931}{18} - \frac{56}{3}\zeta_3\right) + g_s^2 y_t^4\lambda\left(895 - 1296\zeta_3\right) \\
&+ g_s^2 y_t^6\left(-38 + 240\zeta_3\right) + g_s^4 y_b^4\left(-\frac{626}{3} + 32\zeta_3 + 40N_g\right) + g_s^4\lambda y_b^2\left(\frac{1820}{3}\right. \\
&\left.-48\zeta_3 - 64N_g\right) + g_s^4 y_t^2 y_b^2 192 + g_s^4 y_t^2\lambda\left(\frac{1820}{3} - 48\zeta_3 - 64N_g\right) \\
&+ g_s^4 y_t^4\left(-\frac{626}{3} + 32\zeta_3 + 40N_g\right). \qquad (3.2)
\end{aligned}
$$

The purely λ-dependent part of eq. (3.2) has been derived before in [29, 30], the full one-loop and two-loop results in eq. (3.1) are in agreement with [18–20] and in the limit $g_1, g_2, y_b, y_\tau \to 0$ we reproduce the result [4] at three-loop level.

The running of the m^2 parameter is given by

$$\begin{aligned}
\frac{\beta^{(1)}_{m^2}}{m^2} =& y_\tau^2 + y_b^2 3 - g_2^2 \frac{9}{4} - g_1^2 \frac{3}{4} + \lambda 6 + y_t^2 3, \\
\frac{\beta^{(2)}_{m^2}}{m^2} =& - y_\tau^4 \frac{9}{4} - y_b^4 \frac{27}{4} + g_2^2 y_\tau^2 \frac{15}{8} + g_2^2 y_b^2 \frac{45}{8} \\
& + g_2^4 \left(-\frac{385}{32} + \frac{5}{2} N_g \right) + g_1^2 y_\tau^2 \frac{25}{8} + g_1^2 y_b^2 \frac{25}{24} + g_1^2 g_2^2 \frac{15}{16} \\
& + g_1^4 \left(+\frac{157}{96} + \frac{25}{18} N_g \right) - \lambda y_\tau^2 12 - \lambda y_b^2 36 + \lambda g_2^2 36 + \lambda g_1^2 12 \\
& - \lambda^2 30 - y_t^2 y_b^2 \frac{21}{2} + y_t^2 g_2^2 \frac{45}{8} + y_t^2 g_1^2 \frac{85}{24} - y_t^2 \lambda 36 \\
& - y_t^4 \frac{27}{4} + g_s^2 y_b^2 20 + g_s^2 y_t^2 20,
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\frac{\beta^{(3)}_{m^2}}{m^2} =& y_\tau^6\left(-\frac{233}{16}+15\zeta_3\right)+y_b^2y_\tau^4 72+y_b^4y_\tau^2 72+y_b^6\left(\frac{1605}{16}+45\zeta_3\right)\\
&+g_2^2y_\tau^4\left(-\frac{987}{16}+54\zeta_3\right)-g_2^2y_b^2y_\tau^2\frac{27}{2}+g_2^2y_b^4\left(-\frac{3177}{16}+162\zeta_3\right)\\
&+g_2^4y_\tau^2\left(-\frac{255}{128}-\frac{81}{4}\zeta_3-\frac{21}{8}N_g\right)+g_2^4y_b^2\left(-\frac{765}{128}-\frac{243}{4}\zeta_3-\frac{63}{8}N_g\right)\\
&+g_2^6\left(-\frac{39415}{576}+\frac{711}{16}\zeta_3+\frac{2867}{72}N_g+45N_g\zeta_3+\frac{35}{9}N_g^2\right)+g_1^2y_\tau^4\left(\frac{291}{16}-36\zeta_3\right)\\
&-g_1^2y_b^2y_\tau^2\frac{9}{2}+g_1^2y_b^4\left(-\frac{1067}{16}+72\zeta_3\right)+g_1^2g_2^2y_\tau^2\left(-\frac{2331}{64}+45\zeta_3\right)-g_1^2g_2^2y_b^2\frac{865}{64}\\
&+g_1^2g_2^4\left(\frac{2691}{64}-\frac{405}{16}\zeta_3+\frac{21}{4}N_g-3N_g\zeta_3\right)+g_1^4y_\tau^2\left(-\frac{3607}{128}-\frac{15}{4}\zeta_3-\frac{65}{8}N_g\right)\\
&+g_1^4y_b^2\left(-\frac{79207}{3456}-\frac{35}{12}\zeta_3-\frac{155}{72}N_g\right)+g_1^4g_2^2\left(\frac{1053}{32}-\frac{207}{16}\zeta_3+\frac{55}{8}N_g-3N_g\zeta_3\right)\\
&+g_1^6\left(\frac{839}{108}-\frac{51}{16}\zeta_3+\frac{4375}{324}N_g-\frac{95}{9}N_g\zeta_3+\frac{875}{243}N_g^2\right)+\lambda y_\tau^4\left(\frac{261}{4}+72\zeta_3\right)\\
&-\lambda y_b^2y_\tau^2 108+\lambda y_b^4\left(\frac{351}{4}+216\zeta_3\right)+\lambda g_2^2y_\tau^2\left(\frac{189}{8}-108\zeta_3\right)+\lambda g_2^2y_b^2\left(\frac{567}{8}-324\zeta_3\right)\\
&+\lambda g_2^4\left(\frac{11511}{32}-162\zeta_3-\frac{153}{2}N_g\right)+\lambda g_1^2y_\tau^2\left(-\frac{549}{8}+36\zeta_3\right)+\lambda g_1^2y_b^2\left(\frac{393}{8}-132\zeta_3\right)\\
&+\lambda g_1^2g_2^2\left(-\frac{1701}{16}+36\zeta_3\right)+\lambda g_1^4\left(-\frac{1077}{32}-18\zeta_3-\frac{85}{2}N_g\right)+\lambda^2y_\tau^2\frac{99}{2}+\lambda^2y_b^2\frac{297}{2}\\
&+\lambda^2g_2^2\left(-63-108\zeta_3\right)+\lambda^2g_1^2\left(-21-36\zeta_3\right)+\lambda^3 1026+y_t^2y_\tau^4 72+y_t^2y_b^2y_\tau^2\frac{21}{2}\\
&+y_t^2y_b^4\left(\frac{4047}{16}+36\zeta_3\right)+y_t^2g_2^2y_\tau^2\left(-\frac{27}{2}\right)+y_t^2g_2^2y_b^2\left(-\frac{243}{8}-27\zeta_3\right)\\
&+y_t^2g_2^4\left(-\frac{765}{128}-\frac{243}{4}\zeta_3-\frac{63}{8}N_g\right)-y_t^2g_1^2y_\tau^2\frac{9}{2}+y_t^2g_1^2y_b^2\left(-\frac{929}{24}-\zeta_3\right)\\
&+y_t^2g_1^2g_2^2\left(-\frac{3277}{64}+\frac{117}{2}\zeta_3\right)+y_t^2g_1^4\left(-\frac{123103}{3456}-\frac{149}{12}\zeta_3-\frac{635}{72}N_g\right)-y_t^2\lambda y_\tau^2 108\\
&+y_t^2\lambda y_b^2\left(-\frac{315}{2}-216\zeta_3\right)+y_t^2\lambda g_2^2\left(\frac{567}{8}-324\zeta_3\right)+y_t^2\lambda g_1^2\left(-\frac{219}{8}-60\zeta_3\right)\\
&+y_t^2\lambda^2\frac{297}{2}+y_t^4y_\tau^2 72+y_t^4y_b^2\left(\frac{4047}{16}+36\zeta_3\right)+y_t^4g_2^2\left(-\frac{3177}{16}+162\zeta_3\right)\\
&+y_t^4g_1^2\left(-\frac{431}{16}+12\zeta_3\right)+y_t^4\lambda\left(\frac{351}{4}+216\zeta_3\right)+y_t^6\left(\frac{1605}{16}+45\zeta_3\right)\\
&+g_s^2y_b^4\left(\frac{447}{2}-360\zeta_3\right)+g_s^2g_2^2y_b^2\left(-\frac{489}{4}+108\zeta_3\right)+g_s^2g_2^4\left(\frac{135}{4}N_g-36N_g\zeta_3\right)\\
&+g_s^2g_1^2y_b^2\left(-\frac{991}{36}+20\zeta_3\right)+g_s^2g_1^4\left(\frac{55}{4}N_g-\frac{44}{3}N_g\zeta_3\right)+g_s^2\lambda y_b^2\left(-612+576\zeta_3\right)\\
&+g_s^2y_t^2y_b^2\left(41-48\zeta_3\right)+g_s^2y_t^2g_2^2\left(-\frac{489}{4}+108\zeta_3\right)+g_s^2y_t^2g_1^2\left(-\frac{2419}{36}+68\zeta_3\right)\\
&+g_s^2y_t^2\lambda\left(-612+576\zeta_3\right)+g_s^2y_t^4\left(\frac{447}{2}-360\zeta_3\right)\\
&+g_s^4y_b^2\left(\frac{910}{3}-24\zeta_3-32N_g\right)+g_s^4y_t^2\left(\frac{910}{3}-24\zeta_3-32N_g\right).
\end{aligned}
\tag{3.4}
$$

The one-loop and two-loop parts of this result are in agreement with [19, 20]. The purely λ-dependent part can be found in [29, 30]. and for $g_1, g_2, y_b, y_\tau \to 0$ we reproduce the result [4] again.

Now we want to give a numerical evaluation of the β-functions at the scale of the top mass in order to get an idea of the size of the new terms. For $M_t \approx 173.5$ GeV, $M_H \approx 126$ GeV and $\alpha_s = 0.1184$ [50] we get the couplings in the $\overline{\text{MS}}$-scheme at this scale using one-loop matching relations [51–53]:[2] $y_t(M_t) \approx 0.94$, $g_s(M_t) \approx 1.16$, $g_2(M_t) \approx 0.65$, $g_1(M_t) \approx 0.36$ and $\lambda(M_t) \approx 0.13$. The lighter Yukawa couplings can be estimated from the $\overline{\text{MS}}$-masses $m_b \approx 4.18$ GeV and $m_\tau \approx 1.777$ GeV [50] to be $y_b \approx \sqrt{2}\frac{m_b}{v} \approx 0.02$ and $y_\tau \approx \sqrt{2}\frac{m_\tau}{v} \approx 0.01$

For $\beta_\lambda(\mu = M_t)$ we find at one-loop order

$$\begin{aligned}\frac{\beta_\lambda^{(1)}(\mu = M_t)}{(16\pi^2)} = &\underbrace{-1.5 \times 10^{-2}}_{y_t^4} \underbrace{+4.4 \times 10^{-3}}_{\lambda y_t^2} \underbrace{-1.6 \times 10^{-3}}_{g_2^2\lambda} \underbrace{+1.3 \times 10^{-3}}_{\lambda^2} \\ &\underbrace{+6.3 \times 10^{-4}}_{g_2^4} \underbrace{-1.6 \times 10^{-4}}_{\lambda g_1^2} \underbrace{+1.3 \times 10^{-4}}_{g_2^2 g_1^2} \underbrace{2. \times 10^{-5}}_{g_1^4} \\ &\underbrace{+2. \times 10^{-6}}_{\lambda y_b^2} \underbrace{+1.6 \times 10^{-7}}_{\lambda y_\tau^2} \underbrace{-3. \times 10^{-9}}_{y_b^4} \underbrace{-6.3 \times 10^{-11}}_{y_\tau^4} .\end{aligned} \tag{3.5}$$

At two-loop order the largest terms with y_b are of $\mathcal{O}(10^{-7})$ and those with y_τ of $\mathcal{O}(10^{-9})$. Neglecting these small terms we find

$$\begin{aligned}\frac{\beta_\lambda^{(2)}(\mu = M_t)}{(16\pi^2)^2} = &\underbrace{-6.8 \times 10^{-4}}_{g_s^2 y_t^4} \underbrace{+4.1 \times 10^{-4}}_{y_t^6} \underbrace{+2.5 \times 10^{-4}}_{g_s^2 \lambda y_t^2} \underbrace{-4.3 \times 10^{-5}}_{\lambda^2 y_t^2} \\ &\underbrace{+2.2 \times 10^{-5}}_{g_2^2 \lambda y_t^2} \underbrace{+1.5 \times 10^{-5}}_{g_2^2 \lambda^2} \underbrace{-1.4 \times 10^{-5}}_{\lambda^3} \underbrace{+1.1 \times 10^{-5}}_{g_2^6} \\ &\underbrace{+1. \times 10^{-5}}_{g_2^2 y_t^2 g_1^2} \underbrace{+9.6 \times 10^{-6}}_{g_2^4 \lambda} \underbrace{-7. \times 10^{-6}}_{g_2^4 y_t^2} \underbrace{-6.1 \times 10^{-6}}_{\lambda y_t^4} \\ &\underbrace{-5.4 \times 10^{-6}}_{y_t^4 g_1^2} \underbrace{-4.6 \times 10^{-6}}_{g_2^4 g_1^2} \underbrace{+4.2 \times 10^{-6}}_{\lambda y_t^2 g_1^2} \underbrace{-2.6 \times 10^{-6}}_{g_2^2 g_1^4} \\ &\underbrace{+1.9 \times 10^{-6}}_{\lambda g_1^4} \underbrace{+1.6 \times 10^{-6}}_{\lambda^2 g_1^2} \underbrace{-1.4 \times 10^{-6}}_{y_t^2 g_1^4} \underbrace{+1.4 \times 10^{-6}}_{g_2^2 \lambda g_1^2} \\ &\underbrace{-6.2 \times 10^{-7}}_{g_1^6} + (\text{terms} \propto y_b^2, y_\tau^2) .\end{aligned} \tag{3.6}$$

At three-loop level we only give the largest terms and omit small ones of $\mathcal{O}(10^{-7})$:

[2] This has been done with the Mathematica package SMPoleMatching.m by F. Bezrukov which can be downloaded at `http://www.inr.ac.ru/~fedor/SM/download.php`

$$\begin{aligned}\frac{\beta_\lambda^{(3)}(\mu=M_t)}{(16\pi^2)^3} =& \underbrace{5.9\times10^{-5}}_{g_s^2y_t^6}\underbrace{-3.8\times10^{-5}}_{y_t^8}\underbrace{+2.5\times10^{-5}}_{g_s^4y_t^4}\underbrace{-2.3\times10^{-5}}_{g_s^2\lambda y_t^4}\\ &\underbrace{+8.8\times10^{-6}}_{g_s^4\lambda y_t^2}\underbrace{+5.9\times10^{-6}}_{\lambda^2y_t^4}\underbrace{+5.5\times10^{-6}}_{g_2^2y_t^6}\underbrace{-5.1\times10^{-6}}_{\lambda y_t^6}\\ &\underbrace{-4.4\times10^{-6}}_{g_2^8}\underbrace{+4.2\times10^{-6}}_{g_2^6\lambda}\underbrace{-1.9\times10^{-6}}_{g_2^4\lambda y_t^2}\underbrace{+1.7\times10^{-6}}_{g_2^6y_t^2}\\ &\underbrace{+1.5\times10^{-6}}_{g_2^2g_s^2y_t^4}\underbrace{+1.3\times10^{-6}}_{y_t^6g_1^2}\underbrace{-1.3\times10^{-6}}_{g_2^2y_t^4g_1^2}\underbrace{+1.\times10^{-6}}_{g_s^2y_t^4g_1^2}\\ &\underbrace{-9.\times10^{-7}}_{g_2^4g_s^2\lambda}\underbrace{-9.2\times10^{-7}}_{g_2^4\lambda^2}\underbrace{+8.9\times10^{-7}}_{g_2^4g_s^2y_t^2}\underbrace{+8.3\times10^{-7}}_{g_s^2\lambda^2y_t^2}+\text{smaller terms}\,.\end{aligned} \tag{3.7}$$

The dominant contributions contain only g_s, y_t and λ as suggested in [4]. This result therefore explicitly confirms the validity of the approximation made in [4] at the scale of the top mass. On the other hand some terms proportional to powers of g_2^2 are of the same numerical significance as some of the electroweak two-loop contributions and in including them we reduce the theoretical uncertainty in the evolution of λ stemming from β_λ even further. The overall value of the three-loop β-function at this scale $\frac{\beta_\lambda^{(3)}(\mu=M_t)}{(16\pi^2)^3} = 4.2\times10^{-5}$ is about 17% larger than with the electroweak contributions neglected ($\left.\frac{\beta_\lambda^{(3)}(\mu=M_t)}{(16\pi^2)^3}\right|_{g_2,g_1\to0} = 3.6\times10^{-5}$). The whole β-function is dominated by the negative one-loop term $\propto y_t^4$ but with the new terms presented here λ will decrease less which means that these contributions slightly enhance the stability of the electroweak vacuum state in the SM.

For $\beta_{m^2}(\mu=M_t)$ we find at one-loop order

$$\begin{aligned}\frac{\beta_{m^2}^{(1)}(\mu=M_t)}{(16\pi^2)m^2} =& \underbrace{1.7\times10^{-2}}_{y_t^2}\underbrace{-6.\times10^{-3}}_{g_2^2}\underbrace{+4.9\times10^{-3}}_{\lambda}\underbrace{-6.1\times10^{-4}}_{g_1^2}\\ &\underbrace{+7.6\times10^{-6}}_{y_b^2}\underbrace{+6.3\times10^{-7}}_{y_\tau^2}\,.\end{aligned} \tag{3.8}$$

Omitting the small contributions proportional to y_b and y_τ the two-loop terms look like this:

$$\begin{aligned}\frac{\beta_{m^2}^{(2)}(\mu=M_t)}{(16\pi^2)^2m^2} =& \underbrace{9.6\times10^{-4}}_{g_s^2y_t^2}\underbrace{-2.1\times10^{-4}}_{y_t^4}\underbrace{-1.7\times10^{-4}}_{\lambda y_t^2}\\ &\underbrace{+8.4\times10^{-5}}_{g_2^2y_t^2}\underbrace{+7.9\times10^{-5}}_{g_2^2\lambda}\underbrace{+2.1\times10^{-5}}_{g_2^4}\underbrace{+1.6\times10^{-5}}_{y_t^2g_1^2}\underbrace{-2.\times10^{-5}}_{\lambda^2}\\ &\underbrace{+8.\times10^{-6}}_{\lambda g_1^2}\underbrace{+6.6\times10^{-6}}_{g_1^4}\underbrace{+2.\times10^{-6}}_{g_2^2g_1^2}+(\text{terms}\propto y_b^2,y_\tau^2)\,.\end{aligned} \tag{3.9}$$

Again we only give the largest terms at three-loop level and omit small ones of $\mathcal{O}(10^{-7})$:

$$\begin{aligned}\frac{\beta^{(3)}_{m^2}(\mu = M_t)}{(16\pi^2)^3 m^2} = &\underbrace{-5.6 \times 10^{-5}}_{g_s^2 y_t^4} \underbrace{+3.4 \times 10^{-5}}_{g_s^4 y_t^2} \underbrace{+2.7 \times 10^{-5}}_{y_t^6} \underbrace{+1.3 \times 10^{-5}}_{g_2^6} \\ &\underbrace{+9. \times 10^{-6}}_{\lambda y_t^4} \underbrace{-5. \times 10^{-6}}_{g_2^4 y_t^2} \underbrace{-3.9 \times 10^{-6}}_{g_2^2 \lambda y_t^2} \underbrace{-3.5 \times 10^{-6}}_{g_2^4 g_s^2} \underbrace{+3.2 \times 10^{-6}}_{g_s^2 \lambda y_t^2} \underbrace{-1.7 \times 10^{-6}}_{g_2^4 \lambda} \\ &\underbrace{+9.7 \times 10^{-7}}_{g_2^2 g_s^2 y_t^2} \underbrace{+5.7 \times 10^{-7}}_{\lambda^3} \underbrace{+5.7 \times 10^{-7}}_{g_s^2 y_t^2 g_1^2} \underbrace{+5.6 \times 10^{-7}}_{\lambda^2 y_t^2} + \text{smaller terms}\,.\end{aligned} \tag{3.10}$$

We see that especially the term $\propto g_2^6$ is not much smaller than the three dominant terms that have already been computed in [4]. The overall three-loop results with $g_2, g_1 \to 0$ and the electroweak interaction switched on differ only by $1. \times 10^{-6}$ due to cancellations between the new terms.

4 Conclusions

We have computed the three-loop β-functions for the quartic Higgs self-coupling and for the mass parameter m^2 in the unbroken phase of the SM, neglecting only the Yukawa couplings of the first two generations and the mixing of quark generations (an extension that could easily be made but which is numerically negligible). The electroweak contributions are small as expected which confirms the validity of the approximation made in our previous calculation [4]. The full three-loop β-function for the quartic Higgs self-interaction is about 17% larger than without the electroweak terms thus making the SM vacuum state slightly more stable.

Note added: Recently, a similar calculation has independently confirmed the results presented in this paper [54].

Acknowledgements

We thank Johann Kühn for useful discussions and support.

Finally we want to mention that all our calculations have been performed using the thread-based [55] version of FORM [56]. The Feynman diagrams have been drawn with the Latex package Axodraw [57].

This work has been supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics". M.Z. has been supported by the Graduiertenkolleg "Elementarteilchenphysik bei höchsten Energien und höchster Präzission".

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