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On the $\mathcal{O}(\alpha_s^2)$ corrections to $b \to X_u e\bar{\nu}$ inclusive decays

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Abstract

We present $\mathcal{O}(\alpha_s^2)$ QCD corrections to the fully-differential decay rate of a b-quark into inclusive semileptonic charmless final states. Our calculation provides genuine two-loop QCD corrections, beyond the Brodsky-Lepage-Mackenzie (BLM) approximation, to *any* infra-red safe partonic observable that can be probed in $b \to X_u e\bar{\nu}$ decays. Kinematic cuts that closely match those used in experiments can be fully accounted for. To illustrate these points, we compute the non-BLM corrections to moments of the hadronic invariant mass and the hadronic energy with cuts on the lepton energy and the hadronic invariant mass. Our results remove one of the sources of theoretical uncertainty that affect the extraction of the CKM matrix element $|V_{ub}|$ from charmless inclusive B-decays.

Studies of CP violation in B-mesons performed by BELLE and BABAR, firmly established the correctness of the Cabbibo-Kobayashi-Maskawa paradigm at the few percent level. These studies will be continued, when the super-B factory in Japan will come on line. A powerful tool to test the CKM picture is the unitarity of the CKM-matrix. A combined fit to all available data gives $|V_{ub}| = 3.58(13) \times 10^{-3}$ [1]. This number should be compared with the value $|V_{ub}| = 3.38(36) \times 10^{-3}$ extracted from exclusive $B \to he\bar{\nu}$ decays, where the hadron h is either a pion or a ρ-meson, and with $|V_{ub}| = 4.27(38) \times 10^{-3}$ which is obtained from *inclusive* measurements of $B \to X_u e\bar{\nu}$ decays [2]. Although exclusive and inclusive results are not in serious disagreement, they are clearly different and further scrutiny of both exclusive and inclusive determinations of $|V_{ub}|$ is certainly warranted.

The most complicated theoretical issue for both exclusive and inclusive methods is the control of non-perturbative effects. This is hard to do for exclusive decays and important input in this case is provided by *ab initio* lattice QCD calculations of exclusive $B \to \pi, \rho, ..$ transitions. In contrast, in case of *inclusive* semileptonic decays of B-mesons, non-perturbative difficulties can be largely circumvented by the application of local operator product expansion (OPE) [3, 4, 5]. The OPE allows to compute sufficiently inclusive observables related to semileptonic decays of B-mesons, such as the total rate and moments of various kinematic distributions, by correcting distributions and rates of semileptonic decays of b-quarks with a limited number of universal non-perturbative parameters. These non-perturbative parameters can be determined from fits to semileptonic decays of B-mesons to charmed final states $B \to X_c e\bar{\nu}$ [6, 7, 8, 9, 10, 11] and then used in the description of $B \to X_u e\bar{\nu}$ transitions, facilitating the extraction of the CKM matrix element $|V_{ub}|$ from observables in the latter.

While this procedure is well-defined theoretically, it was not used in the determination of $|V_{ub}|$ right away because $B \to X_u e\bar{\nu}$ transitions suffer from a much larger $B \to X_c e\bar{\nu}$ background. One can place severe cuts on the kinematics of final state particles to suppress it; for example, requiring that the hadronic invariant mass is smaller than the mass of the D-meson, $m_D \sim 1.87$ GeV, clearly eliminates the charm background. However, it was realized early on that such cuts lead to problems with the convergence of the operator product expansion and infinitely many terms in the OPE need to be summed up to obtain reliable results. Such a resummation is usually expressed through the so-called shape function [12, 13] which parametrizes the residual motion of a heavy quark inside a heavy meson. A recent discussion of $B \to X_u e\bar{\nu}$ decay in the shape-function region, that includes next-to-next-to-leading order (NNLO) QCD effects, can be found in Ref. [14]. Unfortunately, current uncertainties in the functional form of both leading and sub-leading shape-functions are significant and affect a precise determination of $|V_{ub}|$.

In parallel to the studies of the shape function region, it was suggested that a combination of cuts on hadronic and leptonic invariant masses [15] allows one to extend the phase-space coverage in $B \to X_u e\bar{\nu}$ decays and make the impact of the shape functions smaller. Measurements that use these selection criteria were performed by the BELLE collaboration [16]. Further advances in experimental techniques allowed to achieve an almost complete phase-space coverage in $B \to X_u e\bar{\nu}$ decays. Indeed, in recent experimental measurements it was possible to fully reconstruct the $B\bar{B}$ kinematics from their decay products, thereby allowing to extend selection cuts for the $b \to u$ process into the charm-rich regions and yet, successfully reject the $b \to X_c e\bar{\nu}$ background. For example, two recent measurements by BELLE [17] and BABAR [18] present partial decay rates and a variety of kinematic distributions for $b \to u$ transitions with the cut on the electron energy as low as $E_l > 1$ GeV. These cuts are inclusive enough so that the local OPE expansion can be used with confidence to describe $B \to X_u e\bar{\nu}$ decays.

We summarize now the status of the theoretical description of $B \to X_u e\bar{\nu}$ decays, under the assumption that the local OPE is applicable. The OPE expansion in the inverse b-quark mass m_b is well-established for moments of the hadronic invariant mass and the hadronic energy [3, 4, 5]. The leading order term in the OPE expansion is given by the partonic $b \to u$ transition. The total decay rate for $b \to u$ is known in perturbative QCD through $\mathcal{O}(\alpha_s^2)$ [19] and a large number of kinematic distributions and their moments are known through $\mathcal{O}(\alpha_s)$ [20, 21, 22, 23]. Also, the so-called BLM $\mathcal{O}(\beta_0\alpha_s^2)$ corrections [24], that can be derived by considering the contribution of a massless $q\bar{q}$ pair to the $b \to u$ transition, are known for the decay rate and main kinematic distributions [25, 26, 27]. The only kinematic distribution in $b \to u$ decays that is known beyond the BLM approximation is the electron-neutrino invariant mass distribution, computed in Ref. [28]. While the BLM-approximation is known to account for a significant fraction of the complete $\mathcal{O}(\alpha_s^2)$ correction, the precision of current and, especially, forthcoming measurements of $|V_{ub}|$, the relatively large value of $\alpha_s(m_b)$ and a large variety of kinematic cuts employed in experimental analyses make it very desirable to compute NNLO QCD corrections to the fully-differential $b \to u$ decay rate beyond the BLM approximation. The goal of this paper is to provide such a computation.

The calculation of NNLO QCD corrections to the $b \to ue\bar{\nu}$ decay requires three ingredients: *i*) two-loop amplitudes for the $b \to ue\bar{\nu}$ transition; *ii*) one-loop amplitudes for $b \to uge\bar{\nu}$; *iii*) tree amplitudes for $b \to ugge\bar{\nu}$ and $b \to uq\bar{q}e\bar{\nu}$. The two-loop amplitudes were computed by several authors in recent years [29, 30, 31, 32]. The one-loop amplitudes for $b \to uge\bar{\nu}$ can be extracted from the computation reported in Ref. [33]. Finally, the tree amplitudes for $b \to ugge\bar{\nu}$ and $b \to uq\bar{q}e\bar{\nu}$ are straightforward to calculate and compact results can be obtained by using the spinor-helicity formalism. These amplitudes can be found in Ref. [34].

The well-known challenge for fully-differential NNLO QCD computations is to put these different contributions together in a consistent way. This is not easy to do since individual contributions exhibit infra-red and collinear divergences and correspond to processes with different final-state multiplicities. For the computation reported in this paper, we use a method proposed in Refs. [35, 36] (see also [37]) which combines the idea of sector decomposition [38, 39, 40] with the phase-space partitioning [41] in such a way that singularities are extracted from matrix elements in a process-independent way. This framework leads to a parton level integrator which can be used to compute an arbitrary number of kinematic distributions in a single run of the program. We have recently given a detailed description of the relevant computational techniques in a paper [34] that describes a calculation of NNLO QCD corrections to a related process $t \to be^+\nu$ and so we do not repeat it here. Instead, we focus on the illustration of phenomenological capabilities of the program that are relevant for the description of $b \to u$ transitions.

Numerical results reported below are obtained within the standard framework for perturbative QCD computations. We employ the on-shell renormalization for the b-quark field and the b-quark mass. The strong coupling constant is renormalized in the $\overline{\rm MS}$-scheme. We note that we do not include the charm mass

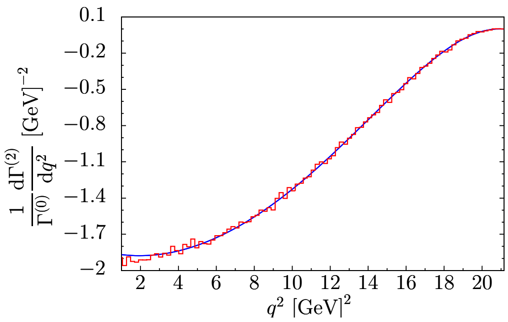

Figure 1: The coefficient of the second order correction to the lepton invariant mass distribution. The solid curve is the analytic result of Ref. [28].

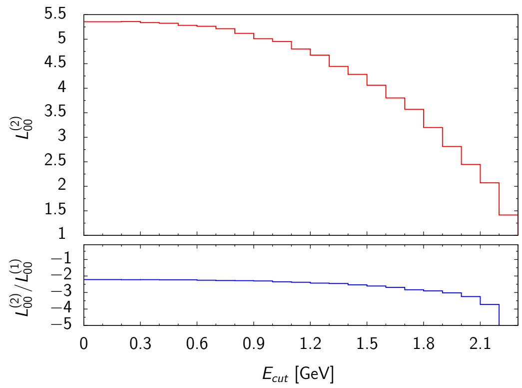

Figure 2: The cumulative histogram that shows $L_{00}^{(2)}$ as a function of the cut on the charged lepton energy $E_l > E_{\rm cut}$. No cut on the hadronic invariant mass is applied.

dependence when we compute contributions of additional $q\bar{q}$ pairs to the decay rate. As was explicitly shown in Ref. [30], contributions of virtual charm loops for physical value of m_c can be obtained, with a good accuracy, from the bottom quark loops by equating charm and bottom masses. We will use this recipe in what follows. We write the differential decay rate for $b \to X_u e\bar{\nu}$ through NNLO in perturbative QCD as

$$\mathrm{d}\Gamma = \mathrm{d}\Gamma^{(0)} + a_s \mathrm{d}\Gamma^{(1)} + a_s^2 \mathrm{d}\Gamma^{(2)} + \mathcal{O}(\alpha_s^3), \quad (1)$$

where $a_s = \alpha_s/\pi$ and α_s is the $\overline{\rm MS}$ strong coupling constant at the scale $\mu = m_b$. By integrating the fully differential decay rate over all the available phase-space for final state particles, we obtain a prediction for the $\mathcal{O}(\alpha_s^2)$ correction to the total decay rate. We write the result of our numerical integration in the following way

$$\Gamma^{(2)} = \Gamma^{(0)}\left(-29.98(8) + 2.143(7)N_f \right. \\ \left. -0.0243 N_h\right), \quad (2)$$

where $N_f = 3$ denotes the number of massless quarks in the theory and $N_h = 2$ denotes the number of quarks whose mass coincides with the b-quark mass. Also, $\Gamma_b^{(0)} = G_F^2 |V_{ub}|^2 m_b^5/(192\pi^3)$ is the total decay rate for $b \to u e\bar{\nu}$ at leading order in perturbative QCD. Comparing our computation to the analytic results presented in Ref. [19], we find agreement for each term shown in Eq.(2) to better than five per mille.

Having reproduced the known result for the NNLO QCD corrections to the total rate, we can now proceed to the discussion of kinematic distributions. Our numerical program is set up in such a way that it can compute various kinematic distributions, both conventional and cumulative, in a single run. To illustrate this, we show in Fig. 1 $\mathrm{d}\Gamma^{(2)}/\mathrm{d}q^2$, where q^2 is the invariant mass of the lepton pair. The solid curve is the result of the analytic calculation from Ref. [28]. The numerical and analytical results perfectly agree for all values of q^2 except in the region $q^2 \sim 0$ where some discrepancy is observed. This discrepancy is not surprising since the analytic results of Ref. [28] were obtained as an expansion around $q^2 = m_b^2$ so that deviations at small q^2 reflect convergence problems of the analytic computation in that region.

To further discuss kinematic distributions, we follow Ref. [20] and define moments of the partonic invariant mass $M_X^2 = (p_b - p_e - p_\nu)^2$ and energy $E_X = E_b - E_e - E_\nu$, in dependence of the lower cut on the electron energy $E_{\rm cut}$ and the upper cut on the partonic invariant mass $M_{\rm cut}$. More specifically, we write

$$L_{ij} = \langle M_X^{2i} E_X^j \theta(E_e - E_{\rm cut})\theta(M_{\rm cut} - M_X)\rangle \quad (3)$$

where $\langle ...\rangle$ denotes the normalized phase-space average for final-state particles in $b \to X_u e\bar{\nu}$

$$\langle \mathcal{F}\rangle \equiv \frac{1}{\Gamma^{(0)}} \int \mathrm{d}\Gamma \mathcal{F}. \quad (4)$$

We note [20] that one can use L_{ij}'s defined in Eq.(3) to obtain moments of the lepton invariant mass q^2.

We write the moments in Eq.(3) as an expansion in the strong coupling constant and explicitly separate the BLM corrections

$$L_{ij} = L_{ij}^{(0)} + a_s L_{ij}^{(1)} + a_s^2 \left(L_{ij}^{(2),\rm BLM} + L_{ij}^{(2)}\right). \quad (5)$$

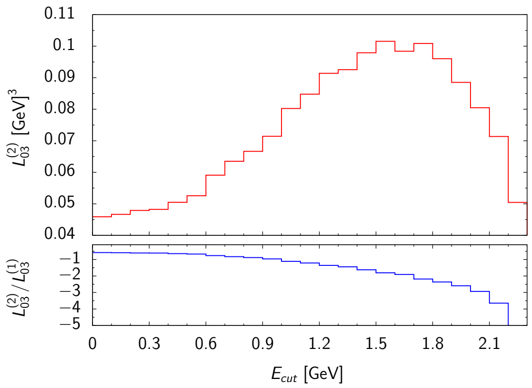

Figure 3: The cumulative histogram that shows $L^{(2)}_{03}$ as a function of the cut on the charged lepton energy $E_l > E_{\rm cut}$. No cut on the hadronic invariant mass is applied.

The BLM correction to these moments is obtained by computing the contributions of a massless $q\bar{q}$ pair $L^{(2),n_f}_{ij}$ and then by rescaling it by the full leading order QCD β-function for three massless flavors

$$L^{(2),{\rm BLM}}_{ij} = -27/2\, L^{(2),n_f=1}_{ij}. \qquad (6)$$

For the numerical calculation of the moments reported below, we use $m_b = 4.6$ GeV as the value of the b-quark pole mass. To specify partonic cuts, we introduce the physical hadronic invariant mass

$$M_H^2 = \bar{\Lambda}^2 + 2m_b\bar{\Lambda}E_X + m_b^2 M_X^2, \qquad (7)$$

where $\bar{\Lambda} = m_{B^\pm} - m_b = 0.6769$ GeV, for our choice of the b-quark mass. We impose a cut on M_H and translate it into a cut on M_X and E_X using Eq.(7). Throughout the paper, we use $M_H < 2.5$ GeV as the cut on the hadronic invariant mass. Also, as we already mentioned, the BLM corrections are well-known. They were discussed previously in the literature (see e.g. Ref. [20]) and, for this reason, we focus on non-BLM corrections in the remainder of this paper.

Our results for the moments are presented in Figs. 2,3 and in Table 1. In Figs. 2,3 we show cumulative histograms for the non-BLM contributions to two moments $L^{(2)}_{0j}$ for $j = 0$ and $j = 3$, with no cut on hadronic invariant mass. In both cases, the x-axis shows the applied cut on the the charged lepton energy. These figures illustrate that our numerical program works as a parton level Monte Carlo integrator and that it can reliably compute large number of infrared safe observables for the $b \to X_u e\bar{\nu}$ decay with various cuts in a *single run*. This should be useful for further studies of charmless decays of B-mesons given, in particular, a large number of kinematic cuts employed in experimental analyses.[1]

To illustrate the dependence of the non-BLM corrections on the applied cuts, we show the ratio of non-BLM contributions $L^{(2)}$ to NLO ones $L^{(1)}$, as a function of the electron energy cut in lower panes of Figs. 2,3. We note that this ratio is renormalization-scale independent. We are interested in this ratio because, if it is independent of $E_{\rm cut}$, we could have found the corrections to the moments without fully-differential NNLO computations. However, it is apparent from Figs. 2,3 that this is not possible and that non-BLM corrections have a different functional dependence on $E_{\rm cut}$ as compared to the NLO ones. In addition, the cut-dependence is strongly moment-dependent and it is more pronounced for higher-j moments.

We will now take a closer look at the numerical values of the computed corrections. To facilitate this, we show in Table 1 our results for leading, next-to-leading and next-to-next-to-leading order *partonic* moments L_{ij} computed with the lepton energy cut of 1 GeV and the hadronic energy cut $M_H < 2.5$ GeV. This set of cuts was previously studied in Ref. [20].

It follows from Table 1 that similar to the total rate BLM and non-BLM corrections have opposite size, so that the total result for NNLO corrections is *smaller* than the BLM corrections taken alone. The non-BLM corrections seem to be more important for lower-moments than for higher moments. Indeed, the ratio $L^{(2)}_{ij}/L^{2,{\rm BLM}}_{ij}$ decreases monotonically by about a factor of 1.6, from 0.1842 to 0.112, for $i = 0$ and j changing from $j = 0$ to $j = 3$. This trend is also visible in the absolute magnitude of the corrections. Taking $\alpha_s(m_b) = 0.24$, we find that for $i = 0, j = 0$, the non-BLM corrections increase the moment by about three percent while for $i = 0, j = 3$, they become as small as one percent.

While these corrections look small compared to the current $\mathcal{O}(10\%)$ uncertainty in the $|V_{ub}|$ determined from inclusive decays, we note that Ref. [42] estimates the total theoretical uncer-

[1]We also note that our numerical program is rather fast. For example, all numerical results reported in this paper, including distributions shown in Figs. 1,2,3 and in Table 1 were obtained in an overnight run on a modest-size computer cluster.

i	j	$L_{ij}^{(0)}$	$L_{ij}^{(1)}$	$L_{ij}^{(2,\mathrm{BLM})}$	$L_{ij}^{(2)}$
0	0	0.87135	-2.261(4)	-27.7(1)	5.1(1)
0	1	0.29306	-0.738(2)	-8.13(1)	1.38(2)
0	2	0.10789	-0.2558(8)	-2.55(1)	0.38(1)
0	3	0.04210	-0.0920(4)	-0.815(2)	0.091(6)
1	0	0.0	0.13110(7)	2.231(1)	-0.638(3)
1	1	0.0	0.05265(3)	0.882(1)	-0.256(1)
1	2	0.0	0.02207(2)	0.365(1)	-0.106(1)
2	0	0.0	$4.973(4) \cdot 10^{-3}$	$6.83(1) \cdot 10^{-2}$	$-9.8(1) \cdot 10^{-3}$
2	1	0.0	$2.144(2) \cdot 10^{-3}$	$2.93(1) \cdot 10^{-2}$	$-4.3(1) \cdot 10^{-3}$
3	0	0.0	$3.452(7) \cdot 10^{-4}$	$4.41(1) \cdot 10^{-3}$	$-4.9(1) \cdot 10^{-4}$

Table 1: Moments of the partonic invariant mass M_X^2 and the partonic energy E_X with the hadronic invariant mass cut $M_H < 2.5$ GeV and the charged lepton energy cut $E_l < 1$ GeV. See text for details.

tainty on $|V_{ub}|$ that can be achieved with various kinematic cuts on M_H, E_e, and q^2 to be close to six percent. The uncertainty in perturbative corrections, which mainly refers to non-BLM $\mathcal{O}(\alpha_s^2)$ effects that we discuss in this paper, is believed [42] to be responsible for 30 to 50% of the full theory uncertainty. Our calculation allows to remove this part of the theory uncertainty by providing explicit results for non-BLM corrections.

For example, one of the scenarios considered in Ref. [42] is a high-cut on the lepton energy $E_l > 2$ GeV; it corresponds to the measurement by the BABAR collaboration reported in Ref. [43]. We find the non-BLM correction to $L_{00}(E_l > 2 \text{ GeV})$ using the cumulative histogram in Fig. 2 and observe that it changes the leading order moment $L_{00}^{(0)}(E_l > 2 \text{ GeV}) = 0.257$ by 6%.[2] Since the experimental measurement corresponds to $|V_{ub}|^2 L_{00}$, a 6% shift in L_{00} due to non-BLM corrections translates into a -3% shift in V_{ub}. We stress that the above number is given to illustrate the magnitude of the expected effect; a precise statement about the impact of non-BLM corrections requires a dedicated analysis along the lines of Ref. [42]. However, it is clear that our computation should help in removing a significant fraction of the full theory error in $|V_{ub}|$ as estimated in [42] for the $E_l > 2$ GeV cut.

We also note that it is customary to consider *normalized* moments, which are defined as $C_{ij} = L_{ij}/L_{00}$. Since both the numerator and the denominator in the definition of C_{ij} receive perturbative corrections, we need to consistently expand C_{ij} in a series in α_s to establish how stable it is against radiative corrections. We find that in case of C_{0j}, the non-BLM corrections are close to one-fifth of the BLM corrections for all values of j and they change the normalized moment by -0.5% for $j = 1$ and by -1.45% for $j = 3$.

The situation changes dramatically for partonic invariant mass moments L_{ij}, with $i \neq 0$. In this case the leading order partonic moments vanish since in the $b \to u e^- \bar{\nu}$ process the partonic invariant mass is zero. As the result, for these moments our NNLO calculation is, essentially, next-to-leading order and the significance of non-BLM corrections increases. As follows from Table 1 for L_{ij} moments with $i \neq 0$ and $j \neq 0$, the non-BLM corrections can be as large as 30%.

To conclude, we presented a computation of $\mathcal{O}(\alpha_s^2)$ corrections to the fully-differential decay rate of charmless semileptonic b decay, $b \to X_u e \bar{\nu}$. Our calculation provides a NNLO QCD description of arbitrary infra-red safe observables and allows arbitrary kinematic cuts including those that closely match the ones employed in experimental analyses. We constructed a parton-level Monte-Carlo integrator which can be used to compute large number of relevant observables and kinematic distributions in a single run of the program. This calculation, together with earlier results on NNLO QCD corrections to fully-differential $b \to c l \bar{\nu}$ transition [44, 45], makes all inclusive semileptonic decays of b-quarks upgraded to that accuracy.

[2]For comparison, we note that the corresponding BLM corrections to $L_{00}^{(0)}$ is -30%.

We hope that these results will contribute to the reduction of the theoretical error on $|V_{ub}|$ and $|V_{cb}|$ that will be achieved in the forthcoming B-physics experiments.

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