

# Four-loop corrections with two closed fermion loops to fermion self energies and the lepton anomalous magnetic moment

Roman Lee<sup>a</sup>, Peter Marquard<sup>b</sup>, Alexander V. Smirnov<sup>c</sup>,  
Vladimir A. Smirnov<sup>d,e</sup>, Matthias Steinhauser<sup>b</sup>

*(a) Budker Institute of Nuclear Physics and Novosibirsk State University  
630090 Novosibirsk, Russia*

*(b) Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT)  
76128 Karlsruhe, Germany*

*(c) Scientific Research Computing Center, Moscow State University  
119992 Moscow, Russia*

*(d) Skobeltsyn Institute of Nuclear Physics, Moscow State University  
119992 Moscow, Russia*

*(e) Institut für Mathematik, Humboldt-Universität zu Berlin  
12489 Berlin, Germany*

## Abstract

We compute the eighth-order fermionic corrections involving two and three closed massless fermion loops to the anomalous magnetic moment of the muon. The required four-loop on-shell integrals are classified and explicit analytical results for the master integrals are presented. As further applications we compute the corresponding four-loop QCD corrections to the mass and wave function renormalization constants for a massive quark in the on-shell scheme.

PACS numbers: 12.20.Ds 12.38.Bx 14.65.-q

# 1 Introduction

In the last about ten years several groups have been active in computing four-loop corrections to various physical quantities. Among them are the order  $\alpha_s^4$  corrections to the  $R$  ratio and the Higgs decay into bottom quarks [1–3], four-loop corrections to moments of the photon polarization function [4–8] which lead to precise results for the charm and bottom quark masses (see, e.g., Ref. [9]), and the free energy density of QCD at high temperatures [10]. The integrals involved in such calculations are either four-loop massless two-point functions or four-loop vacuum integrals with one non-vanishing mass scale. In this paper we take the first steps towards the systematic study of a further class of four-loop single-scale integrals, the so-called on-shell integrals where in the loop massless and massive propagators may be present and the only external momentum is on the mass shell.

On-shell integrals enter a variety of physical quantities, where the anomalous magnetic moments and on-shell counterterms are prominent examples. The first systematic study of two-loop on-shell integrals needed for the evaluation of the on-shell mass and wave function renormalization constants ( $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$ ) for a heavy quark in QCD has been performed in Refs. [11, 12]. Already a few years later, in 1996 the analytical three-loop corrections to the lepton anomalous magnetic moment  $a_l$  became available [13]. This result has been checked in Refs. [14, 15]. In Refs. [14, 16] the three-loop on-shell integrals have been applied to QCD, namely the evaluation of  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$ . The calculation of Ref. [14] has confirmed the numerical result of [17, 18] which has been available before. Both  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$  have also been computed in Ref. [15]. Further application of three-loop on-shell integrals are discussed in Refs. [19, 20]. There is no systematic study of four-loop on-shell integrals available in the literature. Nevertheless, some four-loop results to the anomalous magnetic moment of the muon,  $a_\mu$ , have been computed analytically, in particular contributions from closed electron loops. E.g., the contribution where the photon propagator of the one-loop diagram (see Fig. 1) is dressed by higher order corrections has been considered in several papers [21–27]. Four-loop corrections where one of the two photon propagators of the two-loop diagram is dressed by higher orders has been considered in Ref. [28, 29]. Contributions where both photon propagators get one-loop electron insertions are still missing. This gap will be closed in the present work. Let us mention that all four- and even five-loop results for  $a_l$  are available in the literature in numerical form [27, 30–33] (see also the review articles [34, 35]).

In this paper we take the first step towards the analytical calculation of four-loop on-shell integrals by considering the subclass with two or three closed massless fermion loops, which are marked by a factor  $n_l$ . Thus we are concerned with four-loop terms proportional to  $n_l^3$  and  $n_l^2$  which we consider for three physical quantities: the anomalous magnetic moment of the muon,  $a_\mu$ , the on-shell mass renormalization constant,  $Z_m^{\text{OS}}$ , and the on-shell wave function renormalization constant,  $Z_2^{\text{OS}}$ , for a massive quark. For the latter QCD corrections to the quark two-point functions are computed whereas for the former muon-photon vertex diagrams have to be considered. Some sample Feynman diagrams are

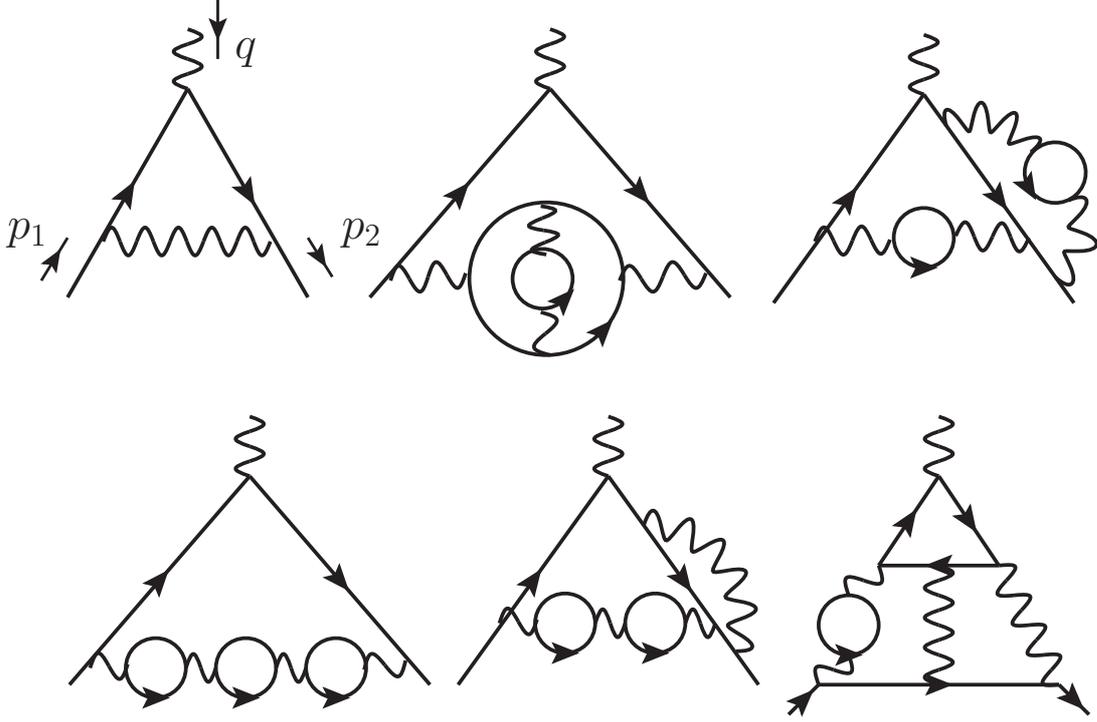


Figure 1: Sample Feynman diagrams for the photon-muon vertex contributing to  $a_\mu$ . Wavy and straight lines represent photons and fermions, respectively. In this paper we consider the contribution where at least two of the closed loops correspond to massless fermions. The last diagram in the second line is a representative of the so-called “light-by-light” contribution.

given in Figs. 1 and 2. The precise definition of these quantities is provided in Sections 3 and 4.

The outline of the paper is as follows: in the next section we provide details of the four-loop on-shell integrals needed for our calculation. In particular, we identify all master integrals and provide analytical results in Appendix A. The renormalization constants  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$  are discussed in Section 3 and Section 4 is devoted to the anomalous magnetic moment of the muon. We discuss the relation between the  $\overline{\text{MS}}$  and on-shell fine structure constant and provide analytical results for  $a_\mu$ . Finally, we conclude in Section 5. Appendix B contains the analytic results for the relation between the fine structure constant defined in the  $\overline{\text{MS}}$  and on-shell scheme.

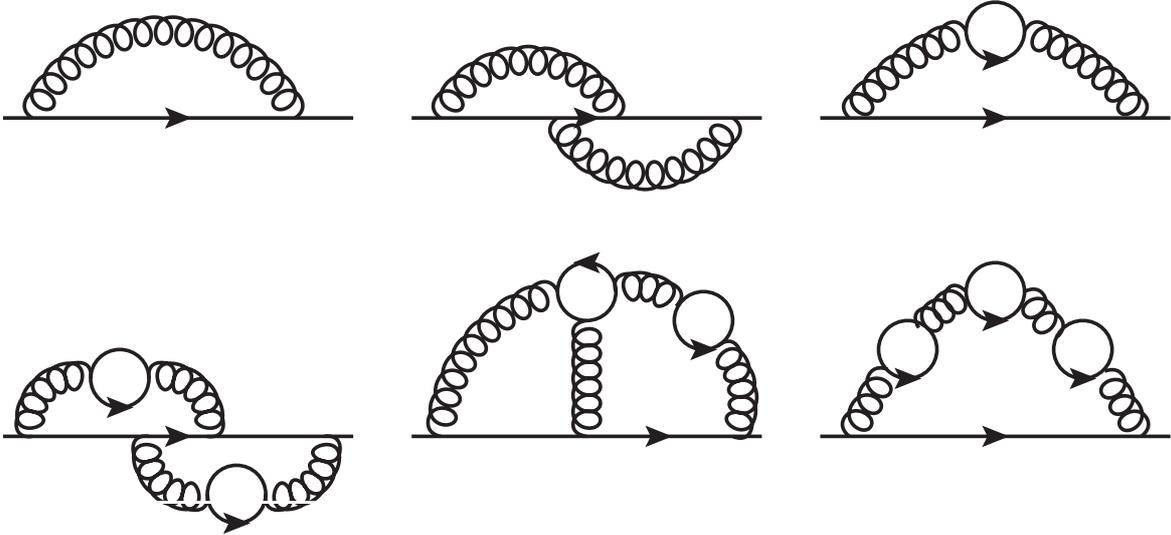


Figure 2: Sample Feynman diagrams for the QCD corrections to the fermion propagator contributing to  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$ . Curly and straight lines represent gluons and fermions, respectively. In this paper we consider the contribution where at least two of the closed loops correspond to massless fermions.

## 2 Four-loop on-shell integrals

In this Section we present the setup used for the calculation and discuss the families of four-loop on-shell integrals needed for the  $n_l^2$  and  $n_l^3$  corrections for  $Z_2^{\text{OS}}$ ,  $Z_m^{\text{OS}}$  and  $a_\mu$ . Since all three cases reduce to the calculation of corrections to the fermion propagator we consider in this Section the corresponding two-point function.

After the generation of the diagrams with `QGRAF` [36] we use `q2e` [37, 38] to translate the output into a `FORM` [39] readable form. In a next step `exp` [37, 38] is applied to map the momenta to one of five families. During the evaluation of the `FORM` code we apply projectors and take traces to end up with integrals which only contain scalar products in the numerator and quadratic denominators.

In the next step we have to reduce all occurring integrals to a minimal set of master integrals. This is done using two different programs in order to have a cross check for the calculation. On the one hand we use `crusher` [40] and on the other hand the `C++` version of `FIRE`.<sup>1</sup> Both programs implements Laporta's algorithm [42] for the solution of integration-by-parts identities [43]. We find complete agreement for the expressions where the physical quantities are expressed in terms of master integrals.

Let us mention that we have performed our calculations for general gauge parameter which drops out once the four-loop results for  $Z_2^{\text{OS}}$ ,  $Z_m^{\text{OS}}$  and  $a_\mu$  are expressed in terms of

<sup>1</sup>The `Mathematica` version of `FIRE` is publicly available [41].

master integrals.<sup>2</sup>

Altogether we end up with 13 master integrals. Seven of them (shown in Fig. 3) are products of one- and two-loop integrals whereas the remaining six integrals (cf. Fig. 4) request a dedicated investigation. We calculate them using the Dimensional Recurrence and Analyticity (DRA) method introduced in [44]. In order to fix the position and order of the poles of the integrals, we use FIESTA [45, 46]. The remaining constants are fixed using the Mellin-Barnes technique [47–51]. In order to express the results in terms of the conventional multiple zeta values we apply the PSLQ algorithm [52] on high-precision numerical results (with several hundreds of decimal digits).<sup>3</sup>

The analytic results for the integrals in Fig. 4 are listed in Appendix A. Results in terms of Gamma functions for the integrals in Fig. 3 are easily obtained recursively using the formulae from the Appendix of Ref. [49]. For convenience also these results are given in Appendix A.

All results have been cross-checked numerically with the help of FIESTA [46] where an accuracy of at least four digits has been achieved.

### 3 Fermionic $n_l^2$ and $n_l^3$ contributions to $Z_m^{\text{OS}}$ and $Z_2^{\text{OS}}$

Both  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$  are obtained from the fermion two-point functions  $\Sigma(q)$  which can be cast in the form

$$\Sigma(q, m_q) = m_q \Sigma_1(q^2, m_q) + (\not{q} - m_q) \Sigma_2(q^2, m_q). \quad (1)$$

Here  $m_q$  represents a generic quark mass whereas bare, on-shell and  $\overline{\text{MS}}$  quark masses are denoted by  $m_q^0$ ,  $M_q$  and  $\bar{m}_q$ .

The derivation of ready-to-use formulae for  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$  is discussed at length in Refs. [14, 15]. Thus, let us for convenience only repeat the final formulae which are applied in our calculations. They read

$$Z_m^{\text{OS}} = 1 + \Sigma_1(M_q^2, M_q), \quad (2)$$

$$(Z_2^{\text{OS}})^{-1} = 1 + 2M_q^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M_q) \Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q). \quad (3)$$

The expressions on the right-hand side are computed by introducing the momentum  $Q$

---

<sup>2</sup>Note that  $Z_m^{\text{OS}}$  and  $a_\mu$  have to be independent of the QCD gauge parameter  $\xi$  whereas we expect that the  $n_l^1$  and  $n_l$ -independent terms of  $Z_2^{\text{OS}}$  do depend on  $\xi$ .

<sup>3</sup>Let us mention that the numerical evaluation of the factorizable four-loop master integrals for  $a_l$  which reduce to the evaluation of the corresponding three-loop master integrals in higher orders of  $\epsilon$  was undertaken in Ref. [53] as a warm-up before a future full four-loop calculation. This was done with the method of [42] based on difference equations. The achieved accuracy of several dozen of decimal digits was not enough for using PSLQ.

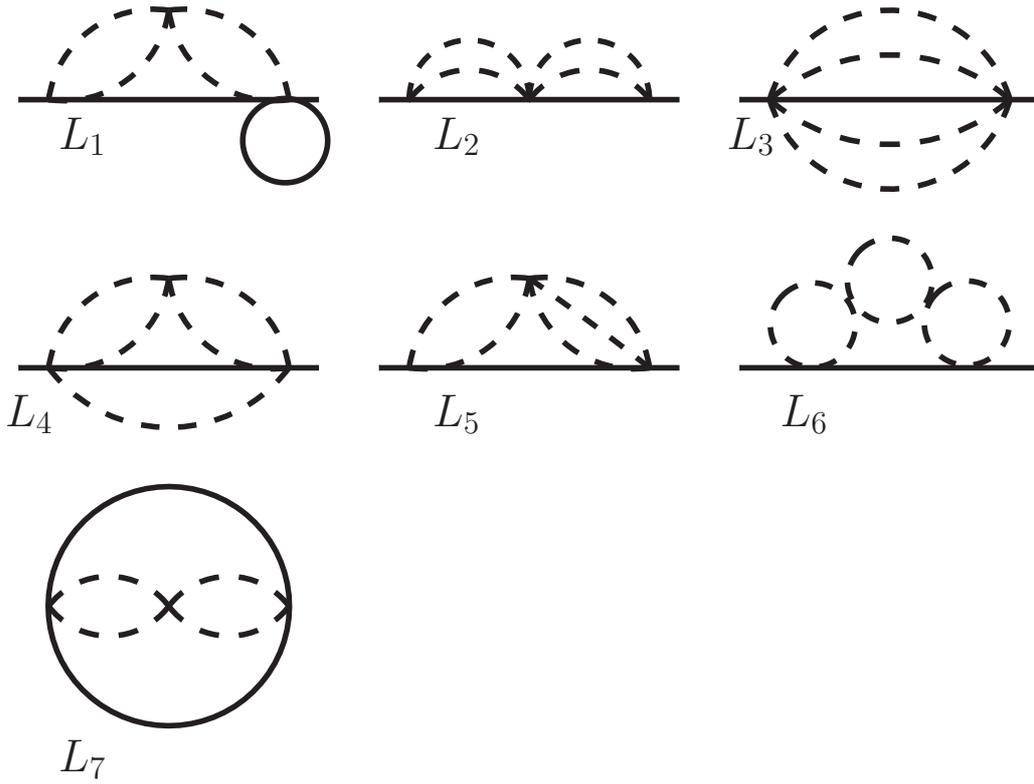


Figure 3: Master integrals for the  $n_l^2$  and  $n_l^3$  contribution which are easily obtained by applying one- and two-loop formulae, see e.g., Ref. [49]. Solid lines carry the mass  $M$  and dashed lines are massless. For  $L_1$  to  $L_6$  we have  $q^2 = M^2$  where  $q$  is the external momentum;  $L_7$  is a vacuum integral.

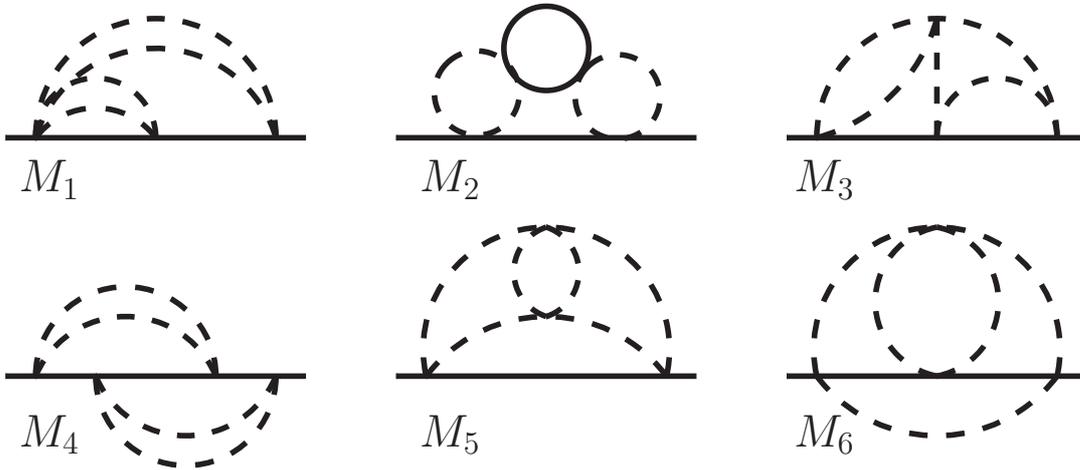


Figure 4: Non-trivial master integrals contributing to the  $n_l^2$  contribution. The same notation as in Fig. 3 has been used.

with  $Q^2 = M_q^2$  via  $q = Q(1+t)$  which leads to the equation

$$\begin{aligned} \text{Tr} \left\{ \frac{\not{Q} + M_q}{4M_q^2} \Sigma(q, M_q) \right\} &= \Sigma_1(q^2, M_q) + t \Sigma_2(q^2, M_q) \\ &= \Sigma_1(M_q^2, M_q) + \left( 2M_q^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M_q) \Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q) \right) t \\ &\quad + \mathcal{O}(t^2). \end{aligned} \quad (4)$$

Hence, to obtain  $Z_m^{\text{OS}}$  one only needs to calculate  $\Sigma_1$  for  $q^2 = M_q^2$ . To calculate  $Z_2^{\text{OS}}$ , one has to compute the first derivative of the self-energy diagrams. Note that the renormalization of the quark mass is taken into account iteratively by explicitly calculating the corresponding counterterm diagrams.

We write the perturbative expansion for  $Z_m^{\text{OS}}$  in terms of the renormalized strong coupling as ( $\gamma_E$  is the Euler-Mascheroni number)

$$\begin{aligned} Z_m^{\text{OS}} &= 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-\epsilon} \delta Z_m^{(1)} + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-2\epsilon} \delta Z_m^{(2)} \\ &\quad + \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-3\epsilon} \delta Z_m^{(3)} + \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-4\epsilon} \delta Z_m^{(4)} + \mathcal{O}(\alpha_s^5). \end{aligned} \quad (5)$$

This allows us to take the ratio between the on-shell and  $\overline{\text{MS}}$  [54–56] mass renormalization constant which is given by

$$\begin{aligned} z_m^{\text{OS}}(\mu) &= \frac{\bar{m}_q(\mu)}{M_q} = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} \\ &= 1 + \frac{\alpha_s(\mu)}{\pi} \delta z_m^{(1)} + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \delta z_m^{(2)} + \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \delta z_m^{(3)} + \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 \delta z_m^{(4)} \\ &\quad + \mathcal{O}(\alpha_s^5) \end{aligned} \quad (6)$$

The coefficients  $\delta z_m^{(i)}$  are by construction finite.

In the case of  $Z_2^{\text{OS}}$  we choose the bare coupling as expansion parameter which in many applications turns out to be convenient. Furthermore, the dependence on  $\mu/M_q$  can be written in factorized form which leads to shorter expressions. Thus we have

$$\begin{aligned} Z_2^{\text{OS}} &= 1 + \frac{\alpha_s^0}{\pi} \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-\epsilon} \delta Z_2^{(1)} + \left( \frac{\alpha_s^0}{\pi} \right)^2 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-2\epsilon} \delta Z_2^{(2)} \\ &\quad + \left( \frac{\alpha_s^0}{\pi} \right)^3 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-3\epsilon} \delta Z_2^{(3)} + \left( \frac{\alpha_s^0}{\pi} \right)^4 \left( \frac{e^{\gamma_E}}{4\pi} \right)^{-4\epsilon} \delta Z_2^{(4)} + \mathcal{O}((\alpha_s^0)^5), \end{aligned} \quad (7)$$

where each term  $\delta Z_2^{(n)}$  contains a factor  $(\mu^2/M_q^2)^{n\epsilon}$ .

We refrain from repeating the one-, two- and three-loop results for  $Z_m^{\text{OS}}$  and  $Z_2^{\text{OS}}$  since analytical expressions for general colour coefficients are available in the literature [14–16].

We split the four-loop coefficient according to the number of closed massless fermion loops and write ( $i \in \{m, 2\}$ )

$$\delta Z_i^{(4)} = \delta Z_i^{(40)} + \delta Z_i^{(41)} n_l + \delta Z_i^{(42)} n_l^2 + \delta Z_i^{(43)} n_l^3. \quad (8)$$

with an analog notation for  $\delta z_m^{(4)}$ .

In the following we present analytical results for  $\delta z_m^{(42)}$ ,  $\delta z_m^{(43)}$ ,  $\delta Z_2^{(42)}$  and  $\delta Z_2^{(43)}$  which read

$$\begin{aligned} \delta z_m^{(43)} = & C_F T^3 \left( \frac{l_M^4}{144} + \frac{13l_M^3}{216} + \left( \frac{89}{432} + \frac{\pi^2}{36} \right) l_M^2 + l_M \left( \frac{\zeta_3}{3} + \frac{1301}{3888} + \frac{13\pi^2}{108} \right) \right. \\ & \left. + \frac{317\zeta_3}{432} + \frac{71\pi^4}{4320} + \frac{89\pi^2}{648} + \frac{42979}{186624} \right), \end{aligned} \quad (9)$$

$$\begin{aligned} \delta z_m^{(42)} = & C_F n_h T^3 \left( \frac{l_M^4}{48} + \frac{13l_M^3}{72} + \frac{125l_M^2}{144} + \frac{2489l_M}{1296} + \frac{5\zeta_3}{144} - \frac{19\pi^4}{480} + \frac{\pi^2}{6} + \frac{128515}{62208} \right) \\ & + C_A C_F T^2 \left( -\frac{11l_M^4}{192} - \frac{91l_M^3}{144} + l_M^2 \left( -\frac{1}{12}\pi^2 a_1 - \frac{\zeta_3}{4} - \frac{\pi^2}{8} - \frac{6539}{2304} \right) \right. \\ & + l_M \left( \frac{a_1^4}{18} + \frac{1}{9}\pi^2 a_1^2 - \frac{11}{18}\pi^2 a_1 + \frac{4a_4}{3} - \frac{37\zeta_3}{16} - \frac{\pi^4}{216} - \frac{29\pi^2}{36} - \frac{15953}{2592} \right) \\ & - \frac{1}{45}a_1^5 + \frac{11a_1^4}{54} - \frac{2}{27}\pi^2 a_1^3 + \frac{11}{27}\pi^2 a_1^2 - \frac{31\pi^4 a_1}{1080} - \frac{103}{108}\pi^2 a_1 + \frac{44a_4}{9} + \frac{8a_5}{3} \\ & \left. - \frac{41\zeta_5}{24} - \frac{13\pi^2 \zeta_3}{48} - \frac{3245\zeta_3}{576} - \frac{4723\pi^4}{51840} - \frac{527\pi^2}{384} - \frac{2708353}{497664} \right) \\ & + C_F^2 T^2 \left( +\frac{11l_M^4}{384} + \frac{97l_M^3}{576} + l_M^2 \left( \frac{1}{6}\pi^2 a_1 + \frac{\zeta_3}{8} - \frac{5\pi^2}{96} - \frac{157}{2304} \right) \right. \\ & + l_M \left( -\frac{1}{9}a_1^4 - \frac{2}{9}\pi^2 a_1^2 + \frac{11}{9}\pi^2 a_1 - \frac{8a_4}{3} - \frac{11\zeta_3}{8} + \frac{11\pi^4}{216} - \frac{21\pi^2}{32} - \frac{50131}{20736} \right) \\ & + \frac{2a_1^5}{45} - \frac{11a_1^4}{27} + \frac{4}{27}\pi^2 a_1^3 - \frac{22}{27}\pi^2 a_1^2 + \frac{31}{540}\pi^4 a_1 + \frac{103}{54}\pi^2 a_1 - \frac{88a_4}{9} - \frac{16a_5}{3} \\ & \left. + \frac{305\zeta_5}{48} + \frac{3\pi^2 \zeta_3}{8} - \frac{2839\zeta_3}{576} + \frac{3683\pi^4}{51840} - \frac{5309\pi^2}{3456} - \frac{2396921}{497664} \right), \end{aligned} \quad (10)$$

$$\begin{aligned} \delta Z_2^{(43)} = & C_F T^3 \left( \frac{\mu^2}{M_q^2} \right)^{4\epsilon} \left( \frac{1}{144\epsilon^4} + \frac{65}{864\epsilon^3} + \frac{89}{192} + \frac{13\pi^2}{432} + \frac{151\zeta_3}{216} + \frac{73669}{31104} + \frac{845\pi^2}{2592} \right. \\ & \left. + \frac{9815\zeta_3}{1296} + \frac{589\pi^4}{4320} + \frac{1157\pi^2}{576} + \frac{2106347}{186624} \right), \end{aligned} \quad (11)$$

$$\delta Z_2^{(42)} = \left( \frac{\mu^2}{M_q^2} \right)^{4\epsilon} \left[ C_F n_h T^3 \left( \frac{1}{36\epsilon^4} + \frac{187}{864\epsilon^3} + \frac{10957}{5184} - \frac{5\pi^2}{108} \right) \right]$$

$$\begin{aligned}
& + \frac{\frac{2}{3}\pi^2 a_1 - \frac{71\zeta_3}{54} - \frac{1013\pi^2}{2592} + \frac{349615}{31104}}{\epsilon} - \frac{10}{9}a_1^4 - \frac{20}{9}\pi^2 a_1^2 + \frac{127}{18}\pi^2 a_1 \\
& - \frac{80a_4}{3} - \frac{21719\zeta_3}{1296} - \frac{\pi^4}{360} - \frac{14027\pi^2}{15552} + \frac{13135057}{186624} \Big) \\
& + C_A C_F T^2 \left( -\frac{11}{192\epsilon^4} - \frac{761}{1152\epsilon^3} + \frac{-\frac{1}{6}\pi^2 a_1 + \frac{\zeta_3}{16} - \frac{13\pi^2}{192} - \frac{64433}{13824}}{\epsilon^2} \right. \\
& + \frac{\frac{5a_1^4}{18} + \frac{5}{9}\pi^2 a_1^2 - \frac{163}{72}\pi^2 a_1 + \frac{20a_4}{3} + \frac{37\zeta_3}{288} - \frac{647\pi^4}{8640} - \frac{1627\pi^2}{1152} - \frac{18287}{768}}{\epsilon} \\
& - \frac{5}{9}a_1^5 + \frac{815a_1^4}{216} - \frac{50}{27}\pi^2 a_1^3 + \frac{815}{108}\pi^2 a_1^2 + \frac{1}{18}\pi^4 a_1 - \frac{281}{18}\pi^2 a_1 \\
& + \frac{815a_4}{9} + \frac{200a_5}{3} - \frac{2079\zeta_5}{32} - \frac{209\pi^2\zeta_3}{144} - \frac{27977\zeta_3}{1728} - \frac{7411\pi^4}{5760} \\
& \left. - \frac{436741\pi^2}{41472} - \frac{60973393}{497664} \right) \\
& + C_F^2 T^2 \left( \frac{11}{384\epsilon^4} + \frac{47}{192\epsilon^3} + \frac{\frac{1}{3}\pi^2 a_1 - \frac{5\zeta_3}{16} - \frac{241\pi^2}{1152} + \frac{2363}{1536}}{\epsilon^2} \right. \\
& + \frac{-\frac{5}{9}a_1^4 - \frac{10}{9}\pi^2 a_1^2 + \frac{163}{36}\pi^2 a_1 - \frac{40a_4}{3} - \frac{773\zeta_3}{72} + \frac{383\pi^4}{1728} - \frac{1181\pi^2}{576} + \frac{2893}{2304}}{\epsilon} \\
& + \frac{10a_1^5}{9} - \frac{815a_1^4}{108} + \frac{100}{27}\pi^2 a_1^3 - \frac{815}{54}\pi^2 a_1^2 - \frac{1}{9}\pi^4 a_1 + \frac{281}{9}\pi^2 a_1 \\
& - \frac{1630a_4}{9} - \frac{400a_5}{3} + \frac{7145\zeta_5}{48} + \frac{187\pi^2\zeta_3}{48} - \frac{50209\zeta_3}{576} + \frac{8413\pi^4}{6480} \\
& \left. - \frac{75089\pi^2}{4608} - \frac{261181}{55296} \right) \Big] , \tag{12}
\end{aligned}$$

where  $l_M = \ln \mu^2/M_q^2$ ,  $\zeta_n$  is Riemann's zeta function,  $a_1 = \ln 2$  and  $a_n = \text{Li}_n(1/2)$  ( $n \geq 1$ ). In the case of QCD the colour factors take the values  $C_A = 3$ ,  $C_F = 4/3$  and  $T = 1/2$ . In Eqs. (10) and (12) the contributions from closed heavy quark loops are marked by  $n_h = 1$  which has been introduced for illustration.

In order to get an impression of the numerical size of the newly calculated terms we evaluate  $z_m^{\text{OS}}$  for  $\mu = M_q$ . After inserting the numerical values for the colour factors we obtain ( $A_s \equiv \alpha_s(M_q)/\pi$ )

$$\begin{aligned}
z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.229 - 0.104 n_h + 1.041 n_l) \\
&+ A_s^3 (-197.816 - 0.827 n_h - 0.064 n_h^2 + 26.946 n_l - 0.022 n_h n_l - 0.653 n_l^2) \\
&+ A_s^4 (-43.465 n_l^2 - 0.017 n_h n_l^2 + 0.678 n_l^3 + \dots) + \mathcal{O}(A_s^5) , \tag{13}
\end{aligned}$$

where the ellipses indicate  $n_l$  independent contributions and terms proportional to  $n_l$  which have not been computed. One observes that the  $n_l^2$  contribution at two loops and the  $n_l^3$  contribution at three loops are quite small. This is in contrast to the linear  $n_l$

terms which can become quite sizeable. E.g., setting  $n_l = 5$ , which corresponds to the case of the top quark, we obtain (for  $n_h = 1$ )

$$\begin{aligned} z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.332 + 5.207 n_l) \\ &\quad + A_s^3 \left( -198.707 + 134.619 n_l - 16.317 n_l^2 \right) \\ &\quad + A_s^4 \left( -1087.060 n_l^2 + 84.768 n_l^3 + \dots \right) + \mathcal{O}(A_s^5) . \end{aligned} \quad (14)$$

At two-loop order the  $n_l$  contribution is only a factor of three smaller than the  $n_l$ -independent term, however, with an opposite sign. At three loops the linear- $n_l$  term has almost the same order of magnitude than the constant contribution but again a different sign. It is remarkable that for  $n_l = 5$  the coefficient of the four-loop  $n_l^2$  term is more than a factor of five larger than the  $n_l$ -independent term at order  $\alpha_s^3$ .

Let us finally compare our results with the approximate expressions obtained in Ref. [57] in the large- $\beta_0$  approximation. In Ref. [57] one finds for the quantity  $M_q/\bar{m}_q(\bar{m}_q)$  the result ( $a_s \equiv \alpha_s(\bar{m}_q)/\pi$ )

$$\begin{aligned} \left. \frac{M_q}{\bar{m}_q(\bar{m}_q)} \right|_{\text{large-}\beta_0} &= 1 + a_s 1.333 + a_s^2 (17.186 - 1.041 n_l) \\ &\quad + a_s^3 (177.695 - 21.539 n_l + 0.653 n_l^2) \\ &\quad + a_s^4 (3046.294 - 553.872 n_l + 33.568 n_l^2 - 0.678 n_l^3) , \end{aligned} \quad (15)$$

where for the renormalization scale  $\mu = \bar{m}_q$  has been chosen. The coefficients of Eq. (15) should be compared with our findings which read

$$\begin{aligned} \frac{M_q}{\bar{m}_q(\bar{m}_q)} &= 1 + a_s 1.333 + a_s^2 (13.443 - 1.041 n_l) \\ &\quad + a_s^3 (190.595 - 26.655 n_l + 0.653 n_l^2) \\ &\quad + a_s^4 (c_0 + c_1 n_l + 43.396 n_l^2 - 0.678 n_l^3) , \end{aligned} \quad (16)$$

where  $c_0$  and  $c_1$  are not yet known. By construction one finds agreement for the coefficient of  $n_l^3$  since it has been used as input in Ref. [57]. As far as the  $n_l^2$  term is concerned the exact coefficient is predicted with an accuracy of about 30%.

## 4 Fermionic $n_l^2$ and $n_l^3$ contributions to $a_\mu$

It is convenient to introduce the form factors  $F_1$  and  $F_2$  of the photon-lepton vertex as

$$\Gamma^\mu(q, p) = F_1(q^2) \gamma^\mu + i \frac{F_2(q^2)}{2M_l} \sigma_{\mu\nu} q^\nu , \quad (17)$$

where  $q$  is the incoming momentum in the photon line and  $M_l$  is the lepton mass. The anomalous magnetic moment is given by

$$a_l = \left( \frac{g-2}{2} \right)_l = F_2(0). \quad (18)$$

In Eq. (17) also the momentum  $p = (p_1 + p_2)/2$  has been introduced where  $p_1^2 = p_2^2 = M_l^2$  are the momenta flowing through the external fermion lines (see Fig. 1 for the directions of the momenta).

The evaluation of  $a_l$  requires that  $\Gamma^\mu(q, p)$  is computed in the limit  $q \rightarrow 0$ . Due to the factor  $q^\nu$  in front of  $F_2$  in Eq. (17) one has to perform an expansion of  $\Gamma^\mu(q, p)$  up to linear terms in  $q$  which can be written as

$$\Gamma^\mu(q, p) = X^\mu(p) + q_\nu Y^{\mu\nu}(p) + \mathcal{O}(q^2), \quad (19)$$

with  $p^2 = M_l^2$ .  $F_2$  is conveniently obtained after the application of a projector given by (see, e.g., Ref. [58])

$$a_l = \frac{1}{2M_l^2(D-1)(D-2)} \text{Tr} \left[ \frac{D-2}{2} (M_l^2 \gamma_\mu - D p_\mu \not{p} - (D-1) M_l p_\mu) X^\mu + \frac{M_l}{4} (\not{p} + M_l) [\gamma_\nu, \gamma_\mu] (\not{p} + M_l) Y^{\mu\nu} \right], \quad (20)$$

and thus  $a_l$  is reduced to the evaluation of on-shell two-point functions as described in Section 2.

We define the loop expansion of  $a_l$  in analogy to Eq. (5) (with  $\alpha_s$  replaced by the fine structure constant) and introduce the same splitting according to the number of massless lepton loops as in Eq. (8).

The Feynman diagrams contributing to  $a_l$  respectively the coefficients of  $\alpha^n$  and  $n_l^k$  can be subdivided to two classes: (i) the one where the external photon couples to the lepton at hand and (ii) the one where it couples to a lepton present in a closed loop. Sample diagrams are given in Fig. 1. In the following we refer to the diagrams of class (ii) as “light-by-light” contribution in analogy to the corresponding hadronic part.

In this paper four-loop corrections contributing to class (i) are evaluated which contain two or three closed massless fermion loops. They are used in order to compute electron loop contributions to  $a_\mu$  neglecting terms of order  $M_e/M_\mu$ .

For the diagrams in class (i) we can proceed as follows: In a first step we renormalize the fine structure constant in the  $\overline{\text{MS}}$  scheme,  $\bar{\alpha}(\mu)$ . The corresponding renormalization constant is easily obtained from the one for  $\alpha_s$  after specifying the colour factors to QED. The  $\overline{\text{MS}}$  scheme has the advantage that the electron mass can be set to zero (which is not the case for the diagrams in class (ii)). After renormalizing the muon mass in the

on-shell scheme we obtain a finite expression for  $a_\mu$  which shows an explicit dependence on  $\ln(\mu^2/M_\mu^2)$ .

In a next step we replace  $\bar{\alpha}(\mu)$  by its on-shell counterpart using the corresponding relation up to three loops. It can best be calculated by considering the photon two point function

$$(q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2) = i \int dx e^{iqx} \langle 0 | j^\mu(x) j^\nu(x) | 0 \rangle \quad (21)$$

and employing the on-shell renormalization condition  $\Pi(q^2 = 0) = 0$ . The form of the renormalization condition reduces the problem to the calculation of two-scale vacuum integrals at three loops. Note, that for the renormalization of the fermion masses in the on-shell scheme the dependence on both masses has to be taken into account. In the limit  $M_e \ll M_\mu$  we obtain (see also Refs. [25, 27, 59])

$$\begin{aligned} \frac{\bar{\alpha}(\mu)}{\alpha} &= 1 + \frac{\alpha}{3\pi} (L_\mu + L_e) + \left(\frac{\alpha}{\pi}\right)^2 \left[ \frac{15}{8} + \frac{L_\mu + L_e}{4} + \frac{(L_\mu + L_e)^2}{9} \right] \\ &+ \left(\frac{\alpha}{\pi}\right)^3 \left( \frac{L_e^3}{27} + \frac{L_\mu L_e^2}{9} + \frac{5L_e^2}{24} + \frac{79L_e}{144} - \frac{695}{648} + \frac{\pi^2}{9} + \frac{7\zeta_3}{64} + \dots \right) + \mathcal{O}(\alpha^4) \end{aligned} \quad (22)$$

with  $L_\mu = \ln(\mu^2/M_\mu^2)$  and  $L_e = \ln(\mu^2/M_e^2)$ . The ellipses in the coefficient of  $(\alpha/\pi)^3$  indicate terms which we left out since they are irrelevant for the  $n_l^2$  contribution discussed in this paper. The complete result containing the exact dependence on  $M_e/M_\mu$  is presented in Appendix B. Note that the result in Eq. (22) can be obtained from the one provided in Ref. [27] where the relation between  $\bar{\alpha}(\mu)$  and  $\alpha$  is given for one massive lepton.

Also in the case of  $a_l$  we refrain from listing the lower-order results which can be found in the literature [13, 32–35]. Rather we concentrate on the new correction terms at four loops. Adopting the notation from Eq. (8) we obtain the following results for the  $n_l^3$  contribution

$$\begin{aligned} a_\mu^{(43)} &= \frac{1}{54} L_{\mu e}^3 - \frac{25}{108} L_{\mu e}^2 + \left( \frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\ &\approx 7.196\,66, \end{aligned} \quad (23)$$

where  $L_{\mu e} = \ln(M_\mu^2/M_e^2)$ . The approximate results have been obtained with the help of [60]  $M_\mu/M_e = 206.7682843(52)$ . The result in Eq. (23) agrees with the one in Ref. [28, 29].

In the case of the  $n_l^2$  contribution we split  $a_\mu^{(42)}$  into two parts. The first one ( $a_\mu^{(42)a}$ ) corresponds to the diagrams containing two closed fermion loops and the second one ( $a_\mu^{(42)b}$ ) originates from diagrams with three closed fermion loops where one of them is a muon and two are electron loops. Thus, we have

$$a_\mu^{(42)} = a_\mu^{(42)a} + a_\mu^{(42)b},$$

with

$$\begin{aligned}
a_\mu^{(42)a} &= L_{\mu e}^2 \left[ \pi^2 \left( \frac{5}{36} - \frac{a_1}{6} \right) + \frac{\zeta_3}{4} - \frac{13}{24} \right] + L_{\mu e} \left[ -\frac{a_1^4}{9} + \pi^2 \left( -\frac{2a_1^2}{9} + \frac{5a_1}{3} - \frac{79}{54} \right) \right. \\
&\quad \left. - \frac{8a_4}{3} - 3\zeta_3 + \frac{11\pi^4}{216} + \frac{23}{6} \right] - \frac{2a_1^5}{45} + \frac{5a_1^4}{9} + \pi^2 \left( -\frac{4a_1^3}{27} + \frac{10a_1^2}{9} \right. \\
&\quad \left. - \frac{235a_1}{54} - \frac{\zeta_3}{8} + \frac{595}{162} \right) + \pi^4 \left( -\frac{31a_1}{540} - \frac{403}{3240} \right) + \frac{40a_4}{3} + \frac{16a_5}{3} - \frac{37\zeta_5}{6} \\
&\quad + \frac{11167\zeta_3}{1152} - \frac{6833}{864} \\
&\approx -3.62427, \tag{24}
\end{aligned}$$

$$\begin{aligned}
a_\mu^{(42)b} &= \left( \frac{119}{108} - \frac{\pi^2}{9} \right) L_{\mu e}^2 + \left( \frac{\pi^2}{27} - \frac{61}{162} \right) L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944} \\
&\approx 0.49405. \tag{25}
\end{aligned}$$

$a_\mu^{(42)b}$  agrees with Ref. [28,29]. Analytical results for  $a_\mu^{(42)a}$  are not present in the literature since corrections originating from diagrams as the third one in the first row of Fig. 1 have not been considered yet. However, we can perform a numerical comparison with the results from Refs. [30,33]<sup>4</sup> which reads

$$a_\mu^{(42)a} \Big|_{\text{num}} = -3.64204(112), \tag{26}$$

There is a good agreement with the analytic result in Eq. (24). The deviation can be explained by corrections of order  $M_e/M_\mu \approx 0.005$  or  $(M_e/M_\mu)^2 \ln^3 M_\mu/M_e \approx 0.004$  [28,29] which are absent in our analytic expressions.

## 5 Conclusions

In this paper the first steps towards the evaluation of four-loop on-shell integrals have been undertaken. As an application within QCD we have computed the contributions involving two massless quark loops to the on-shell renormalization constants  $Z_2^{\text{OS}}$  and  $Z_m^{\text{OS}}$ . As an application in QED we have considered the contribution from four-loop diagrams involving two or three closed electron loops to the anomalous magnetic moment of the muon excluding, however, the light-by-light contribution.

We describe in some detail the techniques and the programs which have been used for the calculation. We are confident that they are generic enough to be applied to the  $n_f^1$  and non-fermionic contribution. The only bottleneck might be the analytic evaluation of the master integrals so that maybe numerical methods have to be applied.

---

<sup>4</sup>In in Ref. [33] the contributions from closed electron and muon loops are always added whereas in our result at least two closed electron loops are present. We are deeply grateful to the authors of Ref. [33] for providing us the results for the contributions containing only electron loops Eq. (26).

## Acknowledgements

We would like to thank M. Nio for useful communications concerning the numerical results for  $a_\mu$ . We also thank K.G. Chetyrkin for carefully reading the manuscript. This work was supported by the DFG through the SFB/TR 9 ‘‘Computational Particle Physics’’. The work of R.L, A.S. and V.S. was also supported by the Russian Foundation for Basic Research through grant 11-02-01196. The Feynman diagrams were drawn with JaxoDraw [61, 62].

## Appendix A: Analytic results for the master integrals

In this appendix we provide the analytic results of all master integrals where we assume an integration measure  $d^D k/(i\pi)^{D/2}$  with  $D = 4 - 2\epsilon$ . Furthermore we write scalar propagators of particles with mass  $M$  in the form  $1/(-k^2 + M^2)$ . For convenience we set  $M = 1$  in the final result since the dependence on  $M$  can easily be restored.

The analytic results for the integrals in Fig. 3 read

$$L_1 = \frac{\Gamma(5 - \frac{3D}{2}) \Gamma(1 - \frac{D}{2}) \Gamma(2 - \frac{D}{2})^2 \Gamma(\frac{D}{2} - 1)^4 \Gamma(3D - 9)}{\Gamma(D - 2)^2 \Gamma(2D - 5)},$$

$$L_2 = \frac{\Gamma(3 - D)^2 \Gamma(2 - \frac{D}{2})^2 \Gamma(\frac{D}{2} - 1)^4 \Gamma(2D - 5)^2}{\Gamma(D - 2)^2 \Gamma(\frac{3D}{2} - 3)^2}, \quad (27)$$

$$L_3 = \frac{\Gamma(5 - 2D) \Gamma(4 - \frac{3D}{2}) \Gamma(\frac{D}{2} - 1)^4 \Gamma(4D - 9)}{\Gamma(2D - 4) \Gamma(\frac{5D}{2} - 5)}, \quad (28)$$

$$L_4 = \frac{\Gamma(6 - 2D) \Gamma(5 - \frac{3D}{2}) \Gamma(2 - \frac{D}{2})^2 \Gamma(\frac{D}{2} - 1)^5 \Gamma(\frac{3D}{2} - 4) \Gamma(4D - 11)}{\Gamma(4 - D) \Gamma(D - 2)^2 \Gamma(2D - 5) \Gamma(\frac{5D}{2} - 6)}, \quad (29)$$

$$L_5 = \frac{\Gamma(6 - 2D) \Gamma(3 - D) \Gamma(2 - \frac{D}{2}) \Gamma(\frac{D}{2} - 1)^5 \Gamma(4D - 11)}{\Gamma(D - 2) \Gamma(\frac{3D}{2} - 3) \Gamma(\frac{5D}{2} - 6)}, \quad (30)$$

$$L_6 = \frac{\Gamma(7 - 2D) \Gamma(2 - \frac{D}{2})^3 \Gamma(\frac{D}{2} - 1)^6 \Gamma(4D - 13)}{\Gamma(D - 2)^3 \Gamma(\frac{5D}{2} - 7)}, \quad (31)$$

$$L_7 = \frac{\Gamma(6 - 2D) \Gamma(5 - \frac{3D}{2})^2 \Gamma(2 - \frac{D}{2})^2 \Gamma(\frac{D}{2} - 1)^4 \Gamma(\frac{3D}{2} - 4)}{\Gamma(10 - 3D) \Gamma(D - 2)^2 \Gamma(\frac{D}{2})}. \quad (32)$$

The analytic results for the integrals in Fig. 4 read

$$\begin{aligned} e^{4\epsilon\gamma_E} M_1 = & \frac{5}{24\epsilon^4} + \frac{25}{24\epsilon^3} + \left( \frac{205}{96} + \frac{17\pi^2}{72} \right) \epsilon^{-2} + \left( -\frac{323}{96} + \frac{85\pi^2}{72} + \frac{79\zeta_3}{18} \right) \epsilon^{-1} + \left( -\frac{55241}{1152} \right. \\ & \left. + \frac{409\pi^2}{288} + \frac{395\zeta_3}{18} + \frac{\pi^4}{8} \right) \epsilon^0 - \left( \frac{733351}{3456} + \frac{4199\pi^2}{288} - \frac{1943\zeta_3}{72} - \frac{5\pi^4}{8} - \frac{499\pi^2\zeta_3}{54} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{407\zeta_5}{6}\epsilon^1 - \left(\frac{14346449}{41472} + \frac{383045\pi^2}{3456} + \frac{19057\zeta_3}{72} + \frac{437\pi^4}{96} - \frac{2495\pi^2\zeta_3}{54} - \frac{2035\zeta_5}{6}\right. \\
& + \frac{2027\pi^6}{11340} - \frac{2285\zeta_3^2}{27}\left.\right)\epsilon^2 - \left(-\frac{391938053}{124416} + \frac{3517963\pi^2}{10368} + \frac{1751323\zeta_3}{864} + \frac{93347\pi^4}{1440}\right. \\
& - \frac{443\pi^2\zeta_3}{216} + \frac{21905\zeta_5}{24} + \frac{2027\pi^6}{2268} - \frac{11425\zeta_3^2}{27} - \frac{2477\pi^4\zeta_3}{90} - \frac{9223\pi^2\zeta_5}{90} + \frac{11681\zeta_7}{42}\left.\right)\epsilon^3 \\
& + O(\epsilon^4), \tag{33}
\end{aligned}$$

$$\begin{aligned}
e^{4\epsilon\gamma_E} M_2 = & -\frac{5}{12\epsilon^4} - \frac{23}{8\epsilon^3} - \left(\frac{433}{48} + \frac{29\pi^2}{36}\right)\epsilon^{-2} - \left(-\frac{297}{32} + \frac{191\pi^2}{24} + \frac{275\zeta_3}{18}\right)\epsilon^{-1} - \left(-\frac{22765}{64}\right. \\
& + \frac{7177\pi^2}{144} - 24\pi^2 a_1 + \frac{2273\zeta_3}{12} + \frac{125\pi^4}{72}\left.\right)\epsilon^0 - \left(-\frac{411105}{128} + \frac{8085\pi^2}{32} - 324\pi^2 a_1\right. \\
& + \frac{105463\zeta_3}{72} + \frac{2747\pi^4}{240} + 80\pi^2 a_1^2 + 40a_1^4 + \frac{1595\pi^2\zeta_3}{54} + \frac{3223\zeta_5}{6} + 960a_4\left.\right)\epsilon^1 \\
& - \left(-\frac{16944559}{768} + \frac{216731\pi^2}{192} - 2706\pi^2 a_1 + \frac{146091\zeta_3}{16} + \frac{43757\pi^4}{1440} + 1080\pi^2 a_1^2\right. \\
& + 540a_1^4 + \frac{16\pi^4 a_1}{3} - \frac{800}{3}\pi^2 a_1^3 - 80a_1^5 + \frac{9785\pi^2\zeta_3}{36} - \frac{52351\zeta_5}{20} + \frac{112339\pi^6}{22680} + \frac{15125\zeta_3^2}{54} \\
& + 12960a_4 + 9600a_5\left.\right)\epsilon^2 - \left(-\frac{68697721}{512} + \frac{589805\pi^2}{128} - 18165\pi^2 a_1 + \frac{4851365\zeta_3}{96}\right. \\
& - \frac{94853\pi^4}{960} + 9020\pi^2 a_1^2 + 4510a_1^4 + 72\pi^4 a_1 - 3600\pi^2 a_1^3 - 1080a_1^5 + \frac{336415\pi^2\zeta_3}{216} \\
& - \frac{8849321\zeta_5}{120} + 32640s_6 - \frac{351599\pi^6}{15120} - 104\pi^4 a_1^2 + 744\pi^2 a_1^4 + \frac{400a_1^6}{3} + 944\pi^2 a_1\zeta_3 \\
& - \frac{375097\zeta_3^2}{36} + \frac{6875\pi^4\zeta_3}{108} + \frac{93467\pi^2\zeta_5}{90} + \frac{652775\zeta_7}{42} + 108240a_4 + 129600a_5 + 96000a_6 \\
& \left. + 1856\pi^2 a_4\right)\epsilon^3 + O(\epsilon^4), \tag{34}
\end{aligned}$$

$$\begin{aligned}
e^{4\epsilon\gamma_E} M_3 = & -\frac{1}{6\epsilon^4} - \frac{7}{6\epsilon^3} - \left(\frac{10}{3} + \frac{13\pi^2}{18}\right)\epsilon^{-2} - \left(-\frac{61}{6} + \frac{73\pi^2}{18} + \frac{118\zeta_3}{9}\right)\epsilon^{-1} - \left(-\frac{851}{4} + \frac{83\pi^2}{18}\right. \\
& + \frac{637\zeta_3}{9} + \frac{37\pi^4}{10}\left.\right)\epsilon^0 - \left(-\frac{14861}{8} - \frac{3467\pi^2}{36} + \frac{1003\zeta_3}{18} + \frac{1121\pi^4}{60} + \frac{1894\pi^2\zeta_3}{27}\right. \\
& + \frac{16018\zeta_5}{15}\left.\right)\epsilon^1 - \left(-\frac{613975}{48} - \frac{25981\pi^2}{24} - \frac{68293\zeta_3}{36} + \frac{83\pi^4}{24} + \frac{9559\pi^2\zeta_3}{27} + \frac{79891\zeta_5}{15}\right. \\
& + \frac{59501\pi^6}{2835} + \frac{17704\zeta_3^2}{27}\left.\right)\epsilon^2 - \left(-\frac{7539347}{96} - \frac{382349\pi^2}{48} - \frac{482627\zeta_3}{24} - \frac{426659\pi^4}{720}\right. \\
& + \frac{3757\pi^2\zeta_3}{54} + \frac{2525\zeta_5}{6} + \frac{43201\pi^6}{420} + \frac{88585\zeta_3^2}{27} + \frac{17204\pi^4\zeta_3}{45} + \frac{206434\pi^2\zeta_5}{45}
\end{aligned}$$

$$+ \frac{1267243\zeta_7}{21} \epsilon^3 + O(\epsilon^4), \quad (35)$$

$$\begin{aligned} e^{4\epsilon\gamma_E} M_4 = & -\frac{1}{3\epsilon^4} - \frac{5}{2\epsilon^3} - \left(\frac{55}{6} + \frac{4\pi^2}{9}\right)\epsilon^{-2} - \left(3 + \frac{19\pi^2}{3} + \frac{56\zeta_3}{9}\right)\epsilon^{-1} - \left(-250 + \frac{1925\pi^2}{36}\right. \\ & - 32\pi^2 a_1 + \frac{464\zeta_3}{3} + \frac{19\pi^4}{45} \Big) \epsilon^0 - \left(-\frac{5091}{2} + \frac{2811\pi^2}{8} - 432\pi^2 a_1 + \frac{14797\zeta_3}{9} + \frac{17\pi^4}{90}\right. \\ & + \frac{320}{3}\pi^2 a_1^2 + \frac{160a_1^4}{3} + \frac{332\pi^2\zeta_3}{27} + \frac{1556\zeta_5}{15} + 1280a_4 \Big) \epsilon^1 - \left(-\frac{55049}{3} + \frac{95693\pi^2}{48}\right. \\ & - 3608\pi^2 a_1 + \frac{24831\zeta_3}{2} - \frac{1084\pi^4}{45} + 1440\pi^2 a_1^2 + 720a_1^4 + \frac{160\pi^4 a_1}{9} - \frac{3200}{9}\pi^2 a_1^3 \\ & - \frac{320a_1^5}{3} + \frac{1046\pi^2\zeta_3}{9} - 8246\zeta_5 + \frac{772\pi^6}{2835} + \frac{2648\zeta_3^2}{27} + 17280a_4 + 12800a_5 \Big) \epsilon^2 \\ & - \left(-\frac{458141}{4} + \frac{329467\pi^2}{32} - 24220\pi^2 a_1 + \frac{938425\zeta_3}{12} - \frac{9979\pi^4}{36} + \frac{36080}{3}\pi^2 a_1^2\right. \\ & + \frac{18040a_1^4}{3} + 240\pi^4 a_1 - 4800\pi^2 a_1^3 - 1440a_1^5 + \frac{19930\pi^2\zeta_3}{27} - \frac{353044\zeta_5}{3} + 44800s_6 \\ & - \frac{76904\pi^6}{945} - 176\pi^4 a_1^2 + 976\pi^2 a_1^4 + \frac{1600a_1^6}{9} + \frac{4160}{3}\pi^2 a_1 \zeta_3 - \frac{155668\zeta_3^2}{9} + \frac{4304\pi^4\zeta_3}{135} \\ & + \frac{7304\pi^2\zeta_5}{45} + \frac{4616\zeta_7}{21} + 144320a_4 + 172800a_5 + 128000a_6 + \frac{6272\pi^2 a_4}{3} \Big) \epsilon^3 \\ & + O(\epsilon^4), \quad (36) \end{aligned}$$

$$\begin{aligned} e^{4\epsilon\gamma_E} M_5 = & -\frac{1}{12\epsilon^4} - \frac{13}{24\epsilon^3} - \left(\frac{15}{16} + \frac{13\pi^2}{36}\right)\epsilon^{-2} - \left(-\frac{1135}{96} + \frac{169\pi^2}{72} + \frac{86\zeta_3}{9}\right)\epsilon^{-1} - \left(-\frac{28699}{192}\right. \\ & + \frac{65\pi^2}{16} + \frac{559\zeta_3}{9} + \frac{149\pi^4}{90} \Big) \epsilon^0 - \left(-\frac{144429}{128} - \frac{14755\pi^2}{288} + \frac{227\zeta_3}{2} + \frac{1937\pi^4}{180}\right. \\ & + \frac{1118\pi^2\zeta_3}{27} + \frac{7604\zeta_5}{15} \Big) \epsilon^1 - \left(-\frac{5327075}{768} - \frac{373087\pi^2}{576} - \frac{45889\zeta_3}{36} + \frac{749\pi^4}{40}\right. \\ & + \frac{7267\pi^2\zeta_3}{27} + \frac{49426\zeta_5}{15} + \frac{14053\pi^6}{1620} + \frac{14063\zeta_3^2}{27} \Big) \epsilon^2 - \left(-\frac{58275695}{1536} - \frac{625859\pi^2}{128}\right. \\ & - \frac{1192693\zeta_3}{72} - \frac{168143\pi^4}{720} + \frac{2951\pi^2\zeta_3}{6} + 5805\zeta_5 + \frac{182689\pi^6}{3240} + \frac{182819\zeta_3^2}{54} \\ & + \frac{51013\pi^4\zeta_3}{270} + \frac{98852\pi^2\zeta_5}{45} + \frac{1021711\zeta_7}{42} \Big) \epsilon^3 + O(\epsilon^4), \quad (37) \end{aligned}$$

$$e^{4\epsilon\gamma_E} M_6 = -\frac{1}{12\epsilon^4} - \frac{17}{24\epsilon^3} - \left(\frac{149}{48} + \frac{13\pi^2}{36}\right)\epsilon^{-2} - \left(\frac{433}{96} + \frac{149\pi^2}{72} + \frac{23\zeta_3}{9}\right)\epsilon^{-1} - \left(-\frac{3817}{64}\right.$$

$$\begin{aligned}
& + \frac{521\pi^2}{144} + \frac{173\zeta_3}{9} + \frac{341\pi^4}{180} \Big) \epsilon^0 - \left( -\frac{97165}{128} - \frac{9419\pi^2}{288} + \frac{1367\zeta_3}{18} + \frac{1667\pi^4}{180} \right. \\
& + \left. \frac{659\pi^2\zeta_3}{27} + \frac{5939\zeta_5}{15} \right) \epsilon^1 - \left( -\frac{4640963}{768} - \frac{80461\pi^2}{192} + \frac{1717\zeta_3}{36} + \frac{833\pi^4}{360} + \frac{3527\pi^2\zeta_3}{27} \right. \\
& + \left. \frac{30599\zeta_5}{15} + \frac{57791\pi^6}{5670} + \frac{734\zeta_3^2}{27} \right) \epsilon^2 - \left( -\frac{61900879}{1536} - \frac{410283\pi^2}{128} - \frac{55357\zeta_3}{24} \right. \\
& - \frac{193957\pi^4}{720} + \frac{8489\pi^2\zeta_3}{54} + \frac{45833\zeta_5}{30} + \frac{1106911\pi^6}{22680} + \frac{14197\zeta_3^2}{54} + \frac{21637\pi^4\zeta_3}{135} \\
& + \left. \frac{75407\pi^2\zeta_5}{45} + \frac{522017\zeta_7}{21} \right) \epsilon^3 + O(\epsilon^4), \tag{38}
\end{aligned}$$

with  $s_6 = \sum_{m=1}^{\infty} \sum_{k=1}^m (-1)^{m+k} / (m^5 k) = 0.98744\dots$

## Appendix B: Relation between $\bar{\alpha}(\mu)$ and $\alpha$

In this Appendix we present the result for the relation between the fine structure constant defined in the  $\overline{\text{MS}}$  and on-shell renormalization scheme involving two massive leptons with masses  $m_1$  and  $m_2$ . We label contributions from leptons with mass  $m_1$  and  $m_2$  by  $n_h$  and  $n_l$ , respectively. Our result reads

$$\begin{aligned}
\frac{\bar{\alpha}(\mu)}{\alpha} - 1 &= \frac{1}{3} l_2 n_l \frac{\alpha}{\pi} + \left\{ \frac{1}{9} l_1 l_2 n_h n_l + \frac{1}{9} l_2^2 n_l^2 + \left( \frac{l_2}{4} + \frac{15}{16} \right) n_l \right\} \left( \frac{\alpha}{\pi} \right)^2 \\
&+ \left\{ \frac{1}{27} l_2^3 n_l^3 + \frac{1}{9} l_1 l_2^2 n_h n_l^2 + n_l^2 \left( \frac{5l_2^2}{24} + \frac{79l_2}{144} + \frac{7\zeta_3}{64} + \frac{\pi^2}{9} - \frac{695}{648} \right) \right. \\
&+ n_l \left( -\frac{1}{3} \pi^2 a_1 - \frac{l_2}{32} + \frac{\zeta_3}{192} + \frac{5\pi^2}{24} + \frac{77}{576} \right) \\
&+ n_h n_l \left[ \frac{79l_1 x^2}{384} - \frac{79l_2 (3x^2 - 8)}{1152} + \frac{5l_1 l_2}{24} + \frac{1}{384} (-128x^4 - 15x^2 - 71) \ln^2(x) \right. \\
&+ \frac{1}{3} (-x^4 + x^3 + x - 1) \text{Li}_2(1-x) + \frac{1}{3} (x^4 + x^3 + x + 1) \text{Li}_2(-x) \\
&+ \frac{(5x^6 + 3x^4 + 3x^2 + 5)}{256x^3} \left( (\text{Li}_2(1-x) + \text{Li}_2(-x)) \ln(x) - 2\text{Li}_3(1-x) - \text{Li}_3(-x) \right) \\
&+ \frac{1}{3} (x^4 + x^3 + x + 1) \ln(x) \ln(x+1) + \frac{(5x^6 + 3x^4 + 3x^2 + 5) \ln^2(x) \ln(x+1)}{512x^3} \\
&+ \left. \frac{405x^3\zeta_3 + 1152\pi^2(x^3 + x) - 5994x^2 + 243x\zeta_3 - 5126}{10368} \right\} \left( \frac{\alpha}{\pi} \right)^3 \\
&+ \left\{ n_h \leftrightarrow n_l, m_1 \leftrightarrow m_2, x \leftrightarrow \frac{1}{x} \right\}, \tag{39}
\end{aligned}$$

with  $x = m_1/m_2$ ,  $l_k = \ln(\mu^2/m_k^2)$  and  $a_1 = \ln 2$ .

## References

- [1] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. **96** (2006) 012003 [hep-ph/0511063].
- [2] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. **101** (2008) 012002 [arXiv:0801.1821 [hep-ph]].
- [3] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn and J. Rittinger, Phys. Rev. Lett. **108** (2012) 222003 [arXiv:1201.5804 [hep-ph]].
- [4] K. G. Chetyrkin, J. H. Kuhn and C. Sturm, Eur. Phys. J. C **48** (2006) 107 [arXiv:hep-ph/0604234].
- [5] R. Boughezal, M. Czakon and T. Schutzmeier, Phys. Rev. D **74** (2006) 074006 [arXiv:hep-ph/0605023].
- [6] C. Sturm, JHEP **0809** (2008) 075 [arXiv:0805.3358 [hep-ph]].
- [7] A. Maier, P. Maierhofer and P. Marquard, Phys. Lett. B **669** (2008) 88 [arXiv:0806.3405 [hep-ph]].
- [8] A. Maier, P. Maierhofer, P. Marquard and A. V. Smirnov, Nucl. Phys. B **824** (2010) 1 [arXiv:0907.2117 [hep-ph]].
- [9] K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, Phys. Rev. D **80** (2009) 074010 [arXiv:0907.2110 [hep-ph]].
- [10] K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, Phys. Rev. D **67** (2003) 105008 [hep-ph/0211321].
- [11] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C **48** (1990) 673.
- [12] D. J. Broadhurst, N. Gray and K. Schilcher, Z. Phys. C **52** (1991) 111.
- [13] S. Laporta and E. Remiddi, Phys. Lett. B **379** (1996) 283 [hep-ph/9602417].
- [14] K. Melnikov and T. v. Ritbergen, Phys. Lett. B **482** (2000) 99 [hep-ph/9912391].
- [15] P. Marquard, L. Mihaila, J. H. Piclum and M. Steinhauser, Nucl. Phys. B **773** (2007) 1 [hep-ph/0702185].
- [16] K. Melnikov and T. van Ritbergen, Nucl. Phys. B **591** (2000) 515 [hep-ph/0005131].
- [17] K. G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. **83** (1999) 4001 [hep-ph/9907509].
- [18] K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. B **573** (2000) 617 [hep-ph/9911434].

- [19] K. Melnikov and T. van Ritbergen, Phys. Rev. Lett. **84** (2000) 1673 [hep-ph/9911277].
- [20] A. G. Grozin, P. Marquard, J. H. Piclum and M. Steinhauser, Nucl. Phys. B **789** (2008) 277 [arXiv:0707.1388 [hep-ph]].
- [21] B. E. Lautrup and E. De Rafael, Nucl. Phys. B **70** (1974) 317 [Erratum-ibid. B **78** (1974) 576].
- [22] T. Kinoshita, H. Kawai and Y. Okamoto, Phys. Lett. B **254** (1991) 235.
- [23] H. Kawai, T. Kinoshita and Y. Okamoto, Phys. Lett. B **260** (1991) 193.
- [24] R. N. Faustov, A. L. Kataev, S. A. Larin and V. V. Starshenko, Phys. Lett. B **254** (1991) 241.
- [25] D. J. Broadhurst, A. L. Kataev and O. V. Tarasov, Phys. Lett. B **298** (1993) 445 [hep-ph/9210255].
- [26] P. A. Baikov and D. J. Broadhurst, [hep-ph/9504398].
- [27] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn, C. Sturm, Nucl. Phys. B **867** (2013) 182 [arXiv:1207.2199 [hep-ph]].
- [28] S. Laporta, Phys. Lett. B **312** (1993) 495 [hep-ph/9306324].
- [29] J. -P. Aguilar, D. Greynat and E. De Rafael, Phys. Rev. D **77** (2008) 093010 [arXiv:0802.2618 [hep-ph]].
- [30] T. Kinoshita and M. Nio, Phys. Rev. D **70** (2004) 113001 [hep-ph/0402206].
- [31] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. D **77** (2008) 053012 [arXiv:0712.2607 [hep-ph]].
- [32] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109** (2012) 111807 [arXiv:1205.5368 [hep-ph]].
- [33] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109** (2012) 111808 [arXiv:1205.5370 [hep-ph]].
- [34] F. Jegerlehner, Springer Tracts Mod. Phys. **226** (2008) 1.
- [35] F. Jegerlehner and A. Nyffeler, Phys. Rept. **477** (2009) 1 [arXiv:0902.3360 [hep-ph]].
- [36] P. Nogueira, J. Comput. Phys. **105** (1993) 279.
- [37] R. Harlander, T. Seidensticker and M. Steinhauser, Phys. Lett. B **426** (1998) 125, arXiv:hep-ph/9712228.

- [38] T. Seidensticker, arXiv:hep-ph/9905298.
- [39] J. A. M. Vermaseren, arXiv:math-ph/0010025.
- [40] P. Marquard, D. Seidel, unpublished.
- [41] A. V. Smirnov, JHEP **0810** (2008) 107 [arXiv:0807.3243 [hep-ph]].
- [42] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033].
- [43] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.
- [44] R. N. Lee, Nucl. Phys. B **830** (2010) 474 [arXiv:0911.0252 [hep-ph]].
- [45] A. V. Smirnov and M. N. Tentyukov, Comput. Phys. Commun. **180** (2009) 735 [arXiv:0807.4129 [hep-ph]].
- [46] A. V. Smirnov, V. A. Smirnov and M. Tentyukov, Comput. Phys. Commun. **182** (2011) 790 [arXiv:0912.0158 [hep-ph]].
- [47] V. A. Smirnov, Phys. Lett. B **460** (1999) 397 [hep-ph/9905323].
- [48] J. B. Tausk, Phys. Lett. B **469** (1999) 225 [hep-ph/9909506].
- [49] V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts Mod. Phys. **250** (2013) 1.
- [50] M. Czakon, Comput. Phys. Commun. **175** (2006) 559 [hep-ph/0511200].
- [51] A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **62** (2009) 445 [arXiv:0901.0386 [hep-ph]].
- [52] Ferguson, H. R. P. and Bailey, D. H. RNR Techn. Rept. (1992) RNR-91-032.
- [53] S. Laporta, Phys. Lett. B **523** (2001) 95 [hep-ph/0111123].
- [54] K. G. Chetyrkin, Phys. Lett. B **404** (1997) 161 [hep-ph/9703278].
- [55] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, Phys. Lett. B **405** (1997) 327 [hep-ph/9703284].
- [56] K. G. Chetyrkin, Nucl. Phys. B **710** (2005) 499 [hep-ph/0405193].
- [57] M. Beneke and V. M. Braun, Phys. Lett. B **348** (1995) 513 [hep-ph/9411229].
- [58] B. Krause, “Zwei- und Dreischleifen-Berechnungen zu elektrischen und magnetischen Dipolmomenten von Elementarteilchen”, PhD thesis, University of Karlsruhe, (1997).
- [59] D. J. Broadhurst, Z. Phys. C **54** (1992) 599.

- [60] P. J. Mohr, B. N. Taylor and D. B. Newell, *Rev. Mod. Phys.* **84** (2012) 1527 [arXiv:1203.5425 [physics.atom-ph]].
- [61] J. A. M. Vermaseren, *Comput. Phys. Commun.* **83** (1994) 45.
- [62] D. Binosi, J. Collins, C. Kaufhold and L. Theussl, *Comput. Phys. Commun.* **180** (2009) 1709 [arXiv:0811.4113 [hep-ph]].