

Kinetic operator effects in $\bar{B} \rightarrow X_c \ell \nu$ at $O(\alpha_s)$

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Abstract

We compute the $O(\alpha_s)$ corrections to the Wilson coefficient of the kinetic operator in inclusive semileptonic B decays. Our analytic calculation agrees with reparameterization invariance and with previous numerical results and paves the way to the calculation of analogous corrections to other power-suppressed coefficients.

1 Introduction

The precision determination of the Cabibbo-Kobayashi-Maskawa quark mixing matrix remains a central goal in the flavor physics program. A new generation of high-luminosity B factories is expected to start operations in a few years and in view of the improved experimental resolution the theoretical uncertainties should be reduced whenever possible. In the case of inclusive semileptonic B decays, which currently provide the most precise determination of $|V_{cb}|$ and $|V_{ub}|$, theoretical uncertainties are already dominant, but there is space for improvement.

As is well known, the theoretical foundation for our understanding of inclusive semileptonic decays $B \rightarrow X_c \ell \nu$ is an Operator Product Expansion (OPE) which ensures that non-perturbative effects are suppressed by at least two powers of the bottom mass m_b . They are parameterized by a limited number of matrix elements of local operators which can be extracted from experimental data. The total inclusive width and the first few moments of the kinematic distributions can be well approximated by a double series in α_s and Λ_{QCD}/m_b [1, 2]. After extracting the most important non-perturbative parameters, including the heavy quark masses, from the moments, one can therefore use them in the OPE expression for the total semileptonic width and determine $|V_{cb}|$ from the comparison with the experimental rate.

It is worth emphasizing that the information obtained from the fits to the moments of $B \rightarrow X_c \ell \nu$ (see [3, 4] for recent accounts) find other important applications. Indeed, the b quark mass and the OPE expectation values obtained from the moments are crucial inputs in the determination of $|V_{ub}|$ from inclusive semileptonic charmless decays, see *e.g.* [5] and refs. therein. The heavy quark masses and the OPE parameters are also relevant for a precise calculation of other inclusive decays like $B \rightarrow X_s \gamma$ [6].

The reliability of the inclusive method rests on our ability to control the higher order contributions in the OPE. If we neglect perturbative corrections, *i.e.* if we work at tree-level, we presently know the $O(1/m_b^2)$ and $O(1/m_b^3)$ contributions [7], while the $O(1/m_b^4)$ and $O(1/m_Q^5)$ effects have been studied in [8]. Unfortunately, new non-perturbative parameters appear at each order in the OPE: as many as nine new expectation values appear at $O(1/m_b^4)$. As a result, only the parameters associated with the $O(1/m_b^{2,3})$ corrections are routinely fitted from experiment. In [8] the parameters associated with $O(1/m_Q^{4,5})$ effects have been estimated in the ground state saturation approximation, finding a relatively small +0.4% net effect on $|V_{cb}|$. Recently, the validity of the ground state saturation has been investigated [9], and it has been shown that the non-factorizable contributions can be in general comparable to the factorizable ones. Additional work is therefore necessary to assess the importance of higher order power effects.

For what concerns the purely perturbative corrections to the free quark decay, they are known at $O(\alpha_s^2)$ in all the relevant cases, namely for the width and the first few moments of the lepton energy and hadronic mass distributions. The complete $O(\alpha_s)$ and $O(\alpha_s^2 \beta_0)$ corrections have been computed some time ago (see [10] and refs. therein), while the remain-

ing two-loop corrections to the width and to the first few moments have been calculated in Refs. [11, 12, 13, 14]. The theoretical uncertainty due to missing purely perturbative effects is now relatively small [14].

The $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ corrections appear to be a potentially more important source of theoretical uncertainty. The $O(\alpha_s)$ corrections to the Wilson coefficient of the kinetic operator have been computed numerically in [15]; they can be also obtained from the parton level $O(\alpha_s)$ result using reparameterization invariance relations [1, 16, 17, 18]. They lead to numerically modest $O(\alpha_s \mu_\pi^2/m_b^2)$ corrections to the width and moments, where μ_π^2 is the matrix element of the kinetic operator. However, in order to have all the $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ effects one should also consider the $O(\alpha_s)$ corrections to the Wilson coefficient of the chromomagnetic operator. A complete $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ calculation has been performed in the simpler case of inclusive radiative decays [19], where the $O(\alpha_s)$ corrections increase the coefficient of μ_G^2 , the matrix element of the chromomagnetic operator, by almost 20% in the rate.

In this paper we present the first part of an analytic calculation of the $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ corrections. We extend the method developed in [19] to semileptonic decays into hadronic final states containing a massive quark and validate it rederiving the $O(\alpha_s \mu_\pi^2/m_b^2)$ corrections and reproducing the reparameterization relations. As we did in [19], we compute the relevant Wilson coefficients at $O(\alpha_s)$ by Taylor expanding off-shell amputated Green functions around the b quark mass shell, and by matching them onto local operators in Heavy Quark Effective Theory (HQET). The extension to semileptonic decays implies new technical difficulties, because one needs to consider the mass of the final quark and the non-vanishing invariant mass of the lepton pair. The integrals involved are therefore less divergent but more complex. We identify a small number of master integrals and express our results in terms of the same functions appearing in the $O(\alpha_s)$ parton calculation. In order to compute contributions to arbitrary moments, we give explicit corrections to the three independent structure functions, namely of the triple differential rate. The $O(\alpha_s \mu_G^2/m_b^2)$ calculation is under way and will be presented in a forthcoming publication.

The outline of this paper is as follows. In section 2 we introduce our notation and review the known $O(1/m_b^2)$ and $O(\alpha_s)$ corrections to the triple differential rate. Section 3 is devoted to a description of the calculation of the $O(\alpha_s \mu_\pi^2/m_b^2)$ contributions. Our analytic results can be found in section 4. In section 5 we review the reparameterization invariance constraints and show that they are satisfied by our results. In section 6 we summarize and conclude. The relevant master integrals and a few technical details are given in the Appendix.

2 Leading Order results

We start recalling the tree-level results and the $O(\alpha_s)$ corrections to the leading, free-quark term of the OPE. We consider the decay of a B meson of four-momentum $p_B = M_B v$ into a lepton pair with momentum q and a hadronic final state containing a charm quark

with momentum $p' = p_B - q$. The hadronic tensor $W^{\mu\nu}$ which determines the hadronic contribution to the differential width is given by the absorptive part of a current correlator in the appropriate kinematic region,

$$W^{\mu\nu}(p_B, q) = \text{Im} \frac{2i}{\pi M_B} \int d^4x e^{-iq \cdot x} \langle \bar{B} | T J_L^{\mu\dagger}(x) J_L^\nu(0) | \bar{B} \rangle, \quad (2.1)$$

where $J_L^\mu = \bar{c} \gamma^\mu P_L b$ is the charged weak current. The correlator is subject to an OPE in terms of local operators, which at the level of the differential rate takes the form of an expansion in inverse powers of the energy release, whose leading term corresponds to the decay of a free quark.

We generally follow the notation of Ref. [10] and express the b -quark decay kinematics in terms of the dimensionless quantities

$$\rho = \frac{m_c^2}{m_b^2}, \quad \hat{u} = \frac{(p - q)^2 - m_c^2}{m_b^2}, \quad \hat{q}^2 = \frac{q^2}{m_b^2}, \quad (2.2)$$

where $p = m_b v$ is the momentum of the b quark and

$$0 \leq \hat{u} \leq \hat{u}_+ = (1 - \sqrt{\hat{q}^2})^2 - \rho \quad \text{and} \quad 0 \leq \hat{q}^2 \leq (1 - \sqrt{\rho})^2. \quad (2.3)$$

We will also employ the energy of the hadronic system normalized to the b mass

$$E = \frac{1}{2}(1 + \rho + \hat{u} - \hat{q}^2). \quad (2.4)$$

The case of tree-level kinematics corresponds to $\hat{u} = 0$; we indicate the corresponding energy of the hadronic final state as

$$E_0 = \frac{1}{2}(1 + \rho - \hat{q}^2). \quad (2.5)$$

The normalized total leptonic energy is

$$\hat{q}_0 = 1 - E \quad \text{from which follows} \quad \hat{u} = 2(1 - E_0 - \hat{q}_0). \quad (2.6)$$

We also introduce a threshold factor

$$\lambda = 4(\hat{q}_0^2 - \hat{q}^2) = 4(E^2 - \rho - \hat{u}). \quad (2.7)$$

In the case of tree-level kinematics, the threshold factor becomes $\lambda_0 = 4(E_0^2 - \rho)$. It is convenient to introduce a short-hand notation for the square root of λ :

$$t = \frac{\sqrt{\lambda}}{2E}, \quad t_0 = \frac{\sqrt{\lambda_0}}{2E_0}. \quad (2.8)$$

It is customary to decompose the hadronic tensor as follows

$$m_b W^{\mu\nu}(p_B, q) = -W_1 g^{\mu\nu} + W_2 v^\mu v^\nu + iW_3 \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma + W_4 \hat{q}^\mu \hat{q}^\nu + W_5 (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu), \quad (2.9)$$

where the structure functions W_i are functions of \hat{q}^2, \hat{q}_0 or equivalently of \hat{q}^2, \hat{u} , v^μ is the four-velocity of the B meson, and $\hat{q}^\mu = q^\mu/m_b$.

In the limit of massless leptons only $W_{1,2,3}$ contribute to the decay rate and one has

$$\frac{d\Gamma}{d\hat{E}_\ell d\hat{q}^2 d\hat{u}} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{16\pi^3} \theta(\hat{u}_+ - \hat{u}) \times \quad (2.10)$$

$$\times \left\{ \hat{q}^2 W_1 - \left[2\hat{E}_\ell^2 - 2\hat{E}_\ell \hat{q}_0 + \frac{\hat{q}^2}{2} \right] W_2 + \hat{q}^2 (2\hat{E}_\ell - \hat{q}_0) W_3 \right\},$$

where $\hat{u}_+ = (1 - \sqrt{\hat{q}^2})^2 - \rho$ represents the kinematic boundary on \hat{u} , and $\hat{E}_\ell = E_\ell/m_b$ is the normalized charged lepton energy. Thanks to the OPE, the structure functions can be expanded in series of α_s and Λ_{QCD}/m_b . There is no term linear in Λ_{QCD}/m_b and therefore

$$W_i = W_i^{(0)} + \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,0)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,0)} + \frac{C_F \alpha_s}{\pi} \left[W_i^{(1)} + \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,1)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,1)} \right] \quad (2.11)$$

where we have neglected terms of higher order in the expansion parameters. μ_π^2 and μ_G^2 are the B -meson matrix elements of the only gauge-invariant dimension 5 operators that can be formed from the b quark and gluon fields [1, 2]. The leading order coefficients are given by

$$W_i^{(0)} = w_i^{(0)} \delta(\hat{u}); \quad w_1^{(0)} = 2E_0, \quad w_2^{(0)} = 4, \quad w_3^{(0)} = 2. \quad (2.12)$$

The tree-level nonperturbative coefficients [2] read

$$W_i^{(\pi,0)} = w_i^{(\pi,0)} \delta(\hat{u}) + w_i^{(\pi,1)} \delta'(\hat{u}) + w_i^{(\pi,2)} \delta''(\hat{u}); \quad (2.13)$$

$$\begin{aligned} w_1^{(\pi,0)} &= \frac{8}{3}(1 - E_0), & w_1^{(\pi,1)} &= \frac{4}{3}E_0(1 - E_0), & w_1^{(\pi,2)} &= \frac{2}{3}E_0\lambda_0; \\ w_2^{(\pi,0)} &= 0, & w_2^{(\pi,1)} &= -8(1 - E_0), & w_2^{(\pi,2)} &= \frac{4}{3}\lambda_0; \\ w_3^{(\pi,0)} &= -2, & w_3^{(\pi,1)} &= -\frac{4}{3}(1 - E_0), & w_3^{(\pi,2)} &= \frac{2}{3}\lambda_0, \end{aligned}$$

and

$$W_i^{(G,0)} = w_i^{(G,0)} \delta(\hat{u}) + w_i^{(G,1)} \delta'(\hat{u}); \quad (2.14)$$

$$\begin{aligned} w_1^{(G,0)} &= -\frac{4}{3}(2 - 5E_0), & w_1^{(G,1)} &= -\frac{4}{3}(E_0 + 3E_0^2 + \frac{1}{2}\lambda_0); \\ w_2^{(G,0)} &= 0, & w_2^{(G,1)} &= \frac{8}{3}(3 - 5E_0); \\ w_3^{(G,0)} &= \frac{10}{3}, & w_3^{(G,1)} &= -\frac{4}{3}(1 + 5E_0). \end{aligned}$$

The perturbative corrections to the free quark decay have been computed in [10] and refs. therein. They read

$$W_i^{(1)} = w_i^{(0)} \left\{ S_i \delta(\hat{u}) - 2(1 - E_0 I_1) \left[\frac{1}{\hat{u}} \right]_+ + \frac{\theta(\hat{u})}{(\rho + \hat{u})} \right\} + R_i \theta(\hat{u}), \quad (2.15)$$

where $S_i = S + \Delta_i$ and

$$S = 2E_0(I_{2,0} - I_{4,0}) - 1 - \frac{1 - \rho - 6\hat{q}^2}{4\hat{q}^2} \ln \rho - \frac{(1 - \rho)^2 - 6\hat{q}^2(1 + \rho) + 5(\hat{q}^2)^2}{4\hat{q}^2} I_{1,0};$$

$$\Delta_1 = -\frac{\rho}{E_0} I_{1,0}; \quad \Delta_2 = \frac{1 - \rho}{4\hat{q}^2} \ln \rho + \left(\frac{(1 - \rho)^2}{4\hat{q}^2} - \frac{1 + \rho}{4} \right) I_{1,0}; \quad \Delta_3 = 0,$$

and the functions R_i are given in Eqs. (2.32-2.34) of Ref. [10].¹ The integrals I_1 , $I_{1,0}$, $I_{2,0}$, and $I_{4,0}$ are given below in Eqs. (A.6,A.7,A.8) and the plus distribution is defined by

$$\left[\frac{1}{\hat{u}} \right]_+ = \lim_{\varepsilon \rightarrow 0} \left[\ln \varepsilon \delta(\hat{u}) + \frac{1}{\hat{u}} \theta(\hat{u} - \varepsilon) \theta(1 - \hat{u}) \right] \quad (2.16)$$

or equivalently by its action on a test function $f(\hat{u})$:

$$\int d\hat{u} f(\hat{u}) \left[\frac{1}{\hat{u}} \right]_+ = \int_0^1 d\hat{u} \frac{f(\hat{u}) - f(0)}{\hat{u}}. \quad (2.17)$$

The upper limit of integration in the rhs of (2.17) can be chosen arbitrarily, but it is convenient to have it larger than the physical boundary, \hat{u}_+ . Ref. [10] uses \hat{u}_+ as upper limit, and the two definitions are related by the simple expression

$$\left[\frac{1}{\hat{u}} \right]_{+,[10]} = \left[\frac{1}{\hat{u}} \right]_+ - \ln \hat{u}_+ \delta(\hat{u}) - \frac{\theta(\hat{u} - \hat{u}_+) \theta(1 - \hat{u})}{\hat{u}}. \quad (2.18)$$

3 The calculation of $O(\alpha_s \mu_\pi^2 / m_b^2)$ effects

The four diagrams in Fig. 1 are our starting point. They contribute to the weak current correlator of Eq. (2.1) and are sufficient to determine both $W_i^{(1)}$ and $W_i^{(\pi,1)}$, while additional diagrams with external background gluons are necessary for the determination of $W_i^{(G,1)}$. The momenta of the external b quarks and W bosons are $p = m_b v + k$ and q , respectively. We write down the corresponding off-shell forward amplitudes, and extract the contributions to $W_{1,2,3}$ by contraction with appropriate tensor projectors. We then Taylor expand around $k = 0$, *i.e.* around the mass-shell of the b quarks, through $O(k^2)$. We always work in $d = 4 - 2\varepsilon$ dimensions and use dimensional regularization for both ultraviolet and infrared divergences. The result of the Taylor expansion is reduced to scalar integrals, which are in turn expressed in terms of 4 independent master integrals, listed in the Appendix, using Integration by Parts (IBP) identities [20].

The forward amplitudes obtained in this way correspond to quark matrix elements of local operators which eventually have to be evaluated in the B meson. In particular, the

¹The variables $\hat{\omega}$, λ_b , and τ of Ref. [10] correspond to $-2E_0$, λ and $(1 - t)/(1 + t)$, respectively.

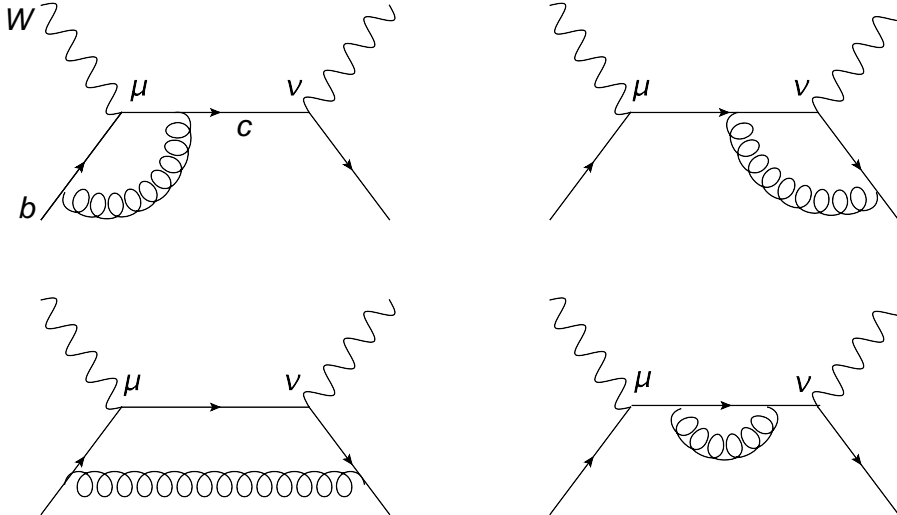


Figure 1: One-loop diagrams contributing to the current correlator.

k -dependent coefficients of the master integrals should be expressed in terms of B meson matrix elements of local operators. To this end it is convenient to use HQET: all $O(k)$ and $O(k^2)$ matrix elements can be expressed in terms of

$$\lambda_1 = \frac{1}{2m_B} \langle \bar{B}(v) | \bar{b}_v (iD)^2 b_v | \bar{B}(v) \rangle, \quad \lambda_2 = -\frac{1}{6m_B} \langle \bar{B}(v) | \bar{b}_v \frac{g_s}{2} G_{\mu\nu} \sigma^{\mu\nu} b_v | \bar{B}(v) \rangle, \quad (3.1)$$

where $D_\mu = \partial_\mu + ig_s G_\mu^a T^a$ is the covariant derivative, $G_{\mu\nu}$ the gluon field tensor, and b_v is the static quark field (see [19] and refs. therein for details). While λ_1 and λ_2 are defined in the asymptotic HQET regime, in practical applications one deals with $\mu_\pi^2 = -\lambda_1 + O(1/m_b)$ and $\mu_G^2 = 3\lambda_2 + O(1/m_b)$, defined in terms of full QCD states at finite m_b . The power corrections to these relations are irrelevant for our calculation. In this paper we consider only the terms proportional to λ_1 .

The next step to compute the physical structure functions W_i consists in taking the imaginary part of the result. There are two kinds of contributions to the imaginary part: the first comes from the imaginary part of a charm propagator, raised to power n , outside of the loop, via

$$\frac{1}{\pi} \text{Im} \frac{1}{[(v - \hat{q})^2 - \rho^2 + i\eta]^n} = \frac{(-1)^n}{n!} \delta^{(n-1)}(\hat{u}); \quad (3.2)$$

the other comes from the imaginary part of a loop integral, and is related to real gluon emission. In the first case the real part of the loop integrals is multiplied by $\delta(\hat{u})$ or its derivatives. It follows that the real parts of the master integrals, together with their derivatives wrt \hat{u} , are only needed at $\hat{u} = 0$, *i.e.* with partonic kinematics. The derivatives of the master integrals wrt \hat{u} can be, in turn, re-expressed in terms of the same master

integrals, for example using

$$\frac{d}{d\hat{u}} \Big|_{\text{fixed } q^2} = \frac{\partial q_0}{\partial \hat{u}} \frac{\partial q_\mu}{\partial q_0} \frac{\partial}{\partial q_\mu} \Big|_{\text{fixed } q^2} = -\frac{1}{2} \frac{(\hat{q}^2 p_\mu - \hat{q}_0 q_\mu)}{(\hat{q}^2 - \hat{q}_0^2)} \frac{\partial}{\partial q_\mu}. \quad (3.3)$$

As a result, the contribution to the final result coming from the real parts of the loop integrals can be expressed in terms of a single combination of dilogarithms and a single logarithm, defined as $I_{4,0}$ and $I_{1,0}$ in Eqs. (A.6,A.7). The latter are the same functions that appear in Eq. (2.15).

All the singularities are located at the threshold, $\hat{u} = 0$. We therefore identify in the master integrals the terms which potentially lead to infrared divergences and employ the identity

$$\hat{u}^{-A+B\epsilon} = \sum_{p=0}^{A-1} \frac{(-1)^p}{p!} \frac{\delta^{(p)}(\hat{u})}{1+p-A+B\epsilon} + \sum_{n=0}^{\infty} \frac{(B\epsilon)^n}{n!} \left[\frac{\ln^n \hat{u}}{\hat{u}^A} \right]_+, \quad (3.4)$$

valid for $A > 0$, where the plus distributions are defined by generalizing Eq. (2.17),

$$\int d\hat{u} \left[\frac{\ln^n \hat{u}}{\hat{u}^m} \right]_+ f(u) = \int_0^1 d\hat{u} \frac{\ln^n \hat{u}}{u^m} \left[f(u) - \sum_{p=0}^{m-1} \frac{u^p}{p!} f^{(p)}(0) \right] \quad (3.5)$$

with $f^{(p)}(u) = \frac{d^p f(u)}{du^p}$. These terms enter the imaginary part of the master integrals and control the infrared divergences due to real gluon emission.

In the calculation of $W_{1,2}$ the problem of the d -dimensional definition of γ_5 can be avoided by simply anticommuting it with all γ^μ matrices. In the case of the parity violating structure function W_3 , however, one needs to proceed with care and adopt a d -dimensional definition for the axial-vector current. One possibility is to follow [21] and employ the replacement

$$\bar{c}\gamma^\mu\gamma_5 b \rightarrow -i \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \bar{c}\gamma_\nu\gamma_\rho\gamma_\sigma b, \quad (3.6)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is a strictly 4-dimensional object. This Levi-Civita tensor is multiplied by another antisymmetric ϵ tensor, necessary to extract the W_3 component of the amplitude, and their product can be expressed as a combination of metric tensors. The latter can then be taken in $4 - 2\epsilon$ dimensions. This definition has several advantages but it violates chiral symmetry and its basic Ward identities. Therefore it requires the introduction of a finite one-loop renormalization constant Z_5

$$\bar{c}\gamma^\mu\gamma_5 b \rightarrow -i \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} Z_5 \bar{c}\gamma_\nu\gamma_\rho\gamma_\sigma b, \quad (3.7)$$

of course in addition to the wave-function renormalization of the external legs. Z_5 is given in [21] and Refs. therein for the massless case,

$$Z_5 = 1 - C_F \frac{\alpha_s}{\pi}, \quad (3.8)$$

and we have checked that this result applies to the case of massive quarks as well. Another possibility, which leads to the same result without an extra finite renormalization, is to anticommute γ_5 to the extreme left of the Dirac string in all diagrams and then to replace it by its four-dimensional definition.

After combining all the diagrams, the infrared divergences are cancelled and the ultraviolet divergences are removed by the b quark wave function renormalization and the charm mass renormalization. The charm mass renormalization also removes all the $\delta'''(\hat{u})$ terms. No renormalization is necessary in the effective theory [19].

4 The $O(\alpha_s \mu_\pi^2/m_b^2)$ results

We now report our results for the $O(\alpha_s)$ corrections to the Wilson coefficients of the kinetic operator. The most singular part of the W_i s has a universal structure

$$B_{(i,\pi)} = \frac{\lambda_0}{3} \left\{ \left[S_i + 3(1 - E_0 I_{1,0}) \right] \delta''(\hat{u}) - 4(1 - E_0 I_{1,0}) \left[\frac{1}{\hat{u}^3} \right]_+ \right\}, \quad (4.1)$$

and the complete results are

$$\begin{aligned} W_1^{(\pi,1)} &= 2E_0 B_{(1,\pi)} + \frac{8}{3}(1 - E_0 I_{1,0})(1 - E_0) \left(E_0 \left[\frac{1}{\hat{u}^2} \right]_+ - 2 \left[\frac{1}{\hat{u}} \right]_+ \right) + R_1^{(\pi)} \theta(\hat{u}) \\ &+ \frac{8E_0}{3} \left[\frac{1 - E_0}{2} S_1 - \lambda_0(1 - E_0 I_{1,0}) - \frac{E_0^2}{\rho} + (3E_0^2 - \rho) I_{1,0} - 2E_0(1 + I_{1,0}) + 3 \right] \delta'(\hat{u}) \\ &+ \frac{8}{3} \left[S_2 - E_0 S_1 + E_0(1 - E_0 I_{1,0}) \left(\frac{\lambda_0}{2} - \frac{(1 - E_0)^2}{\lambda_0} + E_0 \right) + \left(E_0^2 - \frac{3}{4} E_0 - \frac{\rho}{4} \right) I_{1,0} \right. \\ &\left. + \frac{\frac{3}{4} E_0^3 - \frac{\rho}{4} (E_0^2 + 2E_0 - 2\rho(1 - E_0))}{\rho^2} \right] \delta(\hat{u}), \end{aligned} \quad (4.2)$$

$$\begin{aligned} W_2^{(\pi,1)} &= 4B_{(2,\pi)} - 16(1 - E_0 I_{1,0})(1 - E_0) \left[\frac{1}{\hat{u}^2} \right]_+ + R_2^{(\pi)} \theta(\hat{u}) \\ &- 8 \left[(1 - E_0) S_2 + 2 - \frac{8}{3} E_0 + \frac{2}{3} \lambda_0(1 - E_0 I_{1,0}) + \left(\frac{\lambda_0}{2} + \frac{8}{3} \rho - \frac{8}{3} E_0 \right) I_{1,0} + \frac{2E_0^2}{3\rho} \right] \delta'(\hat{u}) \\ &- 8 \left[(1 - E_0 I_{1,0}) \left(2E_0 - \frac{\lambda_0}{3} - 7 \frac{(1 - E_0)^2}{\lambda_0} \right) - \frac{E_0^2 - \rho E_0 + \frac{\rho}{6}}{2\rho^2} - \frac{5}{3} + \left(\frac{2}{3} + \frac{13E_0}{12} \right) I_{1,0} \right] \delta(\hat{u}), \end{aligned} \quad (4.3)$$

$$\begin{aligned} W_3^{(\pi,1)} &= 2B_{(3,\pi)} - 4(1 - E_0 I_{1,0}) \left(\frac{2}{3}(1 - E_0) \left[\frac{1}{\hat{u}^2} \right]_+ - \left[\frac{1}{\hat{u}} \right]_+ \right) + R_3^{(\pi)} \theta(\hat{u}) \\ &- \frac{4}{3} \left[(1 - E_0) S + 2\lambda_0(1 - E_0 I_{1,0}) - 2 \left(I_{1,0} - \frac{E_0}{\rho} \right) (E_0 - \rho) \right] \delta'(\hat{u}) \end{aligned} \quad (4.4)$$

$$-2 \left[S - \left(\frac{4(1-E_0)^2}{\lambda_0} - \frac{4E_0}{3} + \frac{2\lambda_0}{3} + 1 \right) (1-E_0 I_{1,0}) + (1-E_0) I_{1,0} - \frac{E_0(E_0-\rho)}{\rho^2} \right] \delta(\hat{u}).$$

In the above expressions the coefficients of the derivatives of $\delta(\hat{u})$ have been reduced using integration by parts: for instance

$$f(\hat{u}) \delta''(\hat{u}) = f(0) \delta''(\hat{u}) - 2f'(0) \delta'(\hat{u}) + f''(0) \delta(\hat{u}). \quad (4.5)$$

The coefficients of the plus distributions can be similarly reduced by Taylor expanding them around $\hat{u} = 0$, for example:

$$f(\hat{u}) \left[\frac{1}{\hat{u}^2} \right]_+ = f(0) \left[\frac{1}{\hat{u}^2} \right]_+ + f'(0) \left[\frac{1}{\hat{u}} \right]_+ + \frac{f(\hat{u}) - f(0) - \hat{u}f'(0)}{\hat{u}^2}, \quad (4.6)$$

where the last term is regular for $\hat{u} \rightarrow 0$.

The functions $R_i^{(\pi)}$ are related to real gluon emission and are given by

$$\begin{aligned} R_1^{(\pi)} &= \frac{8}{3} E_0 \left(\frac{(1-E_0)(E_0-2\hat{u})}{\hat{u}^2} - \frac{E_0 \lambda_0}{\hat{u}^3} \right) I_{1,0} - \frac{\frac{4}{3} \hat{u}^2 + 2(1-5E)\hat{u} + \frac{28}{3}(E^2-E)}{\lambda} \left(I_1 - \frac{1}{E} \right) \\ &+ \frac{8}{3} \left(\frac{7}{8} E - \frac{3}{8} - \frac{\hat{u}}{4} + \frac{E-z}{\hat{u}} + \frac{E^2 \lambda}{\hat{u}^3} + \frac{E(E-E^2-\lambda)}{\hat{u}^2} \right) I_1 + \frac{16E^2(E-z)}{3\hat{u}^2 z} \\ &- \frac{\hat{u}^2(4E^2-Ez-2z^2)}{6Ez^3} + \frac{-8E^3-3E^2z+5Ez(z+1)+(5-2z)z^2}{3z^3} \\ &+ \frac{4E(2E^2-5Ez+4z^2)}{3\hat{u}z^2} + \frac{\hat{u}(24E^4-12E^3z+4E^2(z-3)z+E(z^2-6z^3)+3z^3)}{6Ez^4}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} R_2^{(\pi)} &= -\frac{16}{3} \frac{E_0 \lambda_0}{\hat{u}^3} (I_{1,0} - I_1) - \frac{16(1-E_0)E_0}{\hat{u}^2} \left(I_{1,0} - \frac{I_1}{3} \right) - \frac{2}{3} \left[16E_0 \frac{4(1-E_0)^2 - \frac{\lambda_0}{4}}{\lambda_0 \hat{u}} \right. \\ &- 64 \frac{(1-E_0)^2 E_0}{\hat{u}} \left(\frac{1}{\lambda_0} - \frac{1}{\lambda} \right) + \frac{70E^2 - E(71\hat{u} + 94) + 10\hat{u}^2 + 43\hat{u} + 24}{\lambda} + 2 \left. \right] I_1 \\ &+ \frac{8(E-1)^2(14(E^2-E) - \hat{u}(15E-2\hat{u}-3))}{\lambda^2} \left(I_1 - \frac{1}{E} \right) + \frac{8E_0 E \hat{u}}{z^4} \\ &+ \frac{2(6E\hat{u} - 8E^2 + \hat{u}(3\hat{u}-4))}{3z^3} + \frac{8(2E^2-7Ez+2z(z+2))}{3z^2} + \frac{128(E-1)^2}{3\lambda} - \frac{16(5E + \hat{u} + 4)}{3\lambda} \\ &+ \left(\frac{(2E-1)\hat{u}^2}{3E^3} + \frac{54E^2-25E+7}{3E^2} - \frac{(22E^2-21E+3)\hat{u}}{6E^3} \right) \left(\frac{4}{\lambda} + \frac{1}{z} \right) \\ &+ \frac{\frac{2\hat{u}^2+\hat{u}}{E} - 10E - 7\hat{u} + 10}{z^2} + \frac{32E(E-z)}{3\hat{u}^2 z}, \end{aligned} \quad (4.8)$$

$$R_3^{(\pi)} = -\frac{8E_0(\lambda_0 - E_0\hat{u} + (1 - \frac{3\hat{u}}{2})\hat{u})}{3\hat{u}^3} I_{1,0} + \left(\frac{8(3E-5E^2+2z)}{3\hat{u}^2} + \frac{8E\lambda}{3\hat{u}^3} - \frac{4 - \frac{8E}{3}}{\hat{u}} - \frac{2}{3} \right) I_1$$

$$\begin{aligned}
& -\frac{4(1-E)(8(E-E^2)-\hat{u}(1-9E+2\hat{u}))}{3\lambda\hat{u}}\left(I_1-\frac{1}{E}\right)+\frac{4E^2\hat{u}}{z^4}-\frac{2(4E^2+3E\hat{u}+\hat{u})}{3z^3} \\
& +\frac{8E^2-20Ez+8z(z+1)}{3\hat{u}z^2}+\frac{16E(E-z)}{3\hat{u}^2z}+\frac{7-3E-2\hat{u}}{3z^2}+\frac{7E-2\hat{u}-1}{3Ez}, \quad (4.9)
\end{aligned}$$

with $z = \hat{u} + \rho$. Of course, the functions $R_i^{(\pi)}$ are regular in the limit $\hat{u} \rightarrow 0$.

The above results can be used in Eq. (2.11) to compute the $O(\alpha_s\mu_\pi^2/m_b^2)$ corrections to the total rate and to the moments of various differential distributions. The phase space integration is rather delicate because of strong cancelations between different singular terms, especially in the presence of a cut on the lepton energy. We have compared the numerical results with those in Tables 1 and 2 of Ref. [15] and found agreement in all cases. In principle, it is also possible to take the limit $\rho \rightarrow 0$ and obtain analytic results for the $B \rightarrow X_u\ell\nu$ decay.

5 Reparameterization Invariance relations

Reparameterization Invariance (RI) [16, 17] connects different orders in the heavy quark expansion. In particular, as we have mentioned in the Introduction, RI links the coefficient of the kinetic operator to the coefficient of the leading, dimension 3 operator. In the total rate this corresponds to a rescaling factor $1 - \mu_\pi^2/2m_b^2$ on the leading power result, which is nothing but the relativistic dilation factor of the lifetime of a moving quark and applies at any order in perturbation theory. The relations for differential distributions and moments tend to be more intricate, see [15], especially in the presence of experimental cuts.

Recently Manohar has derived elegant RI relations [18] that apply directly at the level of the structure functions W_i . They are also valid to all orders in perturbation theory and give the α_s^n coefficient of μ_π^2 in terms of the leading α_s^n coefficient and its derivatives:

$$\begin{aligned}
W_1^{(\pi,n)} &= -W_1^{(n)} + \frac{2}{3}W_2^{(n)} - 2\hat{q}_0\frac{dW_1^{(n)}}{d\hat{u}} + \frac{\lambda}{3}\frac{d^2W_1^{(n)}}{d\hat{u}^2}, \\
W_2^{(\pi,n)} &= \frac{5}{3}W_2^{(n)} - \frac{14}{3}\hat{q}_0\frac{dW_2^{(n)}}{d\hat{u}} - \frac{\lambda}{3}\frac{d^2W_2^{(n)}}{d\hat{u}^2}, \\
W_3^{(\pi,n)} &= -\frac{10}{3}\hat{q}_0\frac{dW_3^{(n)}}{d\hat{u}} - \frac{\lambda}{3}\frac{d^2W_3^{(n)}}{d\hat{u}^2}.
\end{aligned} \quad (5.1)$$

To verify these relations from Eq. (2.15) we need the first two derivatives of the plus distribution of Eq. (2.17). They can be re-expressed in terms of the higher order plus distributions introduced in Eq. (3.5) and of delta functions:

$$\left[\frac{1}{\hat{u}}\right]'_+ = -\left[\frac{1}{\hat{u}^2}\right]_+ + \delta(\hat{u}) - \delta'(\hat{u}), \quad (5.2)$$

$$\left[\frac{1}{\hat{u}}\right]''_+ = 2\left[\frac{1}{\hat{u}^3}\right]_+ - \delta(\hat{u}) + 2\delta'(\hat{u}) - \frac{3}{2}\delta''(\hat{u}), \quad (5.3)$$

where we have neglected terms that do not contribute upon integration in the physical range (2.3). The coefficients $W_i^{(\pi,1)}$ obtained from Eq. (2.15) using the RI relations agree with the results given in the previous Section. Using Eqs. (5.1) one can also verify the relations between moments with and without cuts given in [15].

6 Summary

We have presented an analytic calculation of the $O(\alpha_s)$ corrections to the Wilson coefficient of the kinetic operator in inclusive $B \rightarrow X_c \ell \nu$ semileptonic decays, following and extending the method developed in Ref. [19]. We have confirmed the numerical results presented in [15] and reproduced the RI relations given by Manohar [18]. We have provided several details of the technique we have used; in particular, the Appendix contains all the master integrals.

The calculation represents the first part of a complete study of the perturbative corrections to the coefficients of the power suppressed dimension 5 and 6 operators, and has offered us the opportunity to perform various checks. Our technique is currently employed to compute the $O(\alpha_s \mu_G^2/m_b^2)$ corrections, as done already in the case of $B \rightarrow X_s \gamma$ in [19].

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Appendix

In this Appendix we list the master integrals relevant to our calculation. Whenever appropriate, they are expanded in ϵ up to the order which is necessary in our calculation. As explained above, we need both the imaginary and real parts of the master integrals. However, since their real parts always appear multiplied by $\delta(\hat{u})$ or its derivatives, the real parts and their derivatives wrt \hat{u} are only necessary at $\hat{u} = 0$, *i.e.* with partonic kinematics. We introduce

$$I_{(n_1, n_2, n_3)} = -i\mu^{2\epsilon} \int \frac{d^d k}{\pi^{d/2}} \frac{1}{(k^2 + i\eta)^{n_1} [(k-p)^2 - m_b^2 + i\eta]^{n_2} [(k-p+q)^2 - m_c^2 + i\eta]^{n_3}}. \quad (\text{A.1})$$

Let us also employ $\hat{z} = (p - q)^2/m_b^2 = \hat{u} + \rho$. The massive tadpole integral is always real and is given by

$$I_{(0,1,0)} = m_b^2 \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \left[\frac{1}{\epsilon} + 1 + \epsilon \left(1 + \frac{\pi^2}{12} \right) + O(\epsilon^2) \right], \quad (\text{A.2})$$

with $I_{(0,0,1)}$ given by the same expression with $m_b \leftrightarrow m_c$. We have two distinct two-point functions. The first one develops an imaginary part for $(p - q)^2 > m_c^2$:

$$\begin{aligned} I_{(1,0,1)} &= e^{\gamma_E \epsilon} \Gamma(\epsilon) \left(\frac{\mu^2}{m_c^2} \right)^\epsilon \int_0^1 dx (1-x)^{-\epsilon} \left[1 - \frac{\hat{z}}{\rho} x - i\eta \right]^{-\epsilon} \\ &= \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \left\{ \hat{z}^{\epsilon-1} \hat{u}^{1-2\epsilon} \left[e^{i\pi\epsilon} \left(\frac{1}{\epsilon} + 2 \right) - \frac{1}{2\epsilon} - 1 \right] \right. \\ &\quad \left. + \frac{\hat{z} + \rho}{2\hat{z}} \left[\frac{1}{\epsilon} + 2 - \frac{\hat{u} \ln \hat{z} + 2\rho \ln \rho}{\hat{z} + \rho} \right] + O(\epsilon) \right\}. \end{aligned} \quad (\text{A.3})$$

The value and derivatives wrt \hat{u} of $I_{(1,0,1)}$ at $\hat{u} = 0$ can be readily obtained from the above integral representation:

$$\begin{aligned} I_{(1,0,1)}|_{\hat{u}=0} &= -2\rho \frac{dI_{(1,0,1)}}{d\hat{u}}|_{\hat{u}=0} = \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \rho^{-\epsilon} \left[\frac{1}{\epsilon} + 2 + O(\epsilon) \right], \\ \frac{d^2 I_{(1,0,1)}}{d\hat{u}^2}|_{\hat{u}=0} &= \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \rho^{-2-\epsilon} \left[\frac{1}{\epsilon} + 1 + O(\epsilon) \right], \\ \frac{d^3 I_{(1,0,1)}}{d\hat{u}^3}|_{\hat{u}=0} &= -3 \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \rho^{-3-\epsilon} \left[\frac{1}{\epsilon} + \frac{1}{2} + O(\epsilon) \right]. \end{aligned}$$

The second 2-point function is always real in the kinematic domain we are interested in. It can be directly expanded in ϵ :

$$\begin{aligned} I_{(0,1,1)} &= e^{\gamma_E \epsilon} \Gamma(\epsilon) \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \int_0^1 dx [x\rho + (1-x)(1-x\hat{q}^2)]^{-\epsilon} \\ &= \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \left[\frac{1}{\epsilon} + 2 + \frac{1 - \rho - \hat{q}^2}{2\hat{q}^2} \ln \rho + \frac{E_0 t_0}{\hat{q}^2} \ln \frac{1+t_0}{1-t_0} + O(\epsilon) \right]. \end{aligned} \quad (\text{A.4})$$

The only three-point function is $I_{(1,1,1)}$; we reduce it in the following way

$$\begin{aligned} I_{(1,1,1)} &= -e^{\gamma_E \epsilon} \frac{\epsilon \Gamma(\epsilon)}{m_b^2} \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \int_0^1 dx dy x^{-\epsilon} [x\chi - (1-y)(\hat{u} + i\eta)]^{-1-\epsilon} \\ &= -\frac{1}{m_b^2} \left(\frac{\mu^2}{m_b^2} \right)^\epsilon \int_0^1 \frac{dy}{\chi} \left[\hat{u}^{-2\epsilon} e^{2i\pi\epsilon} \frac{1}{\epsilon} \left(\frac{1}{2\epsilon} + \frac{1}{2} \ln \frac{\chi}{(1-y)^2} + O(\epsilon) \right) \right. \\ &\quad \left. - \frac{1}{2\epsilon} + \ln \bar{\chi} - \frac{1}{2} \ln \chi + O(\epsilon) \right], \end{aligned} \quad (\text{A.5})$$

where $\chi = \hat{z} + y(1 - \hat{q}^2 - \hat{z}) + \hat{q}^2 y^2$, $\bar{\chi} = \rho + y(1 - \hat{q}^2 - \rho) + \hat{q}^2 y^2$.

We need the real and imaginary parts of $I_{(1,1,1)}$, as well as its derivatives wrt \hat{u} at $\hat{u} = 0$. The latter can be computed from the Feynman parameter integral or can be expressed in terms of the master integrals at $\hat{u} = 0$ by applying the derivative wrt \hat{u} at the integrand level in (A.1), see Eq. (3.3). We obtain, up to $O(\epsilon)$ terms,

$$\begin{aligned} I_{(1,1,1)} \Big|_{\hat{u}=0} &= \frac{1}{2m_b^2} \left(\frac{1}{\epsilon} I_{1,0} - I_{4,0} \right), \\ \frac{d}{d\hat{u}} I_{(1,1,1)} \Big|_{\hat{u}=0} &= \frac{1}{\lambda_0 m_c^2} \left[(\rho - E_0) \left(\frac{1}{\epsilon} - 2 \right) + \rho(1 - E_0) \left(\frac{I_{1,0}}{\epsilon} - I_{4,0} \right) + E_0 \ln \rho \right], \\ \frac{d^2}{d\hat{u}^2} I_{(1,1,1)} \Big|_{\hat{u}=0} &= \frac{2}{\lambda_0^2 m_b^2} \left\{ \frac{6\alpha - \beta/\rho^2}{4\epsilon} + \left(\frac{\alpha^2}{2} + \hat{q}^2 \right) \left(\frac{I_{1,0}}{\epsilon} - I_{4,0} \right) + \frac{\beta}{4\rho^2} (\ln \rho + 2) \right. \\ &\quad \left. - \frac{7}{2}\alpha - \frac{1}{2} + \frac{1 - \hat{q}^2(1 + 2\rho I_{1,0})}{2\rho} \right\}, \end{aligned}$$

where $\beta = -8E_0^2(E_0 + \rho) + 4\rho(5E_0 - \rho)$, $\alpha = 2(1 - E_0)$, and

$$I_1 = \int_0^1 dy \frac{1}{\chi(y)} = \frac{\ln \frac{1+t}{1-t}}{\sqrt{\lambda}}, \quad I_{1,0} = \int_0^1 dy \frac{1}{\bar{\chi}(y)} = \frac{\ln \frac{1+t_0}{1-t_0}}{\sqrt{\lambda_0}}, \quad (\text{A.6})$$

$$\begin{aligned} I_{4,0} &= \int_0^1 dy \frac{\ln \bar{\chi}(y)}{\bar{\chi}(y)} = \frac{1}{t_0 E_0} \left[\text{Li}_2(a_1) - \text{Li}_2(a_2) \right. \\ &\quad \left. + \ln \frac{1+t_0}{1-t_0} \ln E_0(1-t_0)^{\frac{1}{4}}(1+t_0)^{\frac{3}{4}} + \ln E_0(1+t_0) \ln \frac{1-E_0(1-t_0)}{1-E_0(1+t_0)} \right] \end{aligned} \quad (\text{A.7})$$

and

$$a_1 = \frac{2t_0/(1+t_0)}{1-E_0(1-t_0)}, \quad a_2 = \frac{2t_0 E_0}{1-E_0(1-t_0)},$$

For the imaginary part we also need the integral

$$I_\chi = \int_0^1 dy \frac{\ln[\chi(y)/(1-y)^2]}{\chi(y)}$$

and its first derivatives at $\hat{u} = 0$:

$$\begin{aligned} I_\chi \Big|_{\hat{u}=0} &= I_{4,0} - 2I_{2,0}, \\ \frac{dI_\chi}{d\hat{u}} \Big|_{\hat{u}=0} &= \frac{2(1-E_0)}{\lambda_0} (I_{4,0} - 2I_{1,0} - 2I_{2,0}) + 2 \frac{(\rho - E_0) \ln \rho}{\lambda_0 \rho}, \\ \frac{d^2 I_\chi}{d\hat{u}^2} \Big|_{\hat{u}=0} &= \frac{(3\hat{q}^2 + 2E_0^2 t_0^2) (I_{4,0} - 3I_{1,0} - 2I_{2,0}) + \hat{q}^2 (I_{1,0} - E_0/\rho)}{\lambda_0^2/4} - \frac{\beta - 6\alpha\rho^2}{\rho^2 \lambda_0^2} (\ln \rho - 1), \end{aligned}$$

where

$$I_{2,0} = \int_0^1 dy \frac{\ln(1-y)}{\bar{\chi}(y)} = \frac{\text{Li}_2(1-E_0(1+t_0)) - \text{Li}_2(1-E_0(1-t_0))}{\sqrt{\lambda_0}}. \quad (\text{A.8})$$

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