

# The Dimensional Recurrence and Analyticity Method for Multicomponent Master Integrals: Using Unitarity Cuts to Construct Homogeneous Solutions

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ABSTRACT: We consider the application of the DRA method to the case of several master integrals in a given sector. We establish a connection between the homogeneous part of dimensional recurrence and maximal unitarity cuts of the corresponding integrals: a maximally cut master integral appears to be a solution of the homogeneous part of the dimensional recurrence relation. This observation allows us to make a necessary step of the DRA method, the construction of the general solution of the homogeneous equation, which, in this case, is a coupled system of difference equations.

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>General setup</b>	<b>2</b>
<b>3</b>	<b>Cut integrals</b>	<b>4</b>
<b>4</b>	<b>A three-loop example</b>	<b>6</b>
<b>5</b>	<b>Conclusion</b>	<b>12</b>
<b>A</b>	<b>Coefficients in the dimensional recurrence relation</b>	<b>12</b>
<b>B</b>	<b>Homogeneous solutions via Saalschutzyan <math>{}_4F_3</math></b>	<b>14</b>

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## 1 Introduction

Recently, a method of evaluating Feynman integrals based on the use of dimensional recurrence relations [1] and analytic properties of Feynman integrals as functions of space-time dimension  $d$  (the DRA method) was introduced [2]. It was successfully applied in a series of calculations [3–9] where master integrals for various families of Feynman integrals were evaluated exactly in  $d$  (in terms of nested sums) and also up to high order in  $\epsilon = 2 - d/2$  (in terms of the conventional multiple zeta values (MZV), using PSLQ). This fast advance of the DRA method was partly due to the availability of a number of magnificent tools and methods: IBP reduction tools, in particular, FIRE [10], the sector decomposition analysis of singularities implemented in FIESTA [11], the application of Mellin-Barnes technique [12–17], the PSLQ algorithm [18]. The DRA method provides results in the form of converging (uniformly in  $d$ ) nested sums with factorized summands. Such a form allows one to evaluate many terms of the  $\epsilon$ -expansion with very high precision. This feature of the DRA method was demonstrated in the evaluation of master integrals for four-loop massless propagators which were previously evaluated in [19] up to transcendentality weight seven. Using the results of DRA method it was possible to perform an evaluation up to weight twelve [9], and it is certainly possible to go further.

However, up to now, all applications of the DRA method concerned cases with only one master integral with a given set of denominators (in a given *sector*). The reason is that the DRA method requires finding the general solution of the homogeneous part of the dimensional recurrence relation. For the case of several master

integrals in one sector this problem becomes very nontrivial. The corresponding homogeneous equation has a matrix form and is equivalent to one difference equation of order higher than one. One may speculate that this problem is, in a sense, artificial and the homogeneous system can be decoupled or, at least, reduced to a triangular form by a proper choice of the master integrals (i.e., by passing to some linear combinations of the integrals with rational coefficients). In this case the high-order difference equation for one master integral should have a hypergeometric-term solution, which can be checked by the Petkovšek’s algorithm Hyper [20]. Unfortunately, in real-life examples, the homogeneous equation appears to have no hypergeometric or d’Alembertian solutions. So, taken as a separate mathematical problem, finding the solution of the homogeneous equation for the case of several master integrals cannot be performed in a systematic way. Therefore, when constructing the homogeneous solution, one has to rely on some additional methods. The goal of this paper is to present a method to find the homogeneous solution using unitarity cuts. The idea is very simple and yet appears to be very useful.

The paper is organized as follows. In the next Section we introduce our notation. In Section 3 we show that the maximal cut of the master integral satisfies the homogeneous part of the dimensional recurrence relation and this property gives a practical tool of finding a solution of the homogeneous part of the dimensional recurrence relations. In Section 4, we illustrate our technique on the evaluation of two master integrals (called  $I_{14}$  and  $I_{15}$  in [21]) for the three-loop static quark potential [21–24]. We reproduce the results presented in [21] and obtained with the help of the Mellin-Barnes representation [12–17] and obtain one more term in  $\epsilon$ -expansion (weight seven).

## 2 General setup

Let us suppose that we are interested in the evaluation of an  $L$ -loop Feynman integral depending on  $E$  linearly independent external momenta  $p_1, \dots, p_E$ . There are  $N = L(L + 1)/2 + LE$  scalar products involving the loop momenta  $l_i$ :

$$s_{ik} = l_i \cdot q_k, \quad i = 1, \dots, L, \quad k = 1, \dots, L + E, \quad (2.1)$$

where  $q_{1, \dots, L} = l_{1, \dots, L}$ ,  $q_{L+1, \dots, L+E} = p_{1, \dots, E}$ . The integral has the form

$$J(\nu; \mathbf{n}) = \int \frac{d^d l_1 \dots d^d l_L}{\pi^{LD/2} \prod_{\alpha=1}^N [D_\alpha + \epsilon_\alpha i 0]^{n_\alpha}} \quad (2.2)$$

where  $\epsilon_\alpha = \pm 1$  and  $\nu = d/2$  is a convenient variable. The quantities  $\epsilon_\alpha i 0 = \pm i 0$  determine the infinitesimal shifts of the denominators poles. The scalar functions  $D_\alpha$  are linear polynomials with respect to  $s_{ik}$ . The functions  $D_\alpha$  are assumed to be linearly independent and to form a complete basis in the sense that any non-zero linear combination of them depends on the loop momenta, and any  $s_{ik}$  can be

expressed in terms of  $D_\alpha$ . The indices  $n_\alpha$  are assumed to be integer, and if  $n_\alpha > 0$  we say that the integral has a denominator  $D_\alpha$ . The integrals having the same set of denominators form a *sector*.

The dimension shifting relation can be written in two equivalent forms [4, 25]:

$$J(\nu - 1; \mathbf{n}) = \tilde{Q}(A_1, \dots, A_N) J(\nu; \mathbf{n}), \quad (2.3)$$

or

$$J(\nu + 1; \mathbf{n}) = \tilde{P}(B_1, \dots, B_N) J(\nu; \mathbf{n}), \quad (2.4)$$

where  $\tilde{Q}(A_1, \dots, A_N)$  and  $\tilde{P}(B_1, \dots, B_N)$  are some polynomials. The operators  $A_\alpha$  and  $B_\alpha$  act as follows:

$$\begin{aligned} A_\alpha J(\nu; n_1, \dots, n_\alpha, \dots, n_N) &= n_\alpha J(\nu; n_1, \dots, n_\alpha + 1, \dots, n_N), \\ B_\alpha J(\nu; n_1, \dots, n_\alpha, \dots, n_N) &= J(\nu; n_1, \dots, n_\alpha - 1, \dots, n_N). \end{aligned} \quad (2.5)$$

In order to obtain the dimensional recurrence relation for some master integral  $J_1(\nu) = J(\nu; \mathbf{n}_1)$ , we have to plug it in Eq. (2.3) and reduce the right-hand side using IBP identities. Observe that the integrals appearing on the right-hand side of Eq. (2.3) belong to the same sector as  $J(\nu; \mathbf{n})$  or simpler (lower) sectors. Therefore, the result of the IBP reduction is also a linear combination of master integrals belonging to the same, or simpler, sectors. Therefore, if there are no other master integrals in the same sector as  $J_1$ , the general form of the dimensional recurrence relation is

$$J_1(\nu + 1) = C(\nu) J_1(\nu) + R(\nu), \quad (2.6)$$

where  $R(\nu)$  contains only simpler master integrals, and  $C(\nu)$  is a rational function. Naturally, the dimensional recurrence relations for simpler master integrals do not depend on  $J_1$ . The homogeneous part of this equation can be easily solved in terms of  $\Gamma$ -functions. The situation is different if there is more than one master integral in a given sector. In this case we will refer to the column of master integrals in a given sector as a *multicomponent master integral* (MMI). The dimensional recurrence relations for MMI form a coupled system of equations which can be written in the matrix notation as

$$\mathbf{J}(\nu + 1) = \mathbb{C}(\nu) \mathbf{J}(\nu) + \mathbf{R}(\nu), \quad (2.7)$$

where  $\mathbf{J} = \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix}$  is an MMI and  $\mathbb{C}(\nu)$  is a matrix with rational elements.

In order to apply the DRA method, we have to find a general solution of the homogeneous equation  $\mathbf{J}_h(\nu + 1) = \mathbb{C}(\nu) \mathbf{J}_h(\nu)$ . This system of difference equations can be reduced to one difference equation of  $k$ -th order, for example, for  $J_{1,h}$ . In particular, for  $k = 2$ , we have

$$J_{1,h}(\nu + 2) + \tilde{C}_1(\nu) J_{1,h}(\nu + 1) + \tilde{C}_2(\nu) J_{1,h}(\nu) = 0, \quad (2.8)$$

where  $\tilde{C}_{1,2}(\nu)$  are rational functions expressed via matrix elements of  $\mathbb{C}(\nu)$ . In general, difference equations of a high order cannot be solved analytically. There is, however, a possibility to check whether the equation has a solution in the form of a hypergeometric term (i.e., such a solution  $f(\nu)$  that  $f(\nu+1)/f(\nu)$  is a rational function). This possibility is based on the Petkovšek's algorithm Hyper [20]. In fact, the (non-)existence of a hypergeometric-term solution allows one to claim also the (non-)existence of a more general solution — d'Alembertian solution. Unfortunately, the application of the Hyper algorithm to real-life examples (in particular, to the one considered in Section 4) proves that such solutions do not exist. Therefore, solving homogeneous matrix difference equations is a very nontrivial mathematical problem.

### 3 Cut integrals

Let us now improve our notation indicating in the arguments of  $J$  also the signs of infinitesimal imaginary shifts in Eq. (2.2):  $J(\nu; \mathbf{n}) \rightarrow J(\nu; \mathbf{n}; \boldsymbol{\epsilon})$ . Below we omit the first argument  $\nu$  where it does not lead to confusion.

Then, an  $\alpha$ -cut integral can be defined as

$$\Delta_\alpha J(\mathbf{n}; \boldsymbol{\epsilon}) = J(\mathbf{n}; \dots, \epsilon_\alpha, \dots) - J(\mathbf{n}; \dots, -\epsilon_\alpha, \dots). \quad (3.1)$$

Similarly, we can define an integral cut over several lines:  $\Delta_{\{\alpha_1, \dots, \alpha_n\}} J(\mathbf{n}; \boldsymbol{\epsilon}) = \Delta_{\alpha_1} \dots \Delta_{\alpha_n} J(\mathbf{n}; \boldsymbol{\epsilon})$ .

For non-positive  $n_\alpha$  the infinitesimal shifts do not change the integral, so,

$$\Delta_\alpha J(\mathbf{n}; \boldsymbol{\epsilon}) = 0 \text{ if } n_\alpha \leq 0.$$

In contrast, for the positive  $n_\alpha$  the cut integral  $\Delta_\alpha J(\mathbf{n}; \boldsymbol{\epsilon})$  is not zero and can be obtained from  $J(\mathbf{n}; \boldsymbol{\epsilon})$  by the replacement

$$[D_\alpha + \epsilon_\alpha i 0]^{-n_\alpha} \rightarrow [D_\alpha + \epsilon_\alpha i 0]^{-n_\alpha} - [D_\alpha - \epsilon_\alpha i 0]^{-n_\alpha} = 2\pi i \epsilon_\alpha \frac{(-1)^{n_\alpha}}{\Gamma(n_\alpha)} \delta^{(n_\alpha-1)}(D_\alpha), \quad (3.2)$$

where  $\delta^{(n)}(x) = \frac{d^n}{dx^n} \delta(x)$  denotes the  $n$ -th derivative of Dirac's  $\delta$  function. For  $n_\alpha = 1$  this prescription reduces to the well-known replacement  $(D_\alpha + i0)^{-1} \rightarrow -2\pi i \delta(D_\alpha)$ .

It is clear that the IBP identities are not sensitive to the specific choice of  $\epsilon_\alpha$  in the sense that the coefficients in these identities do not depend on  $\boldsymbol{\epsilon}$ . We note, however, that the symmetry relations are sensitive to this choice since a symmetry that replaces  $D_\alpha \rightarrow D_\beta$  also replaces  $\epsilon_\alpha \rightarrow \epsilon_\beta$ . Therefore, temporarily we consider the master integrals, identical due to the symmetries, to be different. Then the IBP reduction of an integral  $J(\mathbf{n})$  is also insensitive to the choice of  $\epsilon_\alpha$ , i.e.

$$J(\mathbf{n}, \boldsymbol{\epsilon}) = \sum_i C^i(\mathbf{n}) J_i(\boldsymbol{\epsilon}),$$

where only master integrals  $J_i(\boldsymbol{\epsilon})$  depend on the choice of  $\boldsymbol{\epsilon}$ , but not the coefficients  $C^i(\mathbf{n})$ , which are rational functions of  $\mathbf{n}$ ,  $d$ , and external invariants. The master

integrals entering the right-hand side either belong to the same sector as  $J(\mathbf{n}, \epsilon)$  or to simpler sectors. Applying the  $\Delta_\alpha$  operator to this equation we nullify all integrals without  $D_\alpha$ -denominator. Thus, cutting all the denominators of  $J(\mathbf{n}, \epsilon)$  keeps on the right-hand side only master integrals of the same sector as  $J(\mathbf{n}, \epsilon)$ .

The cut integrals are also the basic tool of the powerful generalized unitarity technique [26, 27] which provides the possibility to construct scattering amplitudes. In fact, this strategy of writing an Ansatz as a linear combination of some basic scalar integrals and constructing the corresponding coefficient functions is very similar to the strategy of solving IBP relations, especially within Baikov's method [28, 29].

The dimensional recurrence relations are also insensitive to the choice of  $\epsilon$ . This is obvious already from the fact that, at the derivation of this relation, we never needed to specify explicitly the shifts  $\pm i0$  in the denominators. Therefore, restoring  $\epsilon$ -dependence in Eq. (2.7), we obtain

$$\mathbf{J}(\nu + 1; \epsilon) = \mathbb{C}(\nu) \mathbf{J}(\nu; \epsilon) + \mathbf{R}(\nu, \epsilon), \quad (3.3)$$

where the inhomogeneous term  $\mathbf{R}(\nu, \epsilon) = \sum_j \mathbf{C}^j(\nu) J_j(\nu; \epsilon)$  includes only integrals of the lower sectors. The matrix  $\mathbb{C}(\nu)$ , as well as  $\mathbf{C}^j(\nu)$ , do not depend on  $\epsilon$ . Taking the cut  $\Delta_{\{\dots\}}$  over all the denominators of  $\mathbf{J}$ , we nullify this term and obtain:

$$\Delta_{\{\dots\}} \mathbf{J}(\nu + 1; \epsilon) = \mathbb{C}(\nu) \Delta_{\{\dots\}} \mathbf{J}(\nu; \epsilon). \quad (3.4)$$

Thus, we arrive at a simple but important observation: *the maximal cut  $\Delta_{\{\dots\}} \mathbf{J}(\nu; \epsilon)$  of a MMI  $\mathbf{J}(\nu; \epsilon)$  is the solution of the homogeneous part of the dimensional recurrence relation for  $\mathbf{J}(\nu; \epsilon)$ .*

Two remarks are in order. First,  $\delta$ -functions in a cut integral may be too restrictive to give a non-zero result for a specific choice of the metric signature. This becomes obvious in the Euclidean case, where the denominators are always positive. But this is also true for Minkovskian metrics as we will see below. Therefore, to satisfy all the restrictions imposed by the  $\delta$ -functions one may have to choose a more general metric signature  $(1, 1, \dots, -1, \dots)$ . Second, a cut integral gives only one solution of the difference equation, while for a  $k$ -th order equation, there are  $k$  independent solutions. For  $k = 2$  we can, in principle, find a second solution in an algorithmic way by the 'constant variation' method. However, we find it possible, and even more convenient, to guess several solutions of the homogeneous equation by examining the Mellin-Barnes representation for the cut integral. The guessed solutions can then be checked to satisfy the homogeneous equation either numerically, or strictly, by using Zeilberger's method of creative telescoping [30, 31].

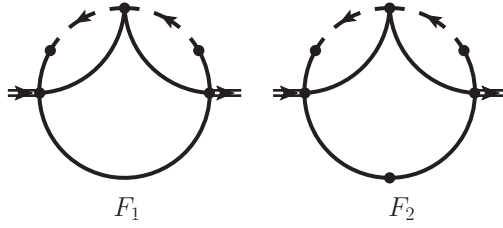


Figure 1. Master integrals  $F_1$  and  $F_2$

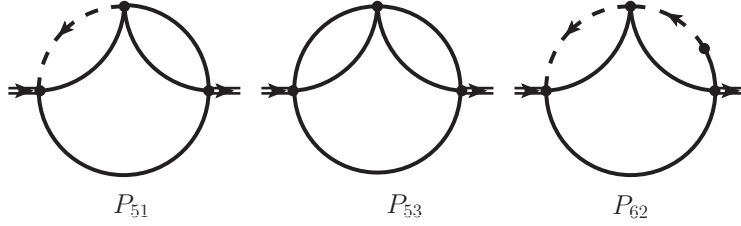


Figure 2. Simpler master integrals  $P_{51}$ ,  $P_{53}$ ,  $P_{62}$

## 4 A three-loop example

Let us evaluate the two master integrals shown in Fig. 1:

$$F_a = \int \int \int \frac{(i\pi^{d/2})^{-3} d^d k d^d l d^d r}{(-k^2)(-r^2)(-l+q)^a(-(k-l)^2)(-l-r)^2(-v \cdot k)(-v \cdot r)}, \quad (4.1)$$

where  $a = 1$  and  $2$ , the external momentum  $q$  is of the form  $(0, \mathbf{q})$ ,  $v = (1, \mathbf{0})$ , and  $-i0$  is implied in all the propagators.

The simpler master integrals are depicted in Fig. 2. Here we follow the labeling of the master integrals applied in our future paper [32]. Moreover, in this labeling,  $F_1 = P_{71}$  and  $F_2 = P_{72}$  but we keep the notation  $F_i$  which is more convenient within the present paper. The dimensional recurrence relation reads:

$$\mathbf{F}(\nu + 1) = \mathbb{C}(\nu) \mathbf{F}(\nu) + \mathbf{R}(\nu), \quad (4.2)$$

where  $\mathbf{F}(\nu) = \begin{pmatrix} F_1(\nu) \\ F_2(\nu) \end{pmatrix}$ ,  $\mathbf{R}(\nu) = \begin{pmatrix} R_1(\nu) \\ R_2(\nu) \end{pmatrix}$  depends on simpler master integrals,

and  $\mathbb{C}(\nu) = \begin{pmatrix} C_{11}(\nu) & C_{12}(\nu) \\ C_{21}(\nu) & C_{22}(\nu) \end{pmatrix}$  is a matrix with rational elements. The functions  $C_{ij}(\nu)$  and  $\mathbf{R}(\nu)$  are presented in the Appendix. Observe that although  $C_{ij}(\nu)$  are quite cumbersome, the determinant of the matrix  $\mathbb{C}(\nu)$  has a simple factorized form:

$$\det \mathbb{C}(\nu) = -\frac{(\nu - 2)(4\nu - 7)^2(4\nu - 5)^2}{16(\nu - 1)^5(2\nu - 3)^2(8\nu - 13)(8\nu - 11)(8\nu - 9)(8\nu - 7)}. \quad (4.3)$$

This seems to be a general situation.

The homogeneous equation reads

$$\mathbf{F}_h(\nu + 1) = \mathbb{C}(\nu) \mathbf{F}_h(\nu). \quad (4.4)$$

The solution of this equation is equivalent to the solution of the second-order difference equation for  $F_{1,h}$

$$F_{1,h}(\nu + 2) + C_1(\nu) F_{1,h}(\nu + 1) + C_2(\nu) F_{1,h}(\nu) = 0, \quad (4.5)$$

where  $C_1$  and  $C_2$  are known functions. As we already mentioned earlier, the solution seems to be out of reach of the conventional mathematical methods based on the use of the Hyper algorithm.

In order to apply the DRA method, we need to find two fundamental solutions of Eq. (4.4), forming a matrix  $\mathbb{F}_h(\nu) = (\mathbf{F}_h^1(\nu), \mathbf{F}_h^2(\nu))$ . Then, using the method described in Section 5 of Ref. [2], we can find the summing factor  $\mathbb{S}(\nu)$ , satisfying the equation

$$\mathbb{S}(\nu) = \mathbb{S}(\nu + 1) \mathbb{C}(\nu). \quad (4.6)$$

As we explained in the previous section, the maximally cut MMI  $\Delta\mathbf{F}$  satisfies the homogeneous equation (4.4), or, equivalently,  $\Delta F_1(\nu)$  satisfies Eq. (4.5). Observe that contracting the lower line of  $F_1$  in Fig. 1 we obtain a scaleless integral which is zero. Therefore, there is no need to cut this line as this cut nullifies no simpler master integrals. In what follows, we also omit the factors  $-2\pi i$  from each cut. Thus, we consider  $F_1$  and perform the replacements  $1/(k^2 + i0) \rightarrow \delta(k^2)$  and  $1/(v \cdot p + i0) \rightarrow \delta(v \cdot p) = \delta(p_0)$  for all the propagators apart from  $1/(-(l + q)^2)$ .

Let us, first, integrate over the loop momenta of the two identical one-loop subdiagrams consisting of one static and two usual propagators

$$J(l) = \int \frac{d^d k}{\pi^{d/2}} \delta(k_0) \delta(k^2) \delta(l^2 - 2l \cdot k), \quad (4.7)$$

where  $\delta(k_0)$  comes from  $1/(v \cdot k + i0) = 1/(k_0 + i0)$ . Here is a subtle point because in Minkowskian metrics we might conclude that this integral is zero due to the kinematical restrictions. Indeed, in Minkowskian space the first two  $\delta$ -functions result in  $k = 0$ , which is incompatible with the last  $\delta$ -function. Let us instead use the metric signature  $(1, 1, -1, -1, \dots)$ , so that  $k^2 = k_0^2 + k_1^2 - k_2^2 - \dots - k_d^2 = k_0^2 + k_1^2 - \vec{k}^2$ . Then a straightforward integration gives

$$J(l) = 2^{2-d} \frac{\Omega(d-2)}{\pi^{d/2}} \frac{(-l^2)^{d-4}}{(\mathbf{l}^2)^{(d-3)/2}}, \quad (4.8)$$

where  $\mathbf{l}^2 = -l_1^2 + \vec{l}^2$ , and  $\Omega(d) = 2\pi^{d/2}/\Gamma(d/2)$  is the volume of the unit hypersphere in Euclidean  $d$ -dimensional space.



To take the final integral

$$\Delta F_1(\nu) = \frac{1}{i^6} \int \frac{d^d l}{\pi^{d/2}} \frac{J(l)^2}{-(l+q)^2} \quad (4.9)$$

we turn to Euclidean space and separate the two terms in the denominator of  $1/(l_0^2 + (\mathbf{l} + \mathbf{q})^2)$  introducing a onefold MB representation. The factor  $\frac{1}{i^6}$  corresponds to six 'time-like' integration variables, two per each loop momenta.

Then the internal integration is taken straightforwardly and we arrive at the following result:

$$\begin{aligned} \Delta F_1(\nu) &= \frac{2^{4-4\nu} \Gamma(6-3\nu)}{\Gamma(\nu-1)^2 \Gamma(8-4\nu, 4\nu - \frac{13}{2})} \\ &\times \int \frac{dz}{2\pi i} \frac{\Gamma(-z) \Gamma(z + \frac{1}{2}) \Gamma(3\nu - \frac{11}{2} - z) \Gamma(z - 4\nu + 8) \Gamma(z + \nu - 1)}{\Gamma(z + 5 - 2\nu)}. \end{aligned} \quad (4.10)$$

It is easy to convert this representation to a linear combination of  ${}_3F_2$  hypergeometric functions.

As we mentioned earlier, this gives us only one solution, while a second-order equation should have two linearly independent solutions. In order to find both solutions, let us observe that there are two series of poles from the right of the integration contour and three series of poles from the left:

$$\begin{aligned} z_1 &= n, & z_2 &= 3\nu - \frac{11}{2} + n, \\ z_3 &= -\frac{1}{2} - n, & z_4 &= 4\nu - 8 - n, & z_5 &= 1 - \nu - n, \end{aligned}$$

where  $n = 0, 1, \dots$ . It turns out that the contribution of any of these series constitute a solution of Eq. (4.5). This can be checked either numerically, or using the Zeilberger's method of creative telescoping [30, 31]. We assume, of course, that the corresponding sums are defined in some region of  $\nu$  where they converge, and then analytically continued to the whole  $\nu$  complex plane. As two independent solutions we choose the contribution of the series of residues at  $z_1$  and  $z_4$ . The solutions have the form

$$F_{1,h}^1(\nu) = \frac{\sqrt{\pi} 2^{4-4\nu} \Gamma(6-3\nu) \Gamma(3\nu - \frac{11}{2})}{\Gamma(5-2\nu) \Gamma(\nu-1) \Gamma(4\nu - \frac{13}{2})} {}_3F_2 \left( \begin{matrix} 8-4\nu, \frac{1}{2}, \nu-1 \\ 5-2\nu, \frac{13}{2}-3\nu \end{matrix} \middle| 1 \right), \quad (4.11)$$

$$F_{1,h}^2(\nu) = \frac{32 \Gamma(6-3\nu) \Gamma(5\nu-9) \Gamma(\frac{5}{2}-\nu)}{2^{4\nu} (8\nu-15) \Gamma(\nu-1)^2 \Gamma(2\nu-3)} {}_3F_2 \left( \begin{matrix} 8-4\nu, \frac{5}{2}-\nu, 4-2\nu \\ 10-5\nu, \frac{17}{2}-4\nu \end{matrix} \middle| 1 \right) \quad (4.12)$$

Analytical properties of  $F_{1,h}^1(\nu)$  and  $F_{1,h}^2(\nu)$  can be found from the above representation. Conventional series representation of the hypergeometric functions  ${}_3F_2$  in Eqs. (4.11), (4.12) converges at  $\Re\nu < 5/2$ . In order to determine the analytical properties of  $F_{1,h}^1(\nu)$  and  $F_{1,h}^2(\nu)$  in the region  $\Re\nu \geq 5/2$ , one has to use the recurrence

relation (4.5). It would be more convenient to use the representation in terms of series converging uniformly in  $d$ . Luckily, both  ${}_3F_2$  in Eqs. (4.11) and (4.12) appear to be nearly-poised, and it is possible to transform them to Saalschutzyan  ${}_4F_3$ , whose series converge uniformly in  $d$ . Explicit expressions of  $F_{1,h}^1(\nu)$  and  $F_{1,h}^2(\nu)$  in terms of Saalschutzyan  ${}_4F_3$  are presented in the Appendix. Therefore, the fundamental matrix of Eq. (4.4) has the form  $\mathbb{F}_h(\nu) = \begin{pmatrix} F_{1,h}^1(\nu) & F_{1,h}^2(\nu) \\ F_{2,h}^1(\nu) & F_{2,h}^2(\nu) \end{pmatrix}$ , where  $F_{2,h}^1(\nu)$  and  $F_{2,h}^2(\nu)$  are obtained from the first equation of the system (4.4):

$$F_{2,h}^1(\nu) = \frac{F_{1,h}^1(\nu+1) - C_{11}(\nu) F_{1,h}^1(\nu)}{C_{12}(\nu)}, \quad F_{2,h}^2(\nu) = \frac{F_{1,h}^2(\nu+1) - C_{11}(\nu) F_{1,h}^2(\nu)}{C_{12}(\nu)}. \quad (4.13)$$

Now, following the recipe formulated in Section 5 of Ref. [2], we obtain the summing factor

$$\mathbb{S}(\nu) = \mathbb{W}(\nu) S(\nu) \begin{pmatrix} F_{2,h}^2(\nu) & -F_{1,h}^2(\nu) \\ -F_{2,h}^1(\nu) & F_{1,h}^1(\nu) \end{pmatrix}, \quad (4.14)$$

where  $S(\nu) = 2^{2\nu}(\nu-2)\Gamma(2\nu-3)^2\Gamma(4\nu-\frac{13}{2}) / \left(\Gamma(2\nu-\frac{7}{2})^2\Gamma(2-\nu)^2\sin(\pi\nu)\right)$  is a solution of the equation  $S(\nu) = S(\nu-1)\det \mathbb{C}(\nu)$  and  $\mathbb{W}(\nu)$  is an arbitrary periodic matrix. Using Eqs. (4.2) and (4.14), we obtain the relation

$$(\mathbb{S}\mathbf{F})(\nu-1) = (\mathbb{S}\mathbf{F})(\nu) + \mathbb{S}(\nu-1)\mathbf{R}(\nu), \quad (4.15)$$

which implies

$$(\mathbb{S}\mathbf{F})(\nu) = \mathbf{W}(\nu) + \sum_{+\infty} \mathbb{S}(\nu-1)\mathbf{R}(\nu), \quad (4.16)$$

where  $\mathbf{W}(\nu)$  is an arbitrary periodic column-vector and the notation  $\sum_{\pm\infty} f(\nu)$  introduced in Ref. [25] means

$$\begin{aligned} \sum_{+\infty} f(\nu) &= -\sum_{n=0}^{\infty} f(\nu+n), \\ \sum_{-\infty} f(\nu) &= \sum_{n=1}^{\infty} f(\nu-n). \end{aligned} \quad (4.17)$$

Now we need to determine  $\mathbf{W}(\nu)$  from the analytical properties of  $(\mathbb{S}\mathbf{F})(\nu)$  which depend on our choice of  $\mathbb{W}(\nu)$ . In particular, if we choose  $\mathbb{W}(\nu) = 1$ , the function  $(\mathbb{S}\mathbf{F})$  has singularities at  $\nu = 2, 2\frac{1}{6}, 2\frac{1}{5}, 2\frac{1}{3}, 2\frac{2}{5}, 2\frac{1}{2}, 2\frac{3}{5}, 2\frac{2}{3}, 2\frac{4}{5}, 2\frac{5}{6}$  on the stripe  $\Re\nu \in [2, 3)$ . In order to cancel these singularities, we can choose  $\mathbb{W}(\nu)$  to be properly degenerate (and sometimes completely vanishing) matrix at the points of singularities, but we should also try to not spoil the behavior of  $(\mathbb{S}\mathbf{F})$  at  $\nu \rightarrow \pm i\infty$ . Therefore, it is very useful to eliminate also the explicit and hidden zeros of  $\mathbb{S}$ , which, at

$\mathbb{W}(\nu) = 1$ , are located at the points  $\nu = 2\frac{1}{8}, 2\frac{3}{8}, 2\frac{5}{8}, 2\frac{7}{8}, \pm i\infty$ . We finally choose

$$\mathbb{W}(\nu) = \frac{(1+c)(1+2c)}{c^2} \begin{pmatrix} 2^5(1-c)(1-2c-4c^2) & 2^5 \frac{1+c}{2c^2-1} (1-2c)^2 \\ -\frac{c}{\sqrt{2}}(1-2c-4c^2) & \frac{c(1-2c)}{\sqrt{2}(2c^2-1)}(1-2c-4c^2) \end{pmatrix}, \quad (4.18)$$

where  $c = \cos(2\pi\nu)$ . With this choice of the summing factor,  $(\mathbb{S}\mathbf{F})$  is holomorphic in the stripe  $\Re\nu \in [2, 3)$  and grows at  $\nu \rightarrow \pm i\infty$  slower than  $\exp(2\pi|\nu|)$ . Taking into account the singularities of  $\sum_{+\infty} \mathbb{S}(\nu-1) \mathbf{R}(\nu)$ , we obtain

$$\mathbf{W}(\nu) = \frac{4\pi^2}{\sin^2(\pi\nu)} \left( \pi - 2 \arctan(4\sqrt{5}) \cos^2(\pi\nu) \right) \begin{pmatrix} 64 \\ -\sqrt{2} \end{pmatrix}. \quad (4.19)$$

Multiplying Eq. (4.16) by  $\mathbb{S}^{-1}(\nu)$ , we obtain

$$\mathbf{F}(\nu) = \mathbb{S}^{-1}(\nu) \mathbf{W}(\nu) + \mathbb{S}^{-1}(\nu) \sum_{+\infty} \mathbb{S}(\nu-1) \mathbf{R}(\nu). \quad (4.20)$$

With quantities  $\mathbb{S}(\nu)$ ,  $\mathbf{W}(\nu)$ ,  $\mathbf{R}(\nu)$  determined by Eqs. (4.14), (4.18), (4.19), and (A.2), the above representation (4.20) gives the final result of the DRA method for the MMI  $\mathbf{F}(\nu) = \begin{pmatrix} F_1(\nu) \\ F_2(\nu) \end{pmatrix}$ .

Let us make two remarks about the two terms in this representation of  $F_{1,2}(\nu)$ . The second term, in fact, does not depend on the explicit form of the summing factor  $\mathbb{S}(\nu)$  because

$$\mathbb{S}^{-1}(\nu) \mathbb{S}(\nu+n) = \begin{cases} \prod_{k=1}^n \mathbb{C}(\nu+k), & n \geq 0 \\ \prod_{k=0}^{-n-1} \mathbb{C}^{-1}(\nu-k), & n < 0 \end{cases}$$

is always a finite product of rational matrices. This product can be evaluated recursively, so that one can organize a numerical evaluation without nested loops. The first term can explicitly be written as a combination of fundamental solutions  $\mathbf{F}_h^1$  and  $\mathbf{F}_h^2$ :

$$\begin{aligned} \mathbb{S}^{-1}(\nu) \mathbf{W}(\nu) &= \begin{pmatrix} F_{1,h}(\nu) \\ F_{2,h}(\nu) \end{pmatrix}, \\ F_{1,h} &= \frac{2^5 \pi^{5/2} (2c-1) (\pi - (c+1) \arctan(4\sqrt{5}))}{(2c+1)(1-c)c^2} \\ &\quad \times \left[ \frac{4c^3 - 2c + 1}{2c^2 - 1} F_{1,h}^1 - (4c^2 + 2c - 1) F_{1,h}^2 \right], \\ F_{2,h} &= \frac{F_{1,h}(\nu+1) - C_{11}(\nu) F_{1,h}(\nu)}{C_{12}(\nu)}, \quad c = \cos(2\pi\nu). \end{aligned} \quad (4.21)$$

Now, taking into account that the evaluation of all the nested sums appearing in representation (4.20) can be organized in one loop, it is easy to calculate  $\mathbf{F}(\nu)$  with high precision and apply the PSLQ algorithm. Then we obtain:

$$\begin{aligned}
F_1(2-\epsilon) &\stackrel{1000}{=} \frac{28\pi^4}{135\epsilon} + \frac{116\pi^2\zeta(3)}{9} + \pi^4 \left( \frac{224}{135} - 4\ln(2) \right) + \frac{226\zeta(5)}{3} \\
&+ \left( -192s_6 + \frac{1808\zeta(5)}{3} - \frac{8\zeta(3)^2}{3} + \frac{928\pi^2\zeta(3)}{9} + 64\pi^2\text{Li}_4\left(\frac{1}{2}\right) + \frac{8}{3}\pi^2\ln^4(2) \right. \\
&\quad \left. - \frac{20}{3}\pi^4\ln^2(2) - 32\pi^4\ln(2) - \frac{428\pi^6}{2835} + \frac{1792\pi^4}{135} \right) \epsilon \\
&+ \left( -768\text{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 128\pi^2\text{Li}_5\left(\frac{1}{2}\right) + 512\pi^2\text{Li}_4\left(\frac{1}{2}\right) - 1536s_6 + \frac{384}{7}s_6\ln(2) \right. \\
&\quad - \frac{384s_{7a}}{7} - \frac{3072s_{7b}}{7} + \frac{4960\zeta(7)}{21} + \frac{35519\pi^2\zeta(5)}{42} + \frac{14464\zeta(5)}{3} - \frac{64\zeta(3)^2}{3} \\
&\quad - \frac{31457\pi^4\zeta(3)}{945} + \frac{7424\pi^2\zeta(3)}{9} - 32\zeta(3)\ln^4(2) + 372\zeta(5)\ln^2(2) + 32\pi^2\zeta(3)\ln^2(2) \\
&\quad - \frac{480}{7}\zeta(3)^2\ln(2) - \frac{3424\pi^6}{2835} + \frac{14336\pi^4}{135} + \frac{16}{15}\pi^2\ln^5(2) + \frac{64}{3}\pi^2\ln^4(2) \\
&\quad \left. - \frac{40}{9}\pi^4\ln^3(2) - \frac{160}{3}\pi^4\ln^2(2) - \frac{3079}{315}\pi^6\ln(2) - 256\pi^4\ln(2) \right) \epsilon^2 + O(\epsilon^3), \quad (4.22)
\end{aligned}$$

$$\begin{aligned}
F_2(2-\epsilon) &\stackrel{1000}{=} -\frac{\pi^4}{\epsilon} - 93\zeta(5) - 14\pi^2\zeta(3) - 2\pi^4\ln(2) \\
&+ \left( -96s_6 + 120\zeta(3)^2 + 32\pi^2\text{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3}\pi^2\ln^4(2) - \frac{10}{3}\pi^4\ln^2(2) - \frac{989\pi^6}{420} \right) \epsilon \\
&+ \left( -384\text{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 64\pi^2\text{Li}_5\left(\frac{1}{2}\right) + \frac{192}{7}s_6\ln(2) - \frac{192s_{7a}}{7} - \frac{1536s_{7b}}{7} \right. \\
&\quad - \frac{32666\zeta(7)}{7} - \frac{40585\pi^2\zeta(5)}{84} + \frac{35047\pi^4\zeta(3)}{630} - 16\zeta(3)\ln^4(2) + 186\zeta(5)\ln^2(2) \\
&\quad \left. + 16\pi^2\zeta(3)\ln^2(2) - \frac{240}{7}\zeta(3)^2\ln(2) + \frac{8}{15}\pi^2\ln^5(2) - \frac{20}{9}\pi^4\ln^3(2) - \frac{3079}{630}\pi^6\ln(2) \right) \epsilon^2 + O(\epsilon^3), \quad (4.23)
\end{aligned}$$

where the notation  $\stackrel{N}{=}$  indicates that the equality holds numerically with at least  $N$  decimal digits,

$$\begin{aligned}
s_6 &= \zeta(-5, -1) + \zeta(6), \\
s_{7a} &= \zeta(-5, 1, 1) + \zeta(-6, 1) + \zeta(-5, 2) + \zeta(-7), \\
s_{7b} &= \zeta(7) + \zeta(5, 2) + \zeta(-6, -1) + \zeta(5, -1, -1),
\end{aligned}$$

and  $\zeta(m_1, \dots, m_k)$  are multiple zeta values

$$\zeta(m_1, \dots, m_k) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^k \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}. \quad (4.24)$$

The terms up to  $\epsilon^1$  are in agreement with the previous results [21].

## 5 Conclusion

We have presented here a method of finding the solution of the homogeneous part of dimensional recurrence relations for multicomponent master integrals. The method is based on the fact that the maximally cut master integral satisfies this homogeneous equation. Strictly speaking, it gives us only one solution, while for a  $k$ -component master integral we need  $k$  linearly independent ones. However, it appears that in the Mellin-Barnes representation of the cut integral each series of poles separately gives rise to a solution. For each individual case, this fact can be checked both numerically and strictly, using Zeilberger’s algorithm of creative telescoping.

As an application of this technique, we have presented the calculation of the two-component master integral  $\begin{pmatrix} P_{71} \\ P_{72} \end{pmatrix}$  given by Eq. (4.20) and entering the three-loop static quark potential. Using this result, we have calculated with a high precision the  $\epsilon$ -expansion up to  $\epsilon^2$ -terms and applied the PSLQ algorithm to express it in terms of conventional constants. Our next natural task is to complete the analytical evaluation of all the master integrals for the three-loop static quark potential, i.e. to evaluate the last three analytically unknown expansion coefficients entering the corresponding result.

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## A Coefficients in the dimensional recurrence relation

Here we present for completeness the quantities  $C_{ij}(\nu)$  and  $R_i(\nu)$  entering Eq. (4.2):

$$\begin{aligned}
C_{11} &= \frac{78656\nu^6 - 709872\nu^5 + 2652380\nu^4 - 5251197\nu^3 + 5809568\nu^2 - 3405384\nu + 826308}{4(\nu - 1)^3(2\nu - 3)^2(8\nu - 13)(8\nu - 11)(8\nu - 9)(8\nu - 7)}, \\
C_{12} &= \frac{(\nu - 2)(13120\nu^4 - 70192\nu^3 + 140108\nu^2 - 123689\nu + 40761)}{8(\nu - 1)^3(2\nu - 3)^2(8\nu - 13)(8\nu - 11)(8\nu - 9)(8\nu - 7)}, \\
C_{21} &= \frac{3(13120\nu^6 - 117424\nu^5 + 434716\nu^4 - 851937\nu^3 + 932032\nu^2 - 539672\nu + 129216)}{8(1 - \nu)^3(2\nu - 3)^2(8\nu - 13)(8\nu - 11)(8\nu - 9)}, \\
C_{22} &= -\frac{(\nu - 2)(6592\nu^4 - 34768\nu^3 + 68324\nu^2 - 59303\nu + 19191)}{16(\nu - 1)^3(2\nu - 3)^2(8\nu - 13)(8\nu - 11)(8\nu - 9)}. \tag{A.1}
\end{aligned}$$

$$\mathbf{R}(\nu) = \begin{pmatrix} R_1(\nu) \\ R_2(\nu) \end{pmatrix},$$

$$\begin{aligned} R_1(\nu) &= -\frac{128}{5(8\nu-14)_8(3\nu-5)_3(2\nu-3)_2} (648214272\nu^{10} - 9064230912\nu^9 \\ &\quad + 56911513696\nu^8 - 211292587888\nu^7 + 513701269195\nu^6 - 854608449763\nu^5 \\ &\quad + 985285600699\nu^4 - 777347268382\nu^3 + 401666882358\nu^2 - 122748402735\nu \\ &\quad + 16847478900) P_{51}(\nu) \\ &\quad - \frac{16(2\nu-3)^2}{(8\nu-13)_7(3\nu-5)_3(\nu-1)} (1133568\nu^6 - 8922240\nu^5 + 29193664\nu^4 \\ &\quad - 50834923\nu^3 + 49690736\nu^2 - 25855817\nu + 5595660) P_{53}(\nu) \\ &\quad - \frac{17440\nu^4 - 94208\nu^3 + 189730\nu^2 - 168891\nu + 56091}{4(\nu-1)^3(4\nu-7)(8\nu-11)(8\nu-9)(8\nu-7)} P_{62}(\nu), \\ R_2(\nu) &= \frac{192}{(8\nu-14)_7(3\nu-5)_3(2\nu-3)_2} (21590784\nu^{10} - 300317184\nu^9 + 1875217824\nu^8 \\ &\quad - 6922120208\nu^7 + 16728915563\nu^6 - 27658192027\nu^5 + 31682083339\nu^4 \\ &\quad - 24828801753\nu^3 + 12740565282\nu^2 - 3865560708\nu + 526619520) P_{51}(\nu) \\ &\quad + \frac{96(2\nu-3)^2}{(8\nu-13)_6(3\nu-5)_3(\nu-1)} (47232\nu^6 - 368208\nu^5 + 1192698\nu^4 - 2055044\nu^3 \\ &\quad + 1986779\nu^2 - 1021999\nu + 218560) P_{53}(\nu) \\ &\quad + \frac{4320\nu^4 - 23000\nu^3 + 45592\nu^2 - 39893\nu + 13008}{4(\nu-1)^3(4\nu-7)(8\nu-11)(8\nu-9)} P_{62}(\nu). \end{aligned} \quad (\text{A.2})$$

The simpler master integrals are

$$\begin{aligned} P_{51}(\nu) &= \frac{\pi^2 \csc(\pi\nu) \csc(3\pi\nu) \Gamma(\nu-1)^2}{\Gamma(5\nu-5)}, \quad (\text{A.3}) \\ P_{53}(\nu) &= \frac{\pi^3 \csc^2(\pi\nu) \csc(3\pi\nu) \Gamma(\nu-1)^3}{\Gamma(4-2\nu) \Gamma(2\nu-2)^2 \Gamma(4\nu-5)}, \\ P_{62}(\nu) &= \frac{\pi^{5/2} 2^{8-6\nu} (2 \cos(2\pi\nu) - 1) \Gamma(\frac{3}{2} - \nu) \Gamma(2\nu - \frac{5}{2})}{(2\nu-3) \Gamma(4\nu - \frac{11}{2}) \sin^3(\pi\nu) \cos(4\pi\nu)} + \frac{\Gamma(2\nu - \frac{5}{2}) \Gamma(\frac{13}{2} - 4\nu)}{2^{1+6\nu} (2\nu-3) \Gamma(\nu - \frac{1}{2})} \\ &\times \sum_{-\infty} \frac{(15536\nu^5 - 75492\nu^4 + 144596\nu^3 - 136177\nu^2 + 62875\nu - 11340) \Gamma(\nu - \frac{1}{2})}{5 \cdot 4^{-3\nu} (5\nu-4)(5\nu-3) \Gamma(2\nu - \frac{1}{2}) \Gamma(\frac{13}{2} - 4\nu)} P_{51}(\nu), \end{aligned}$$

where  $\Sigma_{-\infty}$  is defined in Eq. (4.17). The result for  $P_{62}$  presented here is found using DRA method.

## B Homogeneous solutions via Saalschutzyan ${}_4F_3$

Using Eq. (2.4.2.3) from [33], we obtain

$$F_{1,h}^1(\nu) = \frac{\pi^{3/2} 2^{3-4\nu} \Gamma(2\nu - \frac{5}{2}) {}_4F_3\left(\frac{1}{2}, 4-2\nu, \frac{9}{2}-2\nu, \nu-1 \mid 1\right)}{\cos(3\pi\nu)(\nu-2)^2 \Gamma(\frac{11}{2}-2\nu) \Gamma(\nu-\frac{3}{2}) \Gamma(2\nu-4) \Gamma(4\nu-\frac{13}{2})} + \frac{\pi \Gamma(2-\nu) \Gamma(2\nu-\frac{5}{2}) {}_4F_3\left(2-\nu, \frac{5}{2}-\nu, \nu-\frac{3}{2}, 2\nu-3 \mid 1\right)}{4 \cos(3\pi\nu) \Gamma(\frac{9}{2}-2\nu) \Gamma(\nu-1)^2 \Gamma(4\nu-\frac{13}{2})}, \quad (\text{B.1})$$

$$F_{1,h}^2(\nu) = \frac{2^{10(\nu-2)} \pi \sin(\pi\nu) \Gamma(15-8\nu) {}_4F_3\left(4-2\nu, 4-2\nu, \frac{9}{2}-2\nu, \frac{5}{2}-\nu \mid 1\right)}{\sin(5\pi\nu) \Gamma(\frac{25}{2}-6\nu) \Gamma(2\nu-3) \Gamma(3-\nu)} + \frac{2^{2\nu-5} \pi \sin(\pi\nu) \Gamma(2-\nu) {}_4F_3\left(\frac{1}{2}, 2-\nu, 2-\nu, \frac{5}{2}-\nu \mid 1\right)}{\sin(5\pi\nu) (8\nu-15) \Gamma(\frac{5}{2}-\nu) \Gamma(\nu-1) \Gamma(2\nu-3)}. \quad (\text{B.2})$$

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