TTP12-032

Analysis of the Higgs potentials for two doublets and a singlet.

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Abstract

We consider the most general CP-conserving renormalizable effective scalar potential involving two doublets plus one singlet Higgs and satisfying the electroweak gauge symmetry. After deriving the electroweak-symmetry breaking conditions, we focus on special cases, characterized by specific symmetry properties and/or relations to supersymmetry-inspired extensions of the Standard Model (e.g. n/NMSSM, UMSSM). We then investigate the question of the reconstruction of the potential parameters from the Higgs masses and mixing angles and show that in some specific cases, such as the one of an underlying NMSSM, an accuracy at the order of leading-logarithms is achievable with minimal effort. We finally study a few phenomenological consequences for this latter model. More specifically, we consider how our parameter reconstruction modifies the outcome of two publicly available codes : `micrOMEGAs` and `NMSSMTools`. We observed noteworthy effects in regions of parameter space where Higgs-to-Higgs decays are relevant, impacting the collider searches for light Higgs states and the prediction of the Dark-Matter relic density.

Introduction

The origin of ElectroWeak Symmetry Breaking (EWSB) stands as one of the critical questions in high-energy physics and a central goal of the Large Hadron Collider (LHC) is to reveal its nature. The recent discovery of a new massive boson around 125 GeV [?], reported by both the ATLAS and CMS collaborations [?], and supported by the broad excess seen at TeVatron [?], represents a first step towards the identification of the Higgs boson and the measurement of the underlying Higgs potential, a task which however only the next generation of colliders will probably complete. Although essentially compatible with the Higgs boson of the Standard Model (SM), this new state may already be hinting towards some new physics, in that the peaks of the diphoton and $ZZ \to 4l$ decays differ from what one would expect in the SM. The stronger signal in the $H \to \gamma\gamma$ channel, in particular, seems of importance because this loop-induced process is particularly sensitive to physics beyond the SM. One should also consider the non-observation of events at CMS – although supported by very little statistics – in the $H \to \tau\tau$ channel. Testing the SM-nature of this would-be Higgs state, inspecting possible deviations in its coupling to SM particles shall represent a major undertaking of modern particle physics and a probable probe into the mechanism of EWSB.

The 'Higgs mechanism' [?], involving scalar elementary fields, is the most efficient way to generate masses for the fermions and gauge-bosons. Its implementation within the SM is the minimal one: only one scalar field, transforming as a doublet under $SU(2)_L$, is introduced to break the electroweak (EW) symmetry through its vacuum expectation value (v.e.v.). Nevertheless the Higgs sector is still essentially undetermined and there is no reason to stick to minimality if some benefits should emerge from a more elaborate scalar sector. For instance, introducing a second Higgs doublet allows for an implementation of CP violation through this sector [?]: CP violation appears in this context because some of the parameters in the potential of the Two Higgs Doublet Model (2HDM) can be chosen complex (non-real). Yet the requirements relative to neutral flavor conservation constrain this possibility significantly. Large flavour-changing couplings of neutral Higgs bosons can be avoided in the so-called '2HDM of type II', where the Higgs doublets H_u and H_d, of opposite hypercharges $Y = \pm 1$, enter separately, and respectively, up- and down-type Yukawa terms (at tree level). Another (more exotic) possibility consists in requiring the alignement of the Yukawa coupling matrices in flavor space: see [?]. Although such 2HDM's may hold as autonomous extensions of the SM, they can also be embedded within more elaborate models: Left-Right gauge models and their Grand-Unification Theory (GUT) ramifications – Pati-Salam, $SO(10)$, etc.– offer a first framework for this operation, in which the question of CP-violation was originally central [?].

From another angle, the well-documented 'Hierarchy Problem' [?] underlines the theoretical difficulties for understanding the stability of a Higgs mass at the electroweak scale, with respect to new-physics at very-high energies (GUT, Planck scales). Regarding the SM as the low-energy effective theory of some more-fundamental model, the quadratic sensitivity of scalar squared masses to new-physics masses would lead to a technically unnatural fine-tuning of the Higgs-mass parameter in the more-fundamental theory with the radiative corrections resulting from the integrated-out new-physics states... Unless new-physics appears sufficiently close to the electroweak scale: typically at the TeV scale. Among the proposed solutions, Supersymmetry (SUSY) allows to stabilize a scalar Higgs mass at the electroweak scale, due to the renormalization properties of supersymmetric theories. However, SUSY being obviously not realized in low-energy particle physics, viable SUSY-inspired models need to include SUSY-breaking effects, which are parametrized within the Lagrangian through the so-called 'soft terms', generate e.g. mass terms for all non-SM particles and trigger the Higgs mechanism. This ad-hoc setup could yet remain an acceptable solution to the Hierarchy Problem only if the supersymmetry-breaking scale is near the electroweak scale. Other attractive properties of SUSY-inspired models lie in the possibility of one-step unification, due to the more-convergent running of SM-gauge couplings in the presence of the enlarged SUSY field-content [?], or in the dark-matter (DM) sector, the lightest supersymmetric particle being a stable (or long-lived) and viable candidate in the presence of (approximate) R-parity [?].

Holomorphicity of the superpotential (cancellation of gauge-anomalies) dictates the requirement for at least two $SU(2)_L$ Higgs doublets in a SUSY-inspired model, intervening in a Type II 2HDM fashion, so

that both up-type and down-type masses be generated. The simplest implementation of a SUSY-inspired SM, known as the Minimal Supersymmetric Standard Model (MSSM) [?] confines to this minimal 2HDM requirement. There, the quartic Higgs couplings are determined by the EW gauge couplings, which results in tight constraints on the tree-level mass of the lightest Higgs boson: the latter is indeed bounded from above by the Z^0-boson mass M_Z. Radiative corrections improve this feature and can arrange for fairly heavy Higgs masses provided the SUSY-scale is large enough, see for example [?]. Yet this last necessity tends to conflict with the naturalness-dictated $\lesssim 1$ TeV SUSY-breaking scale. Accommodating for a Higgs state at 125 GeV in the MSSM hence severely constrains the parameter space of this model [?]. Another criticism to this minimal setup, the so-called 'μ-problem' [?], points out the necessity of tuning a supersymmetric mass-term, the conventionally-baptized μ parameter, at the electroweak/TeV scale in order to ensure EWSB: being of supersymmetric origins, this parameter is in principle unrelated to the SUSY-breaking scale and would thus coincide with it out of sheer coincidence.
The introduction of an additional gauge-singlet superfield S addresses both shortcomings of the MSSM. The μ-term can indeed be generated effectively through a $\lambda S H_u \cdot H_d$ term when the singlet takes a v.e.v. s: $\mu_{\text{eff.}} \equiv \lambda s$ [?]. Concerning the lightest Higgs mass, the presence of a new superfield coupling to the Higgs doublets induces additional contributions to the Higgs mass matrix, so that the MSSM limit can be exceeded, already at tree-level [?, ?]. It is also worth to mention that the lightest CP-even Higgs state in this context might well be dominantly of a singlet nature, hence, the singlet decoupling from SM-fermions and gauge bosons, essentially invisible at colliders: the SM-like Higgs state would then be the second lightest and a small mixing effect with the singlet would thus shift its mass towards slightly higher values. In short, radiative corrections are no longer the only mechanism able to generate a SM-like Higgs-state heavier than M_Z in such a singlet-extension.
The simplest version of such a model with singlet-enlarged superfield content is known as the Next-to-Minimal Supersymmetric Standard Model (NMSSM) [?, ?]. It relies on a $\mathbb{Z}_3$ discrete symmetry in order to forbid all dimensional parameters (including μ) in the superpotential, so that the soft-terms provide the only relevant scale in the scalar potential, triggering the EWSB. Several other SUSY-models engaging a singlet in addition to the two Higgs doublets are to be found in the literature, including the nearly Minimal Supersymmetric Standard Model (nMSSM, sometimes MNSSM) [?, ?], $U(1)'$-extended MSSM's, with their simplest version known as the UMSSM [?], models based on the E_6 exceptional group [?], SUSY/compositeness hybrids, such as 'fat Higgs models' [?] or models using the Seiberg Duality [?], etc.
In the present paper, we aim at studying the effective Higgs potential involving 2-doublet+1-singlet Higgs fields. The relations between physical input, represented by the mass matrices and mixing angles, and the parameters of the potential, as well as with the trilinear Higgs couplings, shall be at the center of this discussion, in view of a possible reconstruction of the potential from such input, at, and beyond, leading order (LO). Similar analyses for the 1-doublet setup [?], or the 2-doublet setup, for instance in [?], with the MSSM as a background-model, have already been proposed in the literature. Given that the singlet-extensions of the MSSM offer a natural origin to our 2-doublet+1-singlet setup, we shall refer and return explicitly to such models in the course of our discussion: specific attention will be dedicated in particular to the n/NMSSM or the UMSSM. Most of our discussion should however be generalizable to other models resulting in a 2-doublet+1-singlet Higgs potential[1], as long as matching conditions or/and symmetry properties are satisfied. The first part of the present paper shall be dedicated to the presentation of the general framework, including notations, the discussion of residual symmetries and the pattern of EWSB leading to the Higgs spectrum. In the second part, we shall focus on the question of the reconstruction of the potential from a measurement of the Higgs masses and mixing angles: beyond the general case where a large number of undetermined parameters remain, the possibility of a reconstruction in constrained models will be discussed at leading order. The analysis of the large logarithms appearing in the Coleman-Weinberg [?, ?] approach shall convince us, in particular, that a

[1] We have already referred to Left-Right models and their GUT extensions as an alternative approach to the 2HDM framework. Note that the addition of a SM-gauge singlet is essentially an undemanding requirement and may be arranged within such models as well.

full reconstruction at the order of leading logarithms should be achievable in the $\mathbb{Z}_3$-symmetric case represented by an underlying NMSSM. Concentrating on the NMSSM in the last part, we shall analyse the phenomenological consequences for this model, both in terms of constraints from Higgs-to-Higgs decays and computation of the Dark-Matter relic density. The decay $h_i^0 \to \gamma\gamma$ [?] will also be revisited, although little impact is expected there. This phenomenological analysis will rely on the numerical output of several public codes, including `NMSSMTools_3.2.0` [?,?], `micrOMEGAs_2.4.1` [?,?] and a version of `SloopS` [?,?] adapted to the NMSSM [?].

1 Two Higgs doublet plus one singlet potential

1.1 General parametrization

New-Physics (NP) effects are most conveniently encoded in terms of effective Lagrangians. Under the guidelines of Lorentz and gauge invariance, as well as possible additional symmetries, one can write a list of all the operators, classified according to their mass-dimension. For the two $SU(2)_L$ doublets and the singlet, we shall use the notations (with v_d, v_u and s representing the v.e.v.'s of these fields):

$$H_d = \begin{pmatrix} v_d + (h_d^0 + i a_d^0)/\sqrt{2} \\ H_d^- \end{pmatrix}, \quad H_u = \begin{pmatrix} H_u^+ \\ v_u + (h_u^0 + i a_u^0)/\sqrt{2} \end{pmatrix}, \quad S = s + (h_s^0 + i a_s^0)/\sqrt{2} \tag{1.1}$$

The most general Higgs potential involving these fields and compatible with the electroweak gauge symmetry then reads, when one restricts to renormalizable terms:

$$\begin{aligned} \mathcal{V}^S_{\text{eff.}} &= m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 - \left(m_{12}^2 H_u \cdot H_d + h.c.\right) + \frac{\lambda_1}{2}|H_d|^4 + \frac{\lambda_2}{2}|H_u|^4 + \lambda_3 |H_u|^2 |H_d|^2 \\ &+ \lambda_4 |H_u \cdot H_d|^2 + \left[\frac{\lambda_5}{2}(H_u \cdot H_d)^2 + (\lambda_6 |H_u|^2 + \lambda_7 |H_d|^2) H_u \cdot H_d + h.c.\right] \\ &+ m_S^2 |S|^2 + \kappa^2 |S|^4 + \left[\lambda_T S + \frac{\mu_S^2}{2} S^2 + \frac{A_S}{3} S^3 + \frac{\tilde{A}_S}{3} S|S|^2 + \frac{\kappa_S^2}{4} S^4 + \frac{\tilde{\kappa}_S^2}{4} S^2 |S|^2 + h.c.\right] \\ &+ \left[A_{ud} S H_u \cdot H_d + \tilde{A}_{ud} S^* H_u \cdot H_d + \lambda_M |S|^2 H_u \cdot H_d + \lambda_P^M S^{*\,2} H_u \cdot H_d + \tilde{\lambda}_P^M S^2 H_u \cdot H_d + h.c.\right] \\ &+ \lambda_P^u |S|^2 |H_u|^2 + \lambda_P^d |S|^2 |H_d|^2 + \left[(A_{us} S + \tilde{\lambda}_P^u S^2)|H_u|^2 + (A_{ds} S + \tilde{\lambda}_P^d S^2)|H_d|^2 + h.c.\right] \end{aligned} \tag{1.2}$$

The first two lines comprise the usual 2HDM potential, the third one, the pure-singlet terms and the latter two, the singlet-doublet mixing-terms. $m_{H_u}^2$, $m_{H_d}^2$, λ_1, λ_2, λ_3, λ_4, m_S^2, κ^2, λ_P^u and λ_P^d are (10) real parameters, while m_{12}^2, λ_5, λ_6, λ_7, λ_T, μ_S^2, A_S, $\tilde{A}_S$, κ_S^2, $\tilde{\kappa}_S^2$, A_{ud}, $\tilde{A}_{ud}$, λ_M, λ_P^M, $\tilde{\lambda}_P^M$, A_{us}, A_{ds}, $\tilde{\lambda}_P^u$ and $\tilde{\lambda}_P^d$ are (19) in-principle-complex parameters. One parameter (e.g. λ_T) is superfluous and may be absorbed in a translation of the singlet; three others (m_S^2, $m_{H_u}^2$ and $m_{H_d}^2$) can be traded for the field vacuum expectation values through the minimization conditions. From now on, we will consider, for simplicity, that all the parameters are real, hence barring the possibility of CP-violation. (We will however continue to refer to the 19 potentially non-real parameters as 'complex' parameters.)

1.2 Symmetry classification

By imposing additional symmetries, the form of the potential in Eq.(**??**) can be further constrained at the classical level and the remaining parameters[2] $\lambda_i^{\text{cl.}}$ will be called 'classical' parameters. At the quantum level, all the eliminated terms λ_j^{qm} may reappear, in principle, if the symmetry is broken, either directly by the quantum fluctuations, or spontaneously, when the fields acquire v.e.v.'s. In the later case, symmetry-violation is a relic from higher-dimensional operators at the non-symmetric vacuum, due

[2] We shall use the notation 'λ_i' in order to concisely refer to *any* parameter entering Eq.(**??**).

to the truncation of the potential to dimension ≤ 4 terms. To be definite, if at high energy, beyond a certain scale Λ, the symmetry holds, the potential $\mathcal{V}$ is then well approximated by its classical form (the symmetry-violating effects being negligible) and the $\lambda_i^{\text{cl.}}$ at the scale Λ may be chosen as boundary conditions for the general parameters of Eq.(**??**),

$$\lambda_i^{\text{cl.}} = \lambda(\Lambda)\,; \qquad \lambda_j^{\text{qm}} = 0 \tag{1.3}$$

such that $\mathcal{V} \equiv \mathcal{V}(\lambda_i^{\text{cl.}}(\Lambda))$. At scales $\mu \ll \Lambda$, however, symmetry-violating effects are no longer negligible so that non-trivial values of λ^{qm} are generated by the renormalization group equations.
We shall now enumerate possible symmetries one can impose to the potential of Eq.(**??**):

- Discrete $\mathbb{Z}_n$-symmetries: they are characterized by the transformations $\Phi \mapsto e^{\frac{2\imath\pi}{n}Q_\Phi}\Phi$, where $\Phi = S, H_u, H_d$ and Q_{S,H_u,H_d} are the charges under the discrete symmetry group. They allow for significant selectivity among the complex terms of the general potential, while avoiding the problem of an axion (unless the potential they induce is also accidentally $U(1)$-invariant). Spontaneous breakdown of these symmetries (through Higgs v.e.v.'s) however generically leads to cosmological difficulties, in the form of a domain-wall problem [**?**], which should then be addressed separately.
  1. The complex doublet-terms are controlled by $Q_{H_u} + Q_{H_d}$: $Q_{H_u} + Q_{H_d} \equiv 0\,[n]$ causes no constraint; for even n, $Q_{H_u} + Q_{H_d} \equiv \frac{n}{2}\,[n]$ allows only for λ_5; other choices forbid all the corresponding terms.
  2. Complex mixing-terms are governed by both $Q_{H_u}+Q_{H_d}$ and Q_S. Only in the case $\{Q_{H_u}+Q_{H_d} \equiv 0\,[n], Q_S \equiv 0[n]\}$ are they all allowed by the $\mathbb{Z}_n$-symmetry. Otherwise, the relative choice of Q_S and $Q_{H_u} + Q_{H_d}$ constrains them, with the specific values $Q_S \equiv \pm(Q_{H_u} + Q_{H_d})\,[n]$, $2Q_S \equiv \pm(Q_{H_u} + Q_{H_d})\,[n]$ and up to the exclusion of all these terms.
  3. The complex singlet-terms are governed by Q_S, ranging from conservation of all ($Q_S \equiv 0\,[n]$) to exclusion of all, with the special cases $2Q_S \equiv 0\,[n]$, $3Q_S \equiv 0\,[n]$ and $4Q_S \equiv 0\,[n]$.

  A typical example for such a discrete symmetry and deserving particular attention is that of the $\mathbb{Z}_3$-symmetry with charges $Q_{S,H_u,H_d} = 1$: this corresponds to the case of an underlying NMSSM.Invariance under $\Phi \mapsto e^{\frac{2\imath\pi}{3}}\Phi$ reduces the potential to the form:

$$\begin{aligned}
\mathcal{V}^S_{\mathbb{Z}_3} &= m^2_{H_u}|H_u|^2 + m^2_{H_d}|H_d|^2 + \frac{\lambda_1}{2}|H_d|^4 + \frac{\lambda_2}{2}|H_u|^4 + \lambda_3|H_u|^2|H_d|^2 + \lambda_4|H_u \cdot H_d|^2 \\
&+ m_S^2|S|^2 + \kappa^2|S|^4 + \left[\frac{A_S}{3}S^3 + h.c.\right] \\
&+ \lambda_P^u|S|^2|H_u|^2 + \lambda_P^d|S|^2|H_d|^2 + \left[A_{ud}SH_u \cdot H_d + \lambda_P^M S^{*2}H_u \cdot H_d + h.c.\right]
\end{aligned} \tag{1.4}$$

  The tree-level conditions resulting from the NMSSM read:

$$\begin{array}{llll}
\lambda_1 = \dfrac{g^2+g'^2}{4} = \lambda_2 & ;\, \lambda_3 = \dfrac{g^2-g'^2}{4} & ;\, \lambda_4 = \lambda^2 - \dfrac{g^2}{2} & ;\, \lambda_P^u = \lambda^2 = \lambda_P^d\,; \\
\lambda_P^M = \lambda\kappa & ;\, A_S = \kappa A_\kappa & ;\, A_{ud} = \lambda A_\lambda & ;\, \kappa^2 = \kappa^2
\end{array} \tag{1.5}$$

  Our notations for the SUSY parameters follow those of [**?**], except for the electroweak gauge couplings which we denote as g' and g for, respectively, the hypercharge $U(1)_Y$ and the $SU(2)_L$ symmetry.

- Continuous global symmetries: here we mean essentially global phase transformations $\Phi \mapsto e^{\imath Q_\Phi \alpha}\Phi$, that is $U(1)$-Peccei-Quinn (P.Q.) symmetries [**?**]. Such symmetries are spontaneously broken by the v.e.v.'s of the Higgs fields so that they produce massless axions. They are also chiral in nature, so that anomalies will be generated at the quantum level (unless the field-content is enlarged so as to cancel them). Such symmetries are thus likely to stand only as approximate limiting cases.
  1. $\{Q_{H_u} + Q_{H_d} = 0, Q_S = 0\}$ is automatically satisfied: this is the hypercharge.

2. $\{Q_{H_u}+Q_{H_d}=0, Q_S\neq 0\}$ preserves the doublet potential while constraining drastically the singlet couplings:

$$\begin{aligned}\mathcal{V}_{PQ}^{S-S} &= m_{H_u}^2|H_u|^2+m_{H_d}^2|H_d|^2-\left(m_{12}^2 H_u\cdot H_d+h.c.\right)+\frac{\lambda_1}{2}|H_d|^4+\frac{\lambda_2}{2}|H_u|^4+\lambda_3|H_u|^2|H_d|^2\\ &+ \lambda_4|H_u\cdot H_d|^2+\left[\frac{\lambda_5}{2}(H_u\cdot H_d)^2+(\lambda_6|H_u|^2+\lambda_7|H_d|^2)H_u\cdot H_d+h.c.\right]\\ &+ m_S^2|S|^2+\kappa^2|S|^4+\lambda_P^u|S|^2|H_u|^2+\lambda_P^d|S|^2|H_d|^2+\left(\lambda_M|S|^2H_u\cdot H_d+h.c.\right)\end{aligned} \tag{1.6}$$

3. $\{Q_{H_u}+Q_{H_d}\neq 0, Q_S=0\}$ constrains severely the doublet sector, as well as the mixing terms, while leaving the pure-singlet potential untouched:

$$\begin{aligned}\mathcal{V}_{PQ}^{S-D} &= m_{H_u}^2|H_u|^2+m_{H_d}^2|H_d|^2+\frac{\lambda_1}{2}|H_d|^4+\frac{\lambda_2}{2}|H_u|^4+\lambda_3|H_u|^2|H_d|^2+\lambda_4|H_u\cdot H_d|^2\\ &+ m_S^2|S|^2+\kappa^2|S|^4+\left[\lambda_T S+\frac{\mu_S^2}{2}S^2+\frac{A_S}{3}S^3+\frac{\tilde{A}_S}{3}S|S|^2+\frac{\kappa_S^2}{4}S^4+\frac{\tilde{\kappa}_S^2}{4}S^2|S|^2+h.c.\right]\\ &+ \lambda_P^u|S|^2|H_u|^2+\lambda_P^d|S|^2|H_d|^2+\left[\tilde{\lambda}_P^u S^2|H_u|^2+\tilde{\lambda}_P^d S^2|H_d|^2+h.c.\right]\\ &+ \left[A_{us}S|H_u|^2+A_{ds}S|H_d|^2+h.c\right]\end{aligned} \tag{1.7}$$

4. $\{Q_{H_u}+Q_{H_d}\neq 0, Q_S=-(Q_{H_u}+Q_{H_d})\}$ is the 'usual' Peccei-Quinn symmetry (e.g. [?]) and, without loss of generality, one may choose $(Q_{H_u}=1=Q_{H_d}, Q_S=-2)$. It induces a potential of the same form as that of the $\mathbb{Z}_3$-symmetry (Eq.(**??**)), with the further requirement that A_S and λ_P^M vanish.

$$\begin{aligned}\mathcal{V}_{PQ}^{S} &= m_{H_u}^2|H_u|^2+m_{H_d}^2|H_d|^2+\frac{\lambda_1}{2}|H_d|^4+\frac{\lambda_2}{2}|H_u|^4+\lambda_3|H_u|^2|H_d|^2+\lambda_4|H_u\cdot H_d|^2\\ &+ m_S^2|S|^2+\kappa^2|S|^4+\lambda_P^u|S|^2|H_u|^2+\lambda_P^d|S|^2|H_d|^2+[A_{ud}SH_u\cdot H_d+h.c.]\end{aligned} \tag{1.8}$$

5. $\{Q_{H_u}+Q_{H_d}\neq 0, Q_S=Q_{H_u}+Q_{H_d}\}$ is equivalent to the preceding case with the replacement $S\mapsto\tilde{S}=S^*$.
6. $\{Q_{H_u}+Q_{H_d}\neq 0, Q_S=\frac{1}{2}(Q_{H_u}+Q_{H_d})\}$ is a variant, concerning the singlet-doublet mixing-sector. This is again a subcase of the $\mathbb{Z}_3$-potential (Eq.**??**), with vanishing A_S and A_{ud}: in a coarse understanding of the term, this may be considered as the 'R-symmetric' potential.

$$\begin{aligned}\mathcal{V}_{PQ'}^{S} &= m_{H_u}^2|H_u|^2+m_{H_d}^2|H_d|^2+\frac{\lambda_1}{2}|H_d|^4+\frac{\lambda_2}{2}|H_u|^4+\lambda_3|H_u|^2|H_d|^2+\lambda_4|H_u\cdot H_d|^2\\ &+ m_S^2|S|^2+\kappa^2|S|^4+\lambda_P^u|S|^2|H_u|^2+\lambda_P^d|S|^2|H_d|^2+\left[\lambda_P^M S^{*2}H_u\cdot H_d+h.c.\right]\end{aligned} \tag{1.9}$$

   Note that if one is interested in a SUSY-inspired model, this PQ'-symmetry would a priori forbid the $\lambda SH_u\cdot H_d$ term, resulting in vanishing tree-level conditions for most of the parameters of Eq.(**??**): it is therefore best understood as a R-symmetry at the SUSY level.
7. $\{Q_{H_u}+Q_{H_d}\neq 0, Q_S=-\frac{1}{2}(Q_{H_u}+Q_{H_d})\}$ is equivalent to the preceding choice, with the replacement $S\mapsto\tilde{S}=S^*$.
8. $\left\{Q_{H_u}+Q_{H_d}\neq 0, Q_S\neq\pm\left\{0,\frac{1}{2},1\right\}(Q_{H_u}+Q_{H_d})\right\}$ forbids all the complex terms, hence leading to another, more-constrained subcase of the $\mathbb{Z}_3$-potential:

$$\begin{aligned}\mathcal{V}_{PQ}^{S-C} &= m_{H_u}^2|H_u|^2+m_{H_d}^2|H_d|^2+\frac{\lambda_1}{2}|H_d|^4+\frac{\lambda_2}{2}|H_u|^4+\lambda_3|H_u|^2|H_d|^2+\lambda_4|H_u\cdot H_d|^2\\ &+ m_S^2|S|^2+\kappa^2|S|^4+\lambda_P^u|S|^2|H_u|^2+\lambda_P^d|S|^2|H_d|^2\end{aligned} \tag{1.10}$$

In the following, we shall focus only on $\mathcal{V}_{PQ}^{S}$ and $\mathcal{V}_{PQ'}^{S}$, which both can be viewed as subcases of $\mathcal{V}_{\mathbb{Z}_3}^{S}$.

- $U(1)'$-gauge symmetries: they can be regarded as the gauged-version of the P.Q. symmetries, with the important consequence that the P.Q.-axion is now unphysical. They emerge naturally from $U(1)'$-SUSY models, containing SM-singlets charged under the additional $U(1)'$-gauge symmetry and breaking it spontaneously while acquiring v.e.v.'s. The simplest version of such models, with only one singlet, is called UMSSM [?] and leads back to the $\mathbb{Z}_3$-invariant Higgs potential, but with vanishing A_S and λ_P^M, i.e. $\mathcal{V}^S_{UMSSM} = \mathcal{V}^S_{PQ}$: see Eq.(**??**). The further tree-level conditions are shifted from Eq.(**??**) according to (with Q_{S,H_u,H_d} the Higgs charges under the $U(1)'$-symmetry and $g_{Z'}$, the coupling constant):

$$\lambda_{1,2} \to \lambda_{1,2} + \frac{g_{Z'}^2}{2} Q^2_{H_{u,d}} \;\; ; \;\; \lambda_3 \to \lambda_3 + g_{Z'}^2 Q_{H_u} Q_{H_d} \;\; ; \;\; \lambda_P^{u,d} \to \lambda_P^{u,d} + g_{Z'}^2 Q^2_{H_{u,d}} \;\; ; \;\; \kappa^2 = \frac{g_{Z'}^2}{2} Q_S^2 \quad (1.11)$$

  Note that the SM-fermion sector is also charged under the $U(1)'$-gauge group, so as to ensure invariance of the usual Yukawa terms. To avoid a chiral anomaly of the $U(1)'$ symmetry, an exotic fermion sector will also be necessary.

One may also write tree-level conditions of a different form, not protected by any symmetry: this is for instance the case in the nMSSM, where a $\mathbb{Z}_5^R$ or a $\mathbb{Z}_7^R$ symmetry [?] is imposed at the level of the superpotential, so as to forbid all renormalizable pure singlet-terms, then broken explicitly by gravity effects, in order to arrange for an effective tadpole term (so as to break the resulting P.Q. symmetry), broken also explicitly by the soft-terms. The tree-level potential then differs from the $\mathbb{Z}_3$ case (**??**) by the requirements:

$$\lambda_P^M = \kappa = A_S = 0 \quad ; \quad \lambda_T, m_{12}^2 \neq 0 \quad (1.12)$$

We hence define:

$$\begin{aligned}
\mathcal{V}_T^S &= m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 - \left(m_{12}^2 H_u \cdot H_d + h.c.\right) \\
&+ \frac{\lambda_1}{2}|H_d|^4 + \frac{\lambda_2}{2}|H_u|^4 + \lambda_3 |H_u|^2 |H_d|^2 + \lambda_4 |H_u \cdot H_d|^2 \\
&+ m_S^2 |S|^2 + [\lambda_T S + h.c.] + \lambda_P^u |S|^2 |H_u|^2 + \lambda_P^d |S|^2 |H_d|^2 + [A_{ud} S H_u \cdot H_d + h.c.] \quad (1.13)
\end{aligned}$$

While the absence of a residual symmetry at low-energy is a deliberate feature of the nMSSM (in order to circumvent both axion and domain-wall problems), the resulting lack of protection of the tree-level couplings at low-energy will lead to sizeable consequences for the parameter reconstruction at the loop-level, as we will see later.

1.3 Mass matrices

Spontaneous symmetry breaking is achieved when the scalar fields develop a v.e.v.,

$$\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}, \quad \langle H_d \rangle = \begin{pmatrix} v_d \\ 0 \end{pmatrix}, \quad \langle S \rangle = s \quad (1.14)$$

Imposing the minimization conditions associated with the most general potential in Eq.(**??**), one may trade the parameters $m_{H_d}^2$, $m_{H_u}^2$, m_S^2 for the v.e.v.'s v_u, v_d, s. Introducing the usual definitions $v \equiv$

$\sqrt{v_u^2+v_d^2} \simeq 174$ GeV, $\tan\beta \equiv v_u/v_d$, we can write these relations as[3],

$$\begin{aligned}
m^2_{H_d} &= \Big[A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\Big]\, s\, t_\beta - \Big[2A_{ds}+(\lambda_P^d+2\tilde{\lambda}_P^d)s\Big]\, s \\
&- \lambda_1 v^2 c_\beta^2 - (\lambda_3+\lambda_4+\lambda_5)\, v^2 s_\beta^2 + \left(\lambda_6 v^2 s_\beta^2 - m_{12}^2\right) t_\beta + 3v^2 s_{2\beta}\lambda_7 \\
m^2_{H_u} &= \Big[A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\Big]\, s\, t_\beta^{-1} - \Big[2A_{us}+(\lambda_P^u+2\tilde{\lambda}_P^u)s\Big]\, s \\
&- \lambda_2 v^2 s_\beta^2 - (\lambda_3+\lambda_4+\lambda_5)\, v^2 c_\beta^2 + \left(\lambda_7 v^2 c_\beta^2 - m_{12}^2\right) t_\beta^{-1} + 3v^2 s_{2\beta}\lambda_6 \qquad (1.15) \\
m^2_S &= \Big[A_{ud}+\tilde{A}_{ud}+2(\lambda_P^M+\tilde{\lambda}_M^P+\lambda_M)s\Big]\, \frac{v^2 s_{2\beta}}{2s} - \Big[A_S+\tilde{A}_S+(2\kappa^2+\kappa_S^2+\tilde{\kappa}_S^2)s\Big]\, s \\
&- \Big[(A_{us}+2\tilde{\lambda}_P^u s)s_\beta^2+(A_{ds}+2\tilde{\lambda}_P^d s)c_\beta^2\Big]\, \frac{v^2}{s} - \lambda_P^u v^2 s_\beta^2 - \lambda_P^d v^2 c_\beta^2 - \frac{\lambda_T}{s} - \mu_S^2
\end{aligned}$$

The quadratic terms in $H^\pm_{u,d}$ provide us with the charged Higgs mass matrix:

$$\mathcal{M}^2_{H^\pm} \equiv \left[(A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s)s - \left(\frac{1}{2}(\lambda_4+\lambda_5)s_{2\beta} - \lambda_6 s_\beta^2 - \lambda_7 c_\beta^2\right) v^2 - m_{12}^2\right] \begin{bmatrix} t_\beta^{-1} & 1 \\ 1 & t_\beta \end{bmatrix} \qquad (1.16)$$

Its diagonalization expectedly delivers (massless) charged Goldstone bosons $G^\pm \equiv \cos\beta H_d^\pm - \sin\beta H_u^\pm$ and the physical charged Higgs $H^\pm \equiv \cos\beta H_u^\pm + \sin\beta H_d^\pm$, with mass:

$$m^2_{H^\pm} = \frac{2}{s_{2\beta}}\left[(A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s)s - \left(\frac{1}{2}(\lambda_4+\lambda_5)s_{2\beta} - \lambda_6 s_\beta^2 - \lambda_7 c_\beta^2\right) v^2 - m_{12}^2\right] \qquad (1.17)$$

We turn to the CP-odd squared mass matrix, written in the basis (a_d^0, a_u^0, a_s^0):

$$\begin{aligned}
\mathcal{M}^2_{P\,11} &= \left[\left(A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\right)s + \left(\lambda_6 s_\beta^2+\lambda_7 c_\beta^2-\lambda_5 s_{2\beta}\right)v^2 - m_{12}^2\right] t_\beta \\
\mathcal{M}^2_{P\,22} &= \left[\left(A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\right)s + \left(\lambda_6 s_\beta^2+\lambda_7 c_\beta^2-\lambda_5 s_{2\beta}\right)v^2 - m_{12}^2\right] t_\beta^{-1} \\
\mathcal{M}^2_{P\,33} &= \Big[A_{ud}+\tilde{A}_{ud}+4(\lambda_P^M+\tilde{\lambda}_P^M)s\Big]\, \frac{v^2 s_{2\beta}}{2s} - \left[3A_S+\frac{\tilde{A}_S}{3}+(4\kappa_S^2+\tilde{\kappa}_S^2)s\right] s \qquad (1.18) \\
&\quad - \left[\left(A_{us}+4\tilde{\lambda}_P^u s\right)s_\beta^2 + \left(A_{ds}+4\tilde{\lambda}_P^d s\right)c_\beta^2\right]\frac{v^2}{s} - 2\mu_S^2 - \frac{\lambda_T}{s} \\
\mathcal{M}^2_{P\,12} &= \Big[A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\Big]\, s + \left(\lambda_6 s_\beta^2+\lambda_7 c_\beta^2-\lambda_5 s_{2\beta}\right)v^2 - m_{12}^2 \\
\mathcal{M}^2_{P\,13} &= \Big[A_{ud}-\tilde{A}_{ud}-2(\lambda_P^M-\tilde{\lambda}_P^M)s\Big]\, v s_\beta \\
\mathcal{M}^2_{P\,23} &= \Big[A_{ud}-\tilde{A}_{ud}-2(\lambda_P^M-\tilde{\lambda}_P^M)s\Big]\, v c_\beta
\end{aligned}$$

The neutral Goldstone boson $G^0 \equiv \cos\beta a_d^0 - \sin\beta a_u^0$ can be isolated through the rotation with angle β and we are left with the 2×2 matrix $\mathcal{M}^2_{P'}$ in the basis (a_D^0, a_S^0), with $a_D^0 \equiv \cos\beta a_u^0 + \sin\beta a_d^0$

$$\begin{aligned}
\mathcal{M}^2_{P'\,11} &= \frac{2}{s_{2\beta}}\left[\left(A_{ud}+\tilde{A}_{ud}+(\lambda_P^M+\tilde{\lambda}_P^M+\lambda_M)s\right)s - (\lambda_5 s_{2\beta}-\lambda_6 s_\beta^2-\lambda_7 c_\beta^2)v^2 - m_{12}^2\right] \\
\mathcal{M}^2_{P'\,22} &= \Big[A_{ud}+\tilde{A}_{ud}+4(\lambda_P^M+\tilde{\lambda}_P^M)s\Big]\, \frac{v^2 s_{2\beta}}{2s} - \left[3A_S+\frac{\tilde{A}_S}{3}+(4\kappa_S^2+\tilde{\kappa}_S^2)s\right] s \qquad (1.19) \\
&\quad - \left[\left(A_{us}+4\tilde{\lambda}_P^u s\right)s_\beta^2 + \left(A_{ds}+4\tilde{\lambda}_P^d s\right)c_\beta^2\right]\frac{v^2}{s} - 2\mu_S^2 - \frac{\lambda_T}{s} \\
\mathcal{M}^2_{P'\,12} &= \Big[A_{ud}-\tilde{A}_{ud}-2(\lambda_P^M-\tilde{\lambda}_M^P)s\Big]\, v
\end{aligned}$$

[3]We use the shorthand notations $c_\beta = \cos\beta$, $s_\beta = \sin\beta$, $s_{2\beta} = \sin 2\beta$, $t_\beta = \tan\beta$ etc...

$\mathcal{M}^2_{P'}$ is diagonalized in the subblock of the physical states (a^0_D, a^0_S) by the orthogonal matrix P', to give the two physical CP-odd squared mass $m^2_{a^0_1}$, $m^2_{a^0_2}$, such that

$$\mathrm{diag}(m^2_{a^0_1}, m^2_{a^0_2}) = P'\mathcal{M}^2_{P'}P'^{-1} \tag{1.20}$$

Finally, the CP-even squared mass matrix, in the basis (h^0_d, h^0_u, h^0_S), reads:

$$\begin{aligned}
\mathcal{M}^2_{S\,11} &= \left[\left(A_{ud} + \tilde{A}_{ud} + (\lambda^M_P + \tilde{\lambda}^M_P + \lambda_M)s\right)s + (\lambda_6 s^2_\beta - 3\lambda_7 c^2_\beta)v^2 - m^2_{12}\right] t_\beta + 2\lambda_1 v^2 c^2_\beta \\
\mathcal{M}^2_{S\,22} &= \left[\left(A_{ud} + \tilde{A}_{ud} + (\lambda^M_P + \tilde{\lambda}^M_P + \lambda_M)s\right)s + (\lambda_7 c^2_\beta - 3\lambda_6 s^2_\beta)v^2 - m^2_{12})\right] t^{-1}_\beta + 2\lambda_2 v^2 s^2_\beta \\
\mathcal{M}^2_{S\,33} &= \left[A_{ud} + \tilde{A}_{ud}\right]\frac{v^2 s_{2\beta}}{2s} + \left[A_S + \tilde{A}_S + 2(2\kappa^2 + \kappa^2_S + \tilde{\kappa}^2_S)s\right]s - \left(A_{us}s^2_\beta + A_{ds}c^2_\beta\right)\frac{v^2}{s} - \frac{\lambda_T}{s} \\
\mathcal{M}^2_{S\,12} &= -\left[A_{ud} + \tilde{A}_{ud} + (\lambda^M_P + \tilde{\lambda}^M_P + \lambda_M)s\right]s + \left[(\lambda_3 + \lambda_4 + \lambda_5)s_{2\beta} - 3(\lambda_6 s^2_\beta + \lambda_7 c^2_\beta)\right]v^2 + m^2_{12} \\
\mathcal{M}^2_{S\,13} &= -\left[A_{ud} + \tilde{A}_{ud} + 2(\lambda^M_P + \tilde{\lambda}^M_P + \lambda_M)s\right]vs_\beta + 2\left[A_{ds} + (\lambda^d_P + 2\tilde{\lambda}^d_P)s\right]vc_\beta \\
\mathcal{M}^2_{S\,23} &= -\left[A_{ud} + \tilde{A}_{ud} + 2(\lambda^M_P + \tilde{\lambda}^M_P + \lambda_M)s\right]vc_\beta + 2\left[A_{us} + (\lambda^u_P + 2\tilde{\lambda}^u_P)s\right]vs_\beta
\end{aligned} \tag{1.21}$$

which is diagonalized by a 3×3 orthogonal matrix S, resulting in three CP-even squared masses $m^2_{h^0_1}$, $m^2_{h^0_2}$, $m^2_{h^0_3}$, such that

$$\mathrm{diag}(m^2_{h^0_1}, m^2_{h^0_2}, m^2_{h^0_3}) = S\mathcal{M}^2_S S^{-1} \tag{1.22}$$

We are thus finally left with seven physical Higgs particles, once the three Goldstone bosons G^0, $G^\pm$, giving mass to the $W^\pm$ and Z^0 bosons, have been discarded. In the particular case of the $U(1)'$-gauge symmetry, however, the P.Q.-axion (associated to the vanishing eigenvalue of $\mathcal{M}^2_{P'}$) is also unphysical (giving mass to the Z'-boson, gauge-field of the $U(1)'$ symmetry [?]), so that we are left with only one CP-odd physical mass.

2 Reconstruction of the effective parameters

2.1 Masses and mixing angles as physical input

From an experimental point of view, the 'λ_i' parameters are not directly accessible: they will enter as combinations within the expressions for the Higgs masses and self-couplings. The latter can hopefully be accessed through the experimental measurement of physical quantities. 'Inverting the system', we can therefore trade some λ_i parameters for such physical input. In the simplest case, one would directly use the Higgs masses and their mixing angles, assuming these can be measured (e.g. from fermion/gauge couplings), as the new, physical input. For the 2-doublet+1-singlet system, such quantities provide us with 12 conditions (input measurements) on the λ_i's: the masses of the 2 CP-odd bosons, 3 CP-even and 1 (complex) charged Higgs; the mixing angles from the CP-even (3), CP-odd (1) and the Goldstone (1: β) sectors; finally, the electroweak v.e.v. $v = \sqrt{v_u^2 + v_d^2}$ (from M_W for example). Should those twelve relations prove insufficient to determine all the λ_i's (as is obviously the case for the most general potential), one would have to resort to Higgs self-couplings (or input from another sector) in order to fully determine the parameters.Accessing such self-couplings would require that double or triple Higgs production are kinematically open. This task would most comprehensibly done at future linear colliders. In the meanwhile, the measurements of masses and mixing angles still allow for a partial inversion.

We will assume in the following that all the Higgs-masses have been measured. Note that this hypothesis is somewhat optimistic since singlet-like fields do not couple directly to SM-fermions and gauge-bosons, hence are essentially elusive: only when there is substantial mixing with the doublet states can we expect to access them without having to rely on multi-Higgs couplings. As for the mixing angles, assuming all the Higgs states have been observed in SM decay-channels, one can derive them from the couplings to fermions (note that leptonic decay channels are likely to give cleaner information) and gauge bosons. For a type II model, we have (taken from [?]):

$$\begin{aligned}
h_i^0 t_L t_R^c &= -\frac{Y_t}{\sqrt{2}} S_{i2} \\
h_i^0 b_L b_R^c &= \frac{Y_b}{\sqrt{2}} S_{i1} \\
h_i^0 \tau_L \tau_R^c &= \frac{Y_\tau}{\sqrt{2}} S_{i1} \\
a_i^0 t_L t_R^c &= -i\frac{Y_t}{\sqrt{2}} c_\beta P'_{i1} \\
a_i^0 b_L b_R^c &= i\frac{Y_b}{\sqrt{2}} s_\beta P'_{i1} \\
a_i^0 \tau_L \tau_R^c &= i\frac{Y_\tau}{\sqrt{2}} s_\beta P'_{i1} \\
H^+ b_L t_R^c &= Y_t c_\beta \\
H^- t_L b_R^c &= -Y_b s_\beta \\
H^- \nu_{\tau L} \tau_R^c &= -Y_\tau s_\beta
\end{aligned} \tag{2.1}$$

where,

$$Y_t = \frac{m_t}{v s_\beta}, \quad Y_b = \frac{m_b}{v c_\beta}, \quad Y_\tau = \frac{m_\tau}{v c_\beta} \tag{2.2}$$

and (we mention here only the 1-Higgs to 2-gauge couplings; note that, albeit more difficult to measure, 2-Higgs to 1-gauge as well as quartic couplings shall play a very important role for testing the model):

$$\begin{aligned}
h_i^0 Z_\mu Z_\nu &= g_{\mu\nu} \frac{g'^2 + g^2}{\sqrt{2}} v \left(c_\beta S_{i1} + s_\beta S_{i2} \right) \\
h_i^0 W_\mu^+ W_\nu^- &= g_{\mu\nu} \frac{g^2}{\sqrt{2}} v \left(c_\beta S_{i1} + s_\beta S_{i2} \right)
\end{aligned} \tag{2.3}$$

Combining Higgs couplings to the vector bosons with those to up/down fermions, one can access e.g. S_{i1}/S_{i2}. Moreover, one may be tempted to use Higgs decays into two photons to extract information about the mixing angles: even admitting that such processes are dominated by quark loops, the corresponding relation of branching ratios to mixing angles is already non-trivial and would require an involved extraction procedure for exploitation.

Unitarity relations could also prove useful. For example, a 'missing' matrix element S_{ij} could be reconstructed from

$$\sum_{k=0}^{3} S_{ik} S_{jk} = \delta_{ij} = \sum_{k=0}^{3} S_{ki} S_{kj} \quad i,j = 1,2,3 \tag{2.4}$$

A possible (naive) strategy to reconstruct the mixing angles would be the following: having measured the charged Higgs decay into third generation quarks, one could then deduce t_β, since the ratio $m_{t,b}/v$ is fixed by SM measurements. Then the (doublet) elements S_{i1}, S_{i2}, P'_{i1} could be obtained unambigously from the decays of neutral higgs states into fermions and gauge-bosons. The unitarity relations would finally provide the magnitude of the S_{i3} and P'_{i2} (singlet) elements.

Note finally that, while a full experimental determination of the Higgs mass matrices may seem over-optimistic in the short run, there exists a practical case where we have access to such data: it is that of the output of spectrum generators (e.g. the publicly available `NMSSMTools`, [?]). We will resort to that practical application in the last part of the present paper.

2.2 Partial reconstruction in the general case

Considering the general potential of Eq.(**??**) and discarding any assumption as to an underlying model, a complete reconstruction of the 29 parameters (28 of which being relevant) cannot succeed with only the twelve mass/mixing conditions, hence calls for the measurement of Higgs self-couplings. Yet, information from Eqs.(**??**,**??**,**??**) can already be implemented in a partial reconstruction:

$$\begin{cases} [(A_{ud}+\lambda_P^M s)s-m_{12}^2]\frac{2}{s_{2\beta}} = m^2_{a_i^0}P'^{\,2}_{i1}+\lambda_P^1 \\ (A_{ud}-2\lambda_P^M s)v = m^2_{a_i^0}P'_{i1}P'_{i2}+\lambda_P^{12} \\ -3A_S s+\left(A_{ud}+4\lambda_P^M s\right)\frac{v^2 s_{2\beta}}{2s}-\frac{\lambda_T}{s} = m^2_{a_i^0}P'^{2}_{i2}+\lambda_P^2 \\ \frac{2}{s_{2\beta}}\left[(A_{ud}+\lambda_P^M s)s-\frac{\lambda_4}{2}v^2 s_{2\beta}-m_{12}^2\right] = m^2_{H^\pm}+\lambda_\pm \\ [(A_{ud}+\lambda_P^M s)s-m_{12}^2]t_\beta+2\lambda_1 v^2 c_\beta^2 = m^2_{h_i^0}S_{i1}^2-\lambda_S^1 \\ [(A_{ud}+\lambda_P^M s)s-m_{12}^2]t_\beta^{-1}+2\lambda_2 v^2 s_\beta^2 = m^2_{h_i^0}S_{i2}^2-\lambda_S^2 \\ A_S s+4\kappa^2 s^2+A_{ud}\frac{v^2}{2s}s_{2\beta}-\frac{\lambda_T}{s} = m^2_{h_i^0}S_{i3}^2-\lambda_S^3 \\ -(A_{ud}+\lambda_P^M s)s+(\lambda_3+\lambda_4)v^2 s_{2\beta}+m_{12}^2 = m^2_{h_i^0}S_{i1}S_{i2}-\lambda_S^{12} \\ -(A_{ud}+2\lambda_P^M s)vs_\beta+2\lambda_P^d svc_\beta = m^2_{h_i^0}S_{i1}S_{i3}-\lambda_S^{13} \\ -(A_{ud}+2\lambda_P^M s)vc_\beta+2\lambda_P^u svs_\beta = m^2_{h_i^0}S_{i2}S_{i3}-\lambda_S^{23} \end{cases} \tag{2.5}$$

where $\lambda_P^{1,2,3}, \lambda_P^{12}, \lambda_\pm, \lambda_S^{1,2,3}, \lambda_S^{12,13,23}$ are given by

$$\begin{cases} \lambda_P^1 = -\frac{2}{s_{2\beta}}\left[2\left(\tilde{A}_{ud}+(\tilde{\lambda}_M^P+\lambda_M)s\right)s-(\lambda_5 s_{2\beta}-\lambda_6 s_\beta^2-\lambda_7 c_\beta^2)v^2\right] \\ \lambda_P^{12} = (\tilde{A}_{ud}-2\tilde{\lambda}_M^P s)v \\ \lambda_P^2 = -\left(\tilde{A}_{ud}+4\tilde{\lambda}_P^M s\right)\frac{v^2 s_{2\beta}}{2s}+\left[\frac{\tilde{A}_S}{3}+(4\kappa_S^2+\tilde{\kappa}_S^2)s\right]s+\left[(A_{us}+4\tilde{\lambda}_P^u s)s_\beta^2+(A_{ds}+4\tilde{\lambda}_P^d s)c_\beta^2\right]\frac{v^2}{s}+2\mu_S^2 \\ \lambda_\pm = -\frac{2}{s_{2\beta}}\left[(\tilde{A}_{ud}+(\tilde{\lambda}_P^M+\lambda_M)s)s-\left(\frac{1}{2}\lambda_5 s_{2\beta}-\lambda_6 s_\beta^2-\lambda_7 c_\beta^2\right)v^2\right] \\ \lambda_S^1 = \left[\left(\tilde{A}_{ud}+(\tilde{\lambda}_P^M+\lambda_M)s\right)s+(\lambda_6 s_\beta^2-3\lambda_7 c_\beta^2)v^2\right]t_\beta \\ \lambda_S^2 = \left[\left(\tilde{A}_{ud}+(\tilde{\lambda}_P^M+\lambda_M)s\right)s+(\lambda_7 c_\beta^2-3\lambda_6 s_\beta^2)v^2\right]t_\beta^{-1} \\ \lambda_S^3 = \tilde{A}_{ud}\frac{v^2 s_{2\beta}}{2s}+\tilde{A}_S s+2(\kappa_S^2+\tilde{\kappa}_S^2)s^2-\left(A_{us}s_\beta^2+A_{ds}c_\beta^2\right)\frac{v^2}{s} \\ \lambda_S^{12} = -(\tilde{A}_{ud}+(\tilde{\lambda}_P^M+\lambda_M)s)s+\left[\lambda_5 s_{2\beta}-3(\lambda_6 s_\beta^2+\lambda_7 c_\beta^2)\right]v^2 \\ \lambda_S^{13} = -\left[\tilde{A}_{ud}+2(\tilde{\lambda}_P^M+\lambda_M)s\right]vs_\beta+2(2\tilde{\lambda}_P^d s+A_{ds})vc_\beta \\ \lambda_S^{23} = -\left[\tilde{A}_{ud}+2(\tilde{\lambda}_P^M+\lambda_M)s\right]vc_\beta+2(2\tilde{\lambda}_P^u s+A_{us})vs_\beta \end{cases} \tag{2.6}$$

Our (arbitrary) choice in ordering the parameters within Eqs.(**??**,**??**) was guided by the terms that are relevant at leading order in the n/NMSSM and the UMSSM potentials: beyond $m^2_{H_d}$, $m^2_{H_u}$, m^2_S, which are common to the three models, those are given by

$$\begin{array}{rl} \text{NMSSM} & : \lambda_{1-4},\ \lambda_P^{u,d},\ \lambda_P^M,\ \kappa^2, A_{ud},\ A_S \\ \text{nMSSM} & : \lambda_{1-4},\ m_{12}^2,\ \lambda_P^{u,d},\ \lambda_T,\ A_{ud} \\ \text{UMSSM} & : \lambda_{1-4},\ \lambda_P^{u,d},\ \kappa^2,\ A_{ud} \end{array}$$

These parameters were collected on the left-hand side of Eq.(**??**), while the remaining ones enter the right-hand side through Eq.(**??**).
Note that the relations of Eq.(**??**) hold at any order (since Eq.(**??**) is the most general renormalizable potential satisfying the gauge-symmetry). A practical use of Eq.(**??**) would lie in a model-independent

analysis of a 2-doublet+1singlet potential (in order to discriminate among models, constrain them through precision tests). Then the twelve mass conditions can be used to simplify twelve (arbitrarily chosen) parameters, hence leaving the remaining couplings as the relevant degrees of freedom intervening in / to be determined from the Higgs self-couplings. Not much predictivity should be expected, however, in this general case.

2.3 Reconstruction at the classical level in the constrained models

We focus here on the specific cases inspired by the SUSY models: $\mathcal{V}_T^S$, $\mathcal{V}_{PQ}^S$, $\mathcal{V}_{PQ'}^S$ and $\mathcal{V}_{\mathbb{Z}_3}^S$. Note that such potentials are considered at the classical order: quantum effects and explicit/spontaneous breaking of the symmetries in principle destabilize those potentials to generate the most general one. At this leading order, however, the Eqs.(**??**) vanish, leaving Eqs.(**??**) in a very simple form. Note additionally the further requirements for each potential:

$$\begin{array}{cl} \mathcal{V}_{\mathbb{Z}_3}^S & : m_{12}^2 = \lambda_T = 0 \\ \mathcal{V}_T^S & : A_S = \kappa^2 = \lambda_M^P = 0 \\ \mathcal{V}_{PQ}^S & : A_S = \lambda_M^P = m_{12}^2 = \lambda_T = 0 \\ \mathcal{V}_{PQ'}^S & : A_S = A_{ud} = m_{12}^2 = \lambda_T = 0 \end{array}$$

We end up with eleven classical parameters and eleven conditions[4] for both the potentials $\mathcal{V}_{PQ}^S$ and $\mathcal{V}_{PQ'}^S$. In these cases, all the parameters in the Higgs potential can thus be reconstructed (at leading order) from Higgs masses and mixings: this procedure is explicitly carried out in appendix **??**, Eqs.(**??**,**??**).
In the case of $\mathcal{V}_{\mathbb{Z}_3}^S$, the thirteen classical parameters cannot be fully determined from the twelve conditions. The remaining degree of freedom is conveniently chosen as the singlet v.e.v. s: the reconstruction is also given in appendix **??**, Eqs.(**??**,**??**). Several tracks can be followed in order to determine this remaining degree of freedom. The first one, sticking to the Higgs potential, would consist in relying on trilinear couplings, such as $h_i^0 H^+ H^-$ or $h_i^0 a_j^0 a_j^0$, where the neutral Higgs fields would be largely singlet in nature: kinematical limits and the elusive nature of singlets would tend to disfavor this strategy. Another possibility would be to input information from some other sector (if any): measurement of the higgsino masses in the NMSSM could provide the missing information. Finally, a more predictive option would be to enforce some additional requirement, such as relations among the tree-level couplings: the tree-level relations of the NMSSM, $\frac{\lambda_P^u}{\lambda_P^d} = 1$ or $\frac{\kappa^2(a\lambda_P^u + b\lambda_P^d)}{(\lambda_P^M)^2 \cdot (a+b)} = 1$ (where a, b are real numbers), for instance, or a measure of the P.Q. symmetry breaking, such as $\frac{(a+b)\lambda_P^M}{a\lambda_P^u + b\lambda_P^d} \sim \frac{\kappa}{\lambda}$, may be used as guidelines.
Finally for $\mathcal{V}_T^S$, we have twelve parameters and twelve conditions. Yet a full inversion is not possible either, because CP-even and CP-odd singlet masses are explicitly degenerate in this potential, leaving a bound system. The remaining degree of freedom is again chosen as the singlet v.e.v. s in appendix **??**, Eq.(**??**), but could be replaced by e.g. λ_T, as a measure of the violation of $\mathbb{Z}_3$, for example.
So far, we have considered only the Higgs potentials separately. Moving explicitly to the underlying SUSY models, however, the λ_i's are further constrained by the tree-level relations resulting from their supersymmetric origins: we count 7 parameters in the nMSSM Higgs sector (λ_T, m_{12}^2, $m_{H_u}^2$, $m_{H_d}^2$, m_S^2, λ, A_λ), 7 in the NMSSM as well ($m_{H_u}^2$, $m_{H_d}^2$, m_S^2, λ, A_λ, κ, A_κ) and 6 in the UMSSM ($m_{H_u}^2$, $m_{H_d}^2$, m_S^2, λ, A_λ, $g_{Z'}$; note that we regard the Higgs charges under $U(1)'$ as fixed). Those parameters are then over-constrained by Eq.(**??**) and one should thus consider the remaining conditions as a measurement of the deviation from tree-level conditions due to higher orders (we remind here that the tree-level relations induced by the model of origin among the parameters of the potential are likely to be spoilt by quantum corrections). Depending on the information at our disposal in the remaining spectrum (e.g. SUSY masses), such conditions may be used to estimate the missing parameters (e.g. sfermion masses or trilinear soft couplings) or regarded as precision tests of the model. Note that if the SUSY spectrum is

[4]The explicit presence of a P.Q.-axion, identified as a_1^0, leads to one trivial condition in the CP-odd sector.

sufficiently documented as well, this measurement of the Higgs parameters at leading order, would allow for a (perturbative) computation of all the λ_i's within the specific models at higher orders.

2.4 Reconstruction at the loop level: NMSSM vs. nMSSM

Now we want to apply this formalism to higher order effects. The purpose is simple: it has been shown that, in the MSSM, the bulk of the corrections in Higgs-to-Higgs couplings could be absorbed in writing such couplings in terms of the corrected masses (see for example [?, ?] and the third reference in [?]); could a similar recipe apply to the 2-doublet+1singlet setup? A first strategy is the one presented at the end of the previous subsection: in a definite model, the Higgs spectrum may allow for a determination of the Higgs parameters at leading order; then, provided sufficient information from the other sectors stand at our disposal, reconstructing all the λ_i's at higher order is simply a matter of perturbative calculations. Yet, this approach relies on a heavy machinery and on input which is external to the Higgs sector. We would like to consider cases where input from the Higgs sector only (or almost only) would already improve on the simple tree-level expression for the Higgs self-couplings.

In principle, whatever the potential looked like at the classical level, quantum corrections will generate contributions to all the parameters in the general potential – Eq.(??) – (unless a symmetry protects certain parameters, but we have seen that such symmetries are spontaneously broken by the Higgs v.e.v.'s anyway). Therefore, while the partial-inversion of the general case (Eqs.(??,??)) is still possible, little predictivity or practical use is to be expected from such relations, because the number of undetermined parameters is high. To extract meaningful information, beyond the leading order, from the Higgs spectrum, one would need the corrected potential to retain a sufficiently simple form beyond the classical order.

To be more specific, we consider a tree-level potential of the form (H representing any of the Higgs fields, μ^2, a bilinear, A, a trilinear, and λ_i, a quartic coupling):

$$\mathcal{V}_{\text{tree}} = \mu^2 H^2 + AH^3 + \lambda H^4 \tag{2.7}$$

We now include the radiative corrections, which shift the potential as:

$$\mathcal{V}_{\text{eff}} = (\mu^2 + \delta\mu^2)H^2 + (A + \delta A)H^3 + (\lambda + \delta\lambda)H^4 + \delta\tilde{\mu}^2 H^2 + \delta\tilde{A}H^3 + \delta\tilde{\lambda}H^4 \tag{2.8}$$

where $\delta\mu^2$, δA and $\delta\lambda$ represent corrections to parameters existing at tree-level, while $\delta\tilde{\mu}^2$, $\delta\tilde{A}$ and $\delta\tilde{\lambda}$ denote new couplings which were forbidden by symmetries at tree-level and emerge only at the radiative level. Neglecting numerical coefficients, the corrected Higgs mass m^2 and the trilinear self-coupling g will read (schematically):

$$\begin{cases} m^2 \simeq \mu^2 + \delta\mu^2 + \delta\tilde{\mu}^2 + (A + \delta A + \delta\tilde{A})\langle H\rangle + (\lambda + \delta\lambda + \delta\tilde{\lambda})\langle H\rangle^2 = m^2_{\text{tree}} + O(\delta, \tilde{\delta}) \\ g \simeq A + \delta A + \delta\tilde{A} + (\lambda + \delta\lambda + \delta\tilde{\lambda})\langle H\rangle = g_{\text{tree}} + O(\delta, \tilde{\delta}) \end{cases} \tag{2.9}$$

(with the short-hand notation $\delta/\tilde{\delta}$ for loop induced corrections to parameters present/absent at tree level.) We now assume that we have access to the mass m^2, either from experimental data or from a spectrum generator. Using g_{tree} in the computation of physical quantities (branching ratios, cross-sections) will result in an error of order $O\left(\frac{\delta,\tilde{\delta}}{g}\right)$. If we use the expression for the corrected mass to inverse (partially) the relation between mass and tree level parameters, we obtain: $\delta = \delta_{m^2} + \mathcal{O}(\tilde{\delta})$, where δ_{m^2} symbolises the result of the inversion procedure. The trilinear couplings then provide: $g_{m^2} = g + \mathcal{O}(\tilde{\delta})$, resulting in an error of $\mathcal{O}\left(\frac{\tilde{\delta}}{g}\right)$ at the level of cross sections/branching ratios. Claiming that the inversion procedure carries any improvement with respect to a simple tree-level evaluation holds at the sole condition that radiative corrections δ to tree-level parameters are more important, in magnitude, than the contributions $\tilde{\delta}$ to other operators. Otherwise, even if we identify the parameters subject to large contributions, it is unlikely that the mass-matrices would suffice in determining both these parameters and those intervening at tree-level, unless we input some additional tree-level relations, as in the case of the matching conditions in Eq.(**??**), (**??**) or (**??**).

This discussion shows that, to extract some benefits – beyond the leading order – from the conditions relating masses to effective parameters, we need to identify which terms are potentially subject to large radiative corrections. A simple criterion can be invoked at the one-loop level: it is that of the leading logarithms. To identify those, we simply resort to the Coleman-Weinberg [**?**] one-loop effective potential and analyse the outcome for the special case of the SUSY-inspired models under scrutiny. This method has long been employed for the computation of corrections to the Higgs masses, both in the MSSM [**?**] and in the NMSSM [**?**,**?**] (and references therein). In this approach, the effective corrections to the scalar potential at a scale Λ are determined by the field-dependent tree-level mass matrices $M^2_\Phi(S, H_d, H_u, \ldots)$ of the various fields Φ entering the spectrum, according to (in the $\overline{DR}$-scheme, but note that we shall be interested in the logarithms only):

$$\Delta\mathcal{V}^{\Lambda}_{\text{eff}}(S, H_d, H_u, \ldots) = \frac{1}{64\pi^2}\sum_{\Phi} C_\Phi M^4_\Phi \left[\ln\left(\frac{M^2_\Phi}{\Lambda^2}\right) - \frac{3}{2}\right] \tag{2.10}$$

Here C_Φ, which counts the degrees of freedom, takes the values 1 for real scalar fields, 2 for complex ones, -2 for Majorana fermions, -4 for Dirac fermions and 3 for real gauge-fields. Note that we are interested in the Higgs potential solely, so that we will retain dependence on S, H_d, H_u only, within M^2_Φ. Moreover, we consider no EW-violating effects so that we will not expand the doublet fields H_d, H_u around their v.e.v.'s (except within logarithms). Additionally, the $SU(2)_L$-symmetry can then be invoked to retain only the neutral Higgs fields S, H^0_d, H^0_u (the dependence on the charged Higgs fields can then be restored afterwards in virtue of $SU(2)_L$: only the λ_3 and λ_4 parameters cannot be disentangled in this fashion, but both parameters being present at tree-level in the models we consider, this will be of little consequence for our analysis). We then determine the contributions to the parameters of Eq.(**??**) by letting the singlet take its v.e.v., $S = s + \tilde{S}$, then truncating Eq.(**??**) to renormalizable terms, finally projecting on the couplings of Eq.(**??**).

The results of our analysis of the large logarithms, in the cases of the NMSSM and nMSSM, are provided in appendix **??**. The situation of the NMSSM is quite simple: leading logarithms favor $\mathbb{Z}_3$-conserving terms. We can thus claim, for this model, that the inversion procedure for the $\mathbb{Z}_3$-conserving potential, presented in the previous subsection and Eqs.(**??**,**??**), improves on the tree-level implementation of the couplings and actually includes leading-logarithms automatically. Note that, as defined in Eqs.(**??**,**??**), the effective $\mathbb{Z}_3$-conserving parameters are directly determined in terms of physical quantities, meaning that they do not depend on the renormalization scale Λ : they are simply the parameters of the effective $\mathbb{Z}_3$-conserving potential associated with the physical Higgs spectrum. What we checked explicitly in the Coleman-Weinberg approach (which depends on the renormalization scale Λ) is that this constrained form of an effective potential was legitimate at least up to leading logarithms. Beyond, the effect of the $\mathbb{Z}_3$-violating terms (due to the truncation of the potential to operators of mass-dimension ≤ 4) cannot be neglected. In the case of the nMSSM, however, potentially large logarithms affect non-classical terms. In fact, all the sectors contribute to the $\mathbb{Z}_3$-conserving parameters (including those vanishing at tree level in this model). Additionally, logarithms originating from the nMSSM Higgs sector (the only sector which is sensitive to the breakdown of $\mathbb{Z}_3$ at tree-level) also affect $\mathbb{Z}_3$-violating terms. Inclusion of the leading higher-order effects from the inversion procedure of subsection **??**, Eq.(**??**), thus seems dubious in this case. It seems natural to ascribe this difference of behavior, between nMSSM and NMSSM, to the protection of the parameters by the $\mathbb{Z}_3$-symmetry, which albeit spontaneously broken by the singlet v.e.v., continues to favor $\mathbb{Z}_3$-conserving terms within the NMSSM. We should thus expect a similar property, whatever the $\mathbb{Z}_3$-symmetric model is (SUSY or not), and, beyond the $\mathbb{Z}_3$-symmetry, in any model retaining a symmetry (or approximate symmetry) at low-energy, e.g. PQ or PQ'.