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Vector Correlator in Massless QCD at Order $\mathcal{O}(\alpha_s^4)$ and the QED β-function at five loop

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Abstract: We present a concise summary of recent results for the vector correlator in massless QCD at order $\mathcal{O}(\alpha_s^4)$, with all colour factors being given for a generic colour group. As a direct consequence we arrive at: (i) the full QCD contribution to the QED β–function of order $\alpha^2\,\alpha_s^4$ in the $\overline{\rm MS}$- and MOM-schemes; (ii) the full five-loop result of order α^6 for the β-function of QED with a generic number of single-charged fermions, again for the $\overline{\rm MS}$- and MOM-schemes.

Keywords: Quantum chromodynamics, Perturbative calculations

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1 Introduction

Quantum electrodynamics can be considered to be one of the most successful theoretical concepts, allowing to predict experimental results with unchallenged precision. Here the anomalous magnetic moment of the electron (see, [1] and references therein), positronium spectroscopy [2] or properties of light hydrogenic bound states [3] may be listed as most outstanding examples. At the same time the detailed investigation of its mathematical structure has lead to fundamental field-theoretical concepts, like renormalization or the renormalization group with anomalous dimensions and the β-function as important elements for the analysis of its high energy behaviour [4–6].

As a consequence of Ward identities the β-function of QED is closely related to the vacuum polarization function $\Pi(q^2)$ and, in contrast to non-abelian gauge theories like QCD, no vertex corrections need to be evaluated. Indeed, the coefficient of the n-th term of the perturbative series of the β-function can be directly obtained from the absorptive part of $\Pi(q^2)$ at the same order. The same absorptive part, evaluated including QCD corrections, can be employed to predict the Adler function [7] and thus the total cross section for electron positron annihilation into hadrons or the τ-lepton semileptonic decay rate. The precise experimental information for these fundamental quantities has stimulated dedicated projects for the evaluation of three- [8], four- [9, 10] and recently even five-loop corrections [11–15] for these fundamental quantities. Conversely, it is obvious that the information contained in these five-loop QCD results is sufficient to evaluate the QED vacuum polarization and the β function in the same five-loop order. Partial results in this direction have been presented in [11, 13, 16].

It is the purpose of this work to collect all available results for the vector correlator in massless QCD in a form suitable for a complete evaluation of the QED β function. Two cases will be considered: the QCD contribution to the QED β-function of order $\alpha^2\alpha_s^4$ and the QED β-function of order α^6 for a generic number of charged fermions.

For readers's convenience we provide in the paper the complete results for the vector current correlator, related β-functions and anomalous dimensions, including also lower order contributions. As the latter are known since long, we refer the reader to works [17–19] for corresponding historical discussions and citations. Our full results are also available (in computer-readable form) in `http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp12/ttp12-018`.

2 Preliminaries

Our main object will be the polarization function $\Pi(L, a_s)$ of the flavor singlet vector current defined as

$$(-g_{\alpha\beta}q^2 + q_\alpha q_\beta)\,\Pi(L, a_s) = i \int \mathrm{d}^4x e^{iq\cdot x}\langle 0|\mathrm{T} j_\alpha(x) j_\beta(0)|0\rangle, \tag{2.1}$$

with $j_\alpha = \sum_i \overline{\psi}_i \gamma_\alpha \psi_i$ and $Q^2 = -q^2$, $L = \ln \frac{\mu^2}{Q^2}$, $a_s = \frac{\alpha_s(\mu)}{\pi}$. We work within massless QCD with n_f quark flavours, μ is the normalization point of the $\overline{\mathrm{MS}}$-scheme.

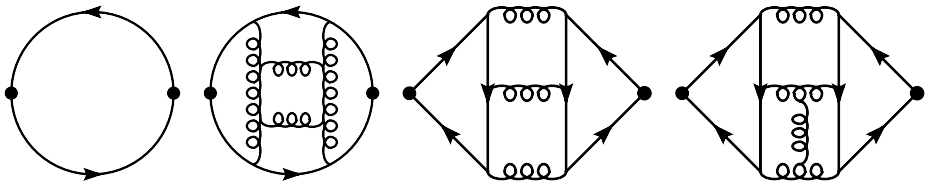

Figure 1: Examples of a lowest order and five-loop non-singlet (a),(b) and singlet (c),(d) diagrams contributing to the vector correlator.

All diagrams contributing to $\Pi(L, a_s)$ can be naturally decomposed into two classes: singlet and non-singlet ones. The non-singlet diagrams are those with both external currents belonging to one and the same quark loop; all other diagrams are referred to as singlet ones (see Fig. 1). As a consequence of this classification the full polarization function can be presented as follows:

$$\Pi(L, a_s) = n_f\,\Pi^{NS}(L, a_s) + n_f^2\,\Pi^{SI}(L, a_s). \tag{2.2}$$

It is worth to note that the knowledge of Π^{NS} and Π^{SI} is enough to construct the polarization function arising from two vector currents with generic flavour structure. Indeed, if $j_\alpha^a = \overline{\psi}\gamma_\alpha t^a \psi$ and $j_\alpha^b = \overline{\psi}\gamma_\alpha t^b \psi$ (t^a and t^b being two arbitrary matrixes in flavor space) then the corresponding polarization function Π^{ab} is, obviously, given by the relation

$$\Pi^{ab} = \mathrm{Tr}(t^a\,t^b)\,\Pi^{NS} + \mathrm{Tr}(t^a)\,\mathrm{Tr}(t^b)\,\Pi^{SI}.$$

For instance, the polarization function corresponding to the electromagnetic vector current

$$j_\mu^{EM} = \sum_i q_i \overline{\psi}_i \gamma_\mu \psi_i$$

(q_i stands for the electric charge of the quark field ψ_i) is given by the expression:

$$\Pi^{EM} = \left(\sum_i q_i^2\right) \Pi^{NS} + \left(\sum_i q_i\right)^2 \Pi^{SI}. \tag{2.3}$$

The absorptive part of Π^{EM} is directly related to the important physical quantity $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ via the relation

$$R(s) = 6\pi \, \left(\Pi^{EM}(-s - i\epsilon) - \Pi^{EM}(-s + i\epsilon)\right). \tag{2.4}$$

Another useful object – so-called Adler function — is defined as [7]

$$D(L, a_s) = -12\,\pi^2 Q^2 \, \frac{\mathrm{d}}{\mathrm{d}Q^2} \Pi(L = \ln \frac{\mu^2}{Q^2}, a_s). \tag{2.5}$$

Note that the choice of the coefficients 6π and $-12\pi^2$ in eqs. (2.4) and (2.5) is conventionally fixed by the requirement that in Born approximation

$$R(s) = \left(\sum_i q_i^2\right) d_R \, (1 + \mathcal{O}(\alpha_s)), \quad D(L, a_s) = n_f \, d_R \, (1 + \mathcal{O}(\alpha_s)),$$

where the d_R is the dimension of the quark representation of the colour gauge group ($d_R = 3$ for QCD).

In spite of the fact, that the vector current in QCD is a scale-invariant object (that is it has zero anomalous dimension), the polarization function is *not* due to the short distance singularities of the T-product in eq. (2.1). The corresponding evolution equation reads (see, e.g. [17])

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s}\right) \Pi(L, a_s) = \gamma(a_s) \tag{2.6}$$

or, equivalently,

$$\frac{\partial}{\partial L} \Pi(L, a_s) = \gamma(a_s) - \beta(a_s) \frac{\partial}{\partial a_s} \Pi(L, a_s). \tag{2.7}$$

Here the anomalous dimension γ is a series in a_s of the form

$$\gamma(a_s) = \frac{1}{16\,\pi^2} \sum_{i \geq 0} \gamma_i a_s^i \tag{2.8}$$

and the QCD β-function is defined as

$$\mu^2 \frac{d}{d\mu^2} a_s = \beta(a_s) \equiv -\sum_{i \geq 0} \beta_i a_s^{i+2}. \tag{2.9}$$

For our analysis we need to know the QCD β-function with three-loop accuracy, the corresponding result is known since long from [20, 21]. Note that the ratio $R(s)$ and the Adler

function are scale-invariant as the rhs of eq. (2.6), being an anomalous dimension does not depend on either Q, μ or L.

In massless QCD eq. (2.7) directly leads to an important relation for the Adler function:

$$D(L, a_s) = 12\pi^2 \left(\gamma(a_s) - \left(\beta(a_s) \frac{\partial}{\partial a_s} \right) \Pi(L, a_s) \right) . \tag{2.10}$$

Eqs. (2.7) and (2.10), are crucial for our ability to compute the Adler function and $R(s)$ at order α_s^4 which, in principle, are determined by *five-loop* diagrams. The simplification arises because the rhs of eq. (2.10) is, in fact, expressible *exclusively* through *four-loop* massless propagators. First, only the four-loop $\mathcal{O}(\alpha_s^3)$ approximation to $\Pi(L, a_s)$ is required as the β-function starts from order α_s^2. Second, the evaluation of any $(L+1)$-loop anomalous dimension in the $\overline{\text{MS}}$-scheme can be reduced, with the help of the R^*-operation [22], to the evaluation of some L-loop masslesss propagators [23]. Finally, four-loop massless propagators can be reduced to 28 master integrals. The reduction is based on evaluating sufficiently many terms of the $1/D$ expansion [24] of the corresponding coefficient functions [25]. The master integrals are known analytically from [26, 27].

Needless to say that any direct way of computing 5-loop diagrams, contributing, for instance, to the Adler function *without* the use of (2.10) is hopelessly difficult and, presumably, will stay so for years and years ahead.

3 Results for the vector correlator

In this section we present all currently available results for the polarization function Π and the anomalous dimension γ which have been utilized in works [12–15] to construct the Adler function and the ratio $R(s)$ in massless QCD to order α_s^4.

3.1 The polarization function Π

By presenting the perturbative expansion of the polarization function Π as follows

$$\Pi^{NS} = \frac{d_R}{16\pi^2} \left(\sum_{i \geq 0} p_i^{NS} \, a_s^i \right) , \quad \Pi^{SI} = \frac{d_R}{16\pi^2} \left(\sum_{i \geq 3} p_i^{SI} \, a_s^i \right) , \tag{3.1}$$

we get

$$p_0^{NS} = \frac{20}{9}, \tag{3.2}$$

$$p_1^{NS} = C_F \left[\frac{55}{12} - 4\zeta_3 \right] , \tag{3.3}$$

$$p_2^{NS} = C_F^2 \left[-\frac{143}{72} - \frac{37}{6}\zeta_3 + 10\zeta_5 \right] + C_F \, C_A \left[\frac{44215}{2592} - \frac{227}{18}\zeta_3 - \frac{5}{3}\zeta_5 \right] + C_F \, T \, n_f \left[-\frac{3701}{648} + \frac{38}{9}\zeta_3 \right] , \tag{3.4}$$

$$
\begin{aligned}
p_3^{NS} = C_F^3 \left[-\frac{31}{192} + \frac{13}{8}\zeta_3 + \frac{245}{8}\zeta_5 - 35\zeta_7\right] + T^2\, n_f^2\, C_F \left[\frac{196513}{23328} - \frac{809}{162}\zeta_3 - \frac{20}{9}\zeta_5\right] \\
+ T\, n_f\, C_F^2 \left[-\frac{7505}{10368} + \frac{1553}{54}\zeta_3 - 4\,\zeta_3^2 + \frac{11}{24}\zeta_4 - \frac{250}{9}\zeta_5\right] \\
+ T\, n_f\, C_F\, C_A \left[-\frac{5559937}{93312} + \frac{41575}{1296}\zeta_3 + \frac{2}{3}\,\zeta_3^2 - \frac{11}{24}\zeta_4 + \frac{515}{27}\zeta_5\right] \\
+ C_F^2\, C_A \left[-\frac{382033}{20736} - \frac{46219}{864}\zeta_3 - \frac{11}{48}\zeta_4 + \frac{9305}{144}\zeta_5 + \frac{35}{2}\zeta_7\right] \\
+ C_F\, C_A^2 \left[\frac{34499767}{373248} - \frac{147473}{2592}\zeta_3 + \frac{55}{6}\,\zeta_3^2 + \frac{11}{48}\zeta_4 - \frac{28295}{864}\zeta_5 - \frac{35}{12}\zeta_7\right], \qquad (3.5)
\end{aligned}
$$

$$
p_3^{SI} = \frac{d^{abc}\, d^{abc}}{d_R} \left\{\frac{431}{1728} - \frac{21}{64}\zeta_3 - \frac{1}{6}\,\zeta_3^2 - \frac{1}{16}\zeta_4 + \frac{5}{16}\zeta_5\right\}. \qquad (3.6)
$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the colour Lie algebra, $d^{abc} = 2\,\mathrm{Tr}(\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\}\frac{\lambda^c}{2})$, T is the trace normalization of the fundamental representation. The exact definitions of the colour structures $d_F^{abcd} d_A^{abcd}$ and $d_F^{abcd} d_F^{abcd}$ appearing below are given in [28]. For QCD (colour gauge group SU(3)):

$$
\begin{aligned}
d_R = 3\,,\; C_F = 4/3\,,\; C_A = 3\,,\; T = 1/2\,, \\
d_F^{abcd} d_A^{abcd} = \frac{15}{2}\,,\; d_F^{abcd} d_F^{abcd} = \frac{5}{12},\quad d^{abc}\, d^{abc} = \frac{40}{3}. \qquad (3.7)
\end{aligned}
$$

For the particular case of the $U(1)$ gauge group the colour factors assume the following values:

$$
d_R = 1\,,\; C_F = 1\,,\; C_A = 0\,,\; T = 1\,,\; d_F^{abcd} d_A^{abcd} = 0\,,\; d_F^{abcd} d_F^{abcd} = 1,\quad d^{abc}\, d^{abc} = 16. \qquad (3.8)
$$

Note that in eqs. (3.2–3.6) we have set to zero $L = \ln\!\left(\mu^2/Q^2\right)$. The full dependence on L can be easily restored from evolution eq. (2.7) and the anomalous dimension γ given below.

3.2 The anomalous dimension γ

On decomposing the anomalous dimension γ into non-singlet and singlet terms

$$
\gamma = n_f\, \gamma^{NS} + n_f^2\, \gamma^{SI},\quad \gamma^{NS} = \frac{d_R}{16\pi^2} \left(\sum_{i\geq 0} \gamma_i^{NS}\, a_s^i\right),\quad \gamma^{SI} = \frac{d_R}{16\pi^2} \left(\sum_{i\geq 3} \gamma_i^{SI}\, a_s^i\right), \qquad (3.9)
$$

we get

$$
\gamma_0^{NS} = \frac{4}{3}, \qquad (3.10)
$$

$$
\gamma_1^{NS} = C_F, \qquad (3.11)
$$

$$
\gamma_2^{NS} = C_F^2 \left[-\frac{1}{8}\right] + C_F\, C_A \left[\frac{133}{144}\right] + C_F\, T\, n_f \left[-\frac{11}{36}\right], \qquad (3.12)
$$

$$\gamma_3^{NS} = C_F^3\left[-\frac{23}{32}\right] + T^2\, n_f^2\, C_F\left[-\frac{77}{972}\right] + T\, n_f\, C_F^2\left[-\frac{169}{216}+\frac{11}{9}\zeta_3\right] + T\, n_f\, C_F\, C_A\left[-\frac{769}{3888}-\frac{11}{9}\zeta_3\right] + C_F^2\, C_A\left[\frac{215}{216}-\frac{11}{18}\zeta_3\right] + C_F\, C_A^2\left[\frac{5815}{15552}+\frac{11}{18}\zeta_3\right], \tag{3.13}$$

$$\begin{aligned}\gamma_4^{NS} = \;& C_F^4\left[\frac{4157}{1536}+\frac{1}{2}\zeta_3\right] + n_f\,\frac{d_F^{abcd}\,d_F^{abcd}}{d_R}\left[-\frac{13}{12}-\frac{4}{3}\zeta_3+\frac{10}{3}\zeta_5\right] \\ &+ \frac{d_F^{abcd}\,d_A^{abcd}}{d_R}\left[\frac{1}{4}-\frac{1}{3}\zeta_3-\frac{5}{3}\zeta_5\right] + T^3\, n_f^3\, C_F\left[\frac{107}{7776}+\frac{1}{54}\zeta_3\right] \\ &+ T^2\, n_f^2\, C_F^2\left[\frac{4961}{10368}-\frac{119}{108}\zeta_3+\frac{11}{24}\zeta_4\right] + T^2\, n_f^2\, C_F\, C_A\left[-\frac{8191}{31104}+\frac{563}{432}\zeta_3-\frac{11}{24}\zeta_4\right] \\ &+ T\, n_f\, C_F^3\left[\frac{2509}{1152}+\frac{67}{24}\zeta_3-\frac{145}{24}\zeta_5\right] + T\, n_f\, C_F^2\, C_A\left[-\frac{66451}{13824}+\frac{2263}{864}\zeta_3-\frac{143}{96}\zeta_4+\frac{85}{16}\zeta_5\right] \\ &+ T\, n_f\, C_F\, C_A^2\left[\frac{22423}{31104}-\frac{9425}{1728}\zeta_3+\frac{143}{96}\zeta_4+\frac{15}{32}\zeta_5\right] + C_F^3\, C_A\left[-\frac{2585}{256}-\frac{71}{12}\zeta_3+\frac{935}{96}\zeta_5\right] \\ &+ C_F^2\, C_A^2\left[\frac{882893}{82944}+\frac{11501}{3456}\zeta_3+\frac{121}{192}\zeta_4-\frac{715}{64}\zeta_5\right] \\ &+ C_F\, C_A^3\left[-\frac{1192475}{497664}+\frac{5609}{3456}\zeta_3-\frac{121}{192}\zeta_4+\frac{275}{128}\zeta_5\right],\end{aligned} \tag{3.14}$$

$$\gamma_3^{SI} = \frac{d^{abc}\,d^{abc}}{d_R}\left\{\frac{11}{144}-\frac{1}{6}\zeta_3\right\}, \tag{3.15}$$

$$\begin{aligned}\gamma_4^{SI} = \frac{d^{abc}\,d^{abc}}{d_R}\Bigg\{ & C_F\left[-\frac{13}{48}-\frac{1}{3}\zeta_3+\frac{5}{6}\zeta_5\right] + C_A\left[\frac{1015}{2304}-\frac{659}{768}\zeta_3+\frac{11}{64}\zeta_4+\frac{5}{64}\zeta_5\right] \\ & + T\, n_f\left[-\frac{55}{576}+\frac{41}{192}\zeta_3-\frac{1}{16}\zeta_4-\frac{5}{48}\zeta_5\right]\Bigg\}.\end{aligned} \tag{3.16}$$

4 QED β-functions in five-loop order for different schemes

4.1 Massless QCD and QED: $\overline{\rm MS}$-scheme

The polarization function Π is known to be directly related to the photon propagator, namely[1]

$$\mathcal{P}_{\alpha\beta}(q) = \left(g_{\alpha\beta}q^2 - q_\alpha\, q_\beta\right)\,\frac{d(Q^2)}{Q^4}, \quad d(Q^2) = \frac{1}{1+e^2\,\Pi^{EM}(Q^2)}. \tag{4.1}$$

The combination $e^2\,d(Q^2)$ is often referred to as "invariant" charge as it is renormalization scale *and scheme* independent due to the corresponding Ward identity. The independence

[1] Without loss of generality we will use the Landau gauge for the photon field.

of the invariant charge on the renormalization scale μ directly leads to the RG equation for the QED coupling constant $A(\mu) = \alpha(\mu)/(4\,\pi) = e(\mu)^2/(16\,\pi^2)$:

$$\mu^2 \frac{d}{d\mu^2} A = \beta^{EM}(A, a_s) = A^2\,(16\pi^2)\,\gamma^{EM}(a_s) = A^2\,d_R \sum_{i\geq 0} \gamma_i^{EM}\,a_s^i, \tag{4.2}$$

with

$$\gamma^{EM}(a_s) \equiv \left(\sum_i q_i^2\right)\gamma^{NS} + \left(\sum_i q_i\right)^2 \gamma^{SI}, \quad \gamma_i^{EM} \equiv \left(\sum_i q_i^2\right)\gamma_i^{NS} + \left(\sum_i q_i\right)^2 \gamma_i^{SI}.$$

The β-function $\beta^{EM}(A, a_s)$ describes the QCD-induced corrections to the running of α in the $\overline{\rm MS}$-scheme.

Using now eqs. (3.10-3.16) and substituting the values of the colour factors corresponding to the SU(3) colour group we find:

$$\begin{aligned}
\beta^{EM}(A, a_s) = A^2 \left(\sum_i q_i^2\right) \Bigg\{ & 4 + 4\,a_s + a_s^2 \left(\frac{125}{12} - \frac{11}{18}\,n_f\right) \\
& + a_s^3 \left(\frac{10487}{432} + \frac{110}{9}\zeta_3 + n_f \left[-\frac{707}{216} - \frac{110}{27}\zeta_3\right] - \frac{77}{972}\,n_f^2\right) \\
& + a_s^4 \left(\frac{2665349}{41472} + \frac{182335}{864}\zeta_3 - \frac{605}{16}\zeta_4 - \frac{31375}{288}\zeta_5 \right. \\
& + n_f \left[-\frac{11785}{648} - \frac{58625}{864}\zeta_3 + \frac{715}{48}\zeta_4 + \frac{13325}{432}\zeta_5\right] \\
& \left. + n_f^2 \left[-\frac{4729}{31104} + \frac{3163}{1296}\zeta_3 - \frac{55}{72}\zeta_4\right] + n_f^3 \left[\frac{107}{15552} + \frac{1}{108}\zeta_3\right]\right)\Bigg\} + \\
& + A^2 \left(\sum_i q_i\right)^2 \Bigg\{ a_s^3 \left(\frac{55}{54} - \frac{20}{9}\zeta_3\right) + a_s^4 \left(\frac{11065}{864} - \frac{34775}{864}\zeta_3 + \frac{55}{8}\zeta_4 + \frac{3875}{216}\zeta_5 \right. \\
& \left. + n_f \left[-\frac{275}{432} + \frac{205}{144}\zeta_3 - \frac{5}{12}\zeta_4 - \frac{25}{36}\zeta_5\right]\right)\Bigg\}.
\end{aligned} \tag{4.3}$$

For the particular cases of 4, 5 and 6 quark flavours eq. (4.3) takes the form (the normalization is chosen in such a way to facilitate the comparison with [19])

$$\beta^{EM}(A, a_s)|_{n_f=4} = 4\,A^2 \left(1.111 + 1.111\,a_s + 2.214\,a_s^2 + 1.210\,a_s^3 - 3.904\,a_s^4\right), \tag{4.4}$$

$$\beta^{EM}(A, a_s)|_{n_f=5} = 4\,A^2 \left(1.222 + 1.222\,a_s + 2.249\,a_s^2 - 1.227\,a_s^3 - 13.429\,a_s^4\right), \tag{4.5}$$

$$\beta^{EM}(A, a_s)|_{n_f=6} = 4\,A^2 \left(1.667 + 1.667\,a_s + 2.813\,a_s^2 - 5.791\,a_s^3 - 32.336 a_s^4\right). \tag{4.6}$$

Another case of interest is pure QED, that is a theory with n_f single-charged fermions minimally coupled to the photon field. In this case the corresponding EM current is identical to the flavour singlet current. The corresponding β-function is, obviously, obtained

from the general formula (4.2) by taking the QED values for the colour factors and setting $q_i = 1$ and $a_s(\mu) = 4\,A(\mu)$:

$$\begin{aligned}
\beta^{QED}(A) =\; & n_f \left[\frac{4\,A^2}{3}\right] + 4\,n_f A^3 - A^4\left[2\,n_f + \frac{44}{9}\,n_f^2\right] \\
+\; & A^5\left[-46\,n_f + \frac{760}{27}\,n_f^2 - \frac{832}{9}\,\zeta_3\,n_f^2 - \frac{1232}{243}\,n_f^3\right] \\
+\; & A^6\Bigg(n_f\left[\frac{4157}{6} + 128\zeta_3\right] + n_f^2\left[-\frac{7462}{9} - 992\zeta_3 + 2720\zeta_5\right] \\
+\; & n_f^3\left[-\frac{21758}{81} + \frac{16000}{27}\zeta_3 - \frac{416}{3}\zeta_4 - \frac{1280}{3}\zeta_5\right] + n_f^4\left[\frac{856}{243} + \frac{128}{27}\zeta_3\right]\Bigg).
\end{aligned} \tag{4.7}$$

If we set $n_f = 1$, then the above result takes the form:

$$\begin{aligned}
\beta^{QED}(A) = & \frac{4}{3}A^2 + 4A^3 - \frac{62}{9}A^4 + A^5\left[-\frac{5570}{243} - \frac{832}{9}\zeta_3\right] \\
& + A^6\left[-\frac{195067}{486} - \frac{800}{3}\zeta_3 - \frac{416}{3}\zeta_4 + \frac{6880}{3}\zeta_5\right]
\end{aligned} \tag{4.8}$$

or, numerically,

$$\begin{aligned}
\beta^{QED}(A) &= \frac{4}{3}A^2 + 4\,A^3 - \frac{62}{9}A^4 + A^5\,(-7.116 - 126.93^{SI}) + A^6\,(776.39 + 729.63^{SI}) \\
&= \frac{4}{3}A^2 + 4\,A^3 - \frac{62}{9}A^4 - 134.045\,A^5 + 1506.02\,A^6,
\end{aligned} \tag{4.9}$$

where in the first line we have explicitly separated non-singlet from singlet contributions.

For future reference it is also useful to present the evolution equation for the QED polarization function:

$$\frac{\partial}{\partial L}\,\Pi^{QED}(L, A) = \gamma^{QED}(A) - \beta^{QED}(A)\frac{\partial}{\partial A}\,\Pi^{QED}(L, A). \tag{4.10}$$

Here

$$\beta^{QED}(A) \equiv A^2\,(16\pi^2)\,\gamma^{QED}(A), \tag{4.11}$$

$$\Pi^{QED}(L, A) \equiv n_f\,\Pi^{NS,QED}(L, A) + n_f^2\,\Pi^{SI,QED}(L, A). \tag{4.12}$$

In addition, $\Pi^{NS,QED}$, $\Pi^{SI,QED}$ and γ^{QED} are Π^{NS}, Π^{SI} and γ respectively with all colour factors substituted according to eq. (3.8) and $a_s = 4\,A$.

4.2 Massless QCD and QED: MOM-scheme

The MOM-scheme for the QED coupling constant is defined by the requirement that at $Q^2 = \mu^2$ the polarization function would vanish. The scheme independence of the invariant charge directly leads to the following relation between $\overline{\rm MS}$-renormalized QED coupling constant $A(\mu)$ and its MOM analog

$$\tilde{A}(\mu) = \frac{A(\mu)}{1 + (4\pi)^2\,A(\mu)\,\Pi(L = 0, a_s(\mu))}. \tag{4.13}$$

The above equation allows to express easily the QED β-function in the MOM-scheme via the corresponding $\overline{\rm MS}$ β-function and the perturbative expansion of the polarization function as given in eqs. (2.3,3.2–3.6,4.3). After differentiating the rhs of (4.13) with respect to μ and using eq. (2.6) we obtain

$$\begin{aligned}\beta^{EM}(\tilde{A},a_s) &= 16\pi^2\,\tilde{A}^2\left[\gamma^{EM}(a_s)-\beta(a_s)\,\frac{\partial}{\partial a_s}\,\Pi^{EM}(L=0,a_s)\right]\\ &= \frac{4}{3}\tilde{A}^2\,D^{EM}(L=0,a_s),\end{aligned}\tag{4.14}$$

where

$$D^{EM}(L,a_s) = -12\,\pi^2 Q^2\,\frac{\rm d}{{\rm d}Q^2}\Pi^{EM}(L=\ln\frac{\mu^2}{Q^2},a_s).\tag{4.15}$$

Explicitly, for the $SU(3)$ colour group we get

$$\begin{aligned}\beta^{EM}(\tilde{A},a_s) &= \frac{4\,\tilde{A}^2}{3}\left(3\sum_i q_i^2\right)\left\{1+a_s+a_s^2\left(\frac{365}{24}-11\zeta_3+\,n_f\left[-\frac{11}{12}+\frac{2}{3}\zeta_3\right]\right)\right.\\
&+\,a_s^3\left(\frac{87029}{288}-\frac{1103}{4}\zeta_3+\frac{275}{6}\zeta_5+\,n_f\left[-\frac{7847}{216}+\frac{262}{9}\zeta_3-\frac{25}{9}\zeta_5\right]+n_f^2\left[\frac{151}{162}-\frac{19}{27}\zeta_3\right]\right)\\
&+\,a_s^4\left(\frac{144939499}{20736}-\frac{5693495}{864}\zeta_3+\frac{5445}{8}\,\zeta_3^2+\frac{65945}{288}\zeta_5-\frac{7315}{48}\zeta_7\right.\\
&+\,n_f\left[-\frac{13044007}{10368}+\frac{12205}{12}\zeta_3-55\,\zeta_3^2+\frac{29675}{432}\zeta_5+\frac{665}{72}\zeta_7\right]\\
&\left.\left.+\,n_f^2\left[\frac{1045381}{15552}-\frac{40655}{864}\zeta_3+\frac{5}{6}\,\zeta_3^2-\frac{260}{27}\zeta_5\right]+\,n_f^3\left[-\frac{6131}{5832}+\frac{203}{324}\zeta_3+\frac{5}{18}\zeta_5\right]\right)\right\}\\
&+\,\frac{4\,\tilde{A}^2}{3}\left(\sum_i q_i\right)^2\left\{a_s^3\left(\frac{55}{72}-\frac{5}{3}\zeta_3\right)+a_s^4\left(\frac{5795}{192}-\frac{8245}{144}\zeta_3-\frac{55}{4}\,\zeta_3^2+\frac{2825}{72}\zeta_5\right.\right.\\
&\left.\left.+\,n_f\left[-\frac{745}{432}+\frac{65}{24}\zeta_3+\frac{5}{6}\,\zeta_3^2-\frac{25}{12}\zeta_5\right]\right)\right\}.\end{aligned}\tag{4.16}$$

For particular cases of 4, 5 and 6 quark flavours eq. (4.16) reads

$$\beta^{EM}(\tilde{A},a_s)|_{n_f=4} = 4\,\tilde{A}^2\left(1.111+1.111\,a_s+1.694\,a_s^2+2.881\,a_s^3+28.132\,a_s^4\right),\tag{4.17}$$

$$\beta^{EM}(\tilde{A},a_s)|_{n_f=5} = 4\,\tilde{A}^2\left(1.222+1.222\,a_s+1.723\,a_s^2-0.879\,a_s^3+10.703\,a_s^4\right),\tag{4.18}$$

$$\beta^{EM}(\tilde{A},a_s)|_{n_f=6} = 4\,\tilde{A}^2\left(1.667+1.667\,a_s+2.156\,a_s^2-6.995\,a_s^3-13.994\,a_s^4\right).\tag{4.19}$$

In the same way one could derive the pure QED β-function $\beta^{QED}(\tilde{A})$ in the MOM-scheme (it was first introduced by Gell-Mann and Low in [5] under the name "ψ function"). Indeed, an analog of eq. (4.13) now assumes the following form:

$$\tilde{A}(\mu) = \frac{A(\mu)}{1+(4\pi)^2\,A(\mu)\,\Pi^{QED}(L=0,A)}.\tag{4.20}$$

The resulting formula for the QED β-function in the MOM-scheme reads (we have used relation (4.10) and definitions (4.11,4.12))

$$\begin{aligned}\beta^{QED}(\tilde{A}) &= 16\pi^2\,\tilde{A}^2\left[\gamma^{QED}(A) - \beta^{QED}(A)\,\frac{\partial}{\partial A}\,\Pi^{QED}(L=0,A)\right]\\ &= \frac{4}{3}\tilde{A}^2\,D^{QED}(L=0,A),\end{aligned} \tag{4.21}$$

where $D^{QED}(L,A) \equiv -12\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\,\Pi^{QED}(L=\ln\frac{\mu^2}{Q^2},A)$. In addition, the coupling constant $A(\mu)$ appearing in the rhs of eq. (4.21) should be expressed in terms of $\tilde{A}(\mu)$ using an inversion of eq. (4.20). Explicitly, for the U(1) gauge group we arrive at the following result:

$$\begin{aligned}\beta^{QED}(\tilde{A}) =&\; n_f\left[\frac{4\,\tilde{A}^2}{3}\right] + 4\,n_f\tilde{A}^3 + \tilde{A}^4\left(-2\,n_f + n_f^2\left[-\frac{184}{9}+\frac{64}{3}\zeta_3\right]\right)\\ &+\tilde{A}^5\left(-46\,n_f + n_f^2\left[104+\frac{512}{3}\zeta_3-\frac{1280}{3}\zeta_5\right] + n_f^3\left[128-\frac{256}{3}\zeta_3\right]\right)\\ &+\tilde{A}^6\left(n_f\left[\frac{4157}{6}+128\zeta_3\right] + n_f^2\left[-1004-\frac{2944}{3}\zeta_3-5760\zeta_5+8960\zeta_7\right]\right.\\ &\left.+\;n_f^3\left[-\frac{27064}{27}-\frac{26240}{9}\zeta_3+1024\,\zeta_3^2+2560\zeta_5\right] + n_f^4\left[-\frac{8576}{9}+\frac{3584}{9}\zeta_3+\frac{5120}{9}\zeta_5\right]\right).\end{aligned} \tag{4.22}$$

After setting $n_f = 1$ we get:

$$\begin{aligned}\beta^{QED}(\tilde{A}) &= \frac{4}{3}\tilde{A}^2 + 4\,\tilde{A}^3 + \tilde{A}^4\left[-\frac{202}{9}+\frac{64}{3}\zeta_3\right] + \tilde{A}^5\left[186+\frac{256}{3}\zeta_3-\frac{1280}{3}\zeta_5\right]\\ &\quad+\tilde{A}^6\left[-\frac{122387}{54}-\frac{10112}{3}\zeta_3+1024\,\zeta_3^2-\frac{23680}{9}\zeta_5+8960\zeta_7\right]\\ &= \frac{4}{3}A^2 + 4\,A^3 + 3.199\,A^4 + A^5\,(-26.918 - 126.929^{SI}) + A^6\,(1054.41 + 413.592^{SI})\\ &= \frac{4}{3}A^2 + 4\,A^3 + 3.199\,A^4 - 153.847A^5 + 1467.998\;A^6,\end{aligned} \tag{4.23}$$

where the singlet contribution has again been identified separately.

5 Discussion

ℓ	1	2	3	4
	-	ζ_3	$\zeta_3, \zeta_4, \zeta_5$	$\zeta_3, \zeta_4, \zeta_5, \zeta_3^2, \zeta_6, \zeta_7$

Table 1: Possible irrational structures which are allowed to appear in ℓ-loop massless propagators.

Let us discuss the structure of trancendentalities appearing in our results. It follows from work [26] that the variety of ζ-constants entering into the $\overline{\rm MS}$-renormalized (euclidian) massless propagators depends on the loop order according to Table 1. Table 2 provides the same information about possible irrational numbers which could show up in anomalous dimensions. An examination of eqs. (3.2–3.6, 3.10–3.16) immediately reveals that the

ℓ	1,2	3	4	5
	-	ζ_3	$\zeta_3, \zeta_4, \zeta_5$	$\zeta_3, \zeta_4, \zeta_5, \zeta_3^2, \zeta_6, \zeta_7$

Table 2: Possible irrational structures which are allowed to appear in ℓ-loop anomalous dimensions and β-functions.

observed pattern of trancendentalities is significantly more limited than what is allowed by Tables 1 and 2. Indeed, the four-loop anomalous dimension γ_3 contains no ζ_4 and no ζ_5 while the three-loop polarization function contains ζ_5 but does not comprise ζ_4. Let us move up one loop. The situation is getting even more puzzling: the five-loop anomalous dimension γ_4 does contain ζ_4 but still does not include ζ_3^2, ζ_6 and ζ_7. The four-loop polarization function contains ζ_4 but is free from ζ_6. Unfortunately, we are not aware about the existence of any reason behind these remarkable facts, except for one observation, namely the absence of ζ_4 in the MOM β-functions (4.16) and (4.22).

Indeed, according to eqs. (4.14) and (4.21) the constant ζ_4 does not appear in these two β-functions since it does not appear in the Adler function. However, the puzzle of the absence of ζ_4 in $\mathcal{O}(\alpha_s^3)$ contribution to the Adler function has been recently fully explained in [26]. The explanation is based on a quite peculiar structure of irrational contributions to each four-loop master integral.

Why this absence continues to hold at five loops is still a mystery (at least for us). In all probability it is connected with some regularities of *five-loop* master integrals. But here starts terra incognita ...

6 Conclusions

We have presented four new results, namely the QED β-functions β^{EM} and β^{QED} in the $\overline{\rm MS}$- and MOM-schemes.

We have described the status of results for the vector correlator in massless QCD. These are not completely new as they have been used to produce the Adler function and $R(s)$ in works [12–15]. Nevertheless, we believe that the separate presentation of the polarization function and its anomalous dimension is both useful and instructive. First, it reflects the real way how the calculations have been done. Second, it clearly demonstrates puzzling regularities of the structure of irrational terms contributing to Π and γ. Third, it makes trivial the construction of the QED β-function in the $\overline{\rm MS}$- and MOM-schemes (cmp. with the somewhat unnecessary complicated "inverse engineering" employed in [19] to reconstruct the function $\beta^{EM}(A, a_s)$ at four loop level).

The calculations of Π and γ have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using parallel MPI-based [29] as well as thread-based [30] versions of FORM [31]. For the evaluation of colour factors we have used the FORM program COLOR [32]. The diagrams have been generated with QGRAF [33]. The figures have been drawn with the help of Axodraw [34] and JaxoDraw [35].

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Finally, we want to note that the result (4.8) for the function $\beta^{QED}(A)$ with $n_f = 1$ was first reported by one of the present authors in September 2011 during the 10-th International Symposium on Radiative Corrections, RADCOR 2011 (see the 10-th page of the file `http://www.icts.res.in/media/uploads/Program/Files/chet.pdf`).

References

[1] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, *Tenth-Order QED Contribution to the Electron g-2 and an Improved Value of the Fine Structure Constant*, arXiv:1205.5368.

[2] S. G. Karshenboim, *Precision physics of simple atoms: QED tests, nuclear structure and fundamental constants*, *Phys. Rept.* **422** (2005) 1–63, [hep-ph/0509010].

[3] M. I. Eides, H. Grotch, and V. A. Shelyuto, *Theory of light hydrogenic bound states*, *Springer Tracts Mod. Phys.* **222** (2007) 1–262.

[4] E. Stueckelberg and A. Petermann, *La normalisation des constantes dans la theorie des quanta*, *Helv. Phys. Acta.* **26** (1953) 499–520.

[5] M. Gell-Mann and F. Low, *Quantum electrodynamics at small distances*, *Phys.Rev.* **95** (1954) 1300–1312.

[6] N. Bogolyubov and D. Shirkov, *Charge renormalization group in quantum field theory*, *Nuovo Cim.* **3** (1956) 845–863.

[7] S. L. Adler, *Some simple vacuum polarization phenomenology: $e^+e^- \to$ hadrons: The mu - mesic atom x-ray discrepancy and (g-2) of the muon*, *Phys. Rev.* **D10** (1974) 3714.

[8] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, *Higher Order Corrections to $\sigma_{tot}(e^+e^- \to$ Hadrons) in Quantum Chromodynamics*, *Phys. Lett.* **B85** (1979) 277.

[9] S. G. Gorishny, A. L. Kataev, and S. A. Larin, *The O (alpha-s**3) corrections to sigma-tot (e+ e- → hadrons) and Gamma (tau- → tau-neutrino + hadrons) in QCD*, *Phys. Lett.* **B259** (1991) 144–150.

[10] L. R. Surguladze and M. A. Samuel, *Total hadronic cross-section in e+ e- annihilation at the four loop level of perturbative QCD*, *Phys. Rev. Lett.* **66** (1991) 560–563.

[11] P. Baikov, K. Chetyrkin, and J. H. Kuhn, *The Cross section of e+ e- annihilation into hadrons of order alpha(s)**4 n(f)**2 in perturbative QCD*, *Phys.Rev.Lett.* **88** (2002) 012001, [hep-ph/0108197].

[12] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Order α_s^4 QCD Corrections to Z and τ Decays*, *Phys. Rev. Lett.* **101** (2008) 012002, [arXiv:0801.1821].

[13] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order alpha$_s^4$ in a General Gauge Theory*, *Phys. Rev. Lett.* **104** (2010) 132004, [arXiv:1001.3606].

[14] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, and J. Rittinger, *Complete $\mathcal{O}(\alpha_s^4)$ QCD Corrections to Hadronic Z Decays*, *Phys. Rev. Lett.* **108** (May, 2012) 222003.

[15] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, and J. Rittinger, *Adler Function, Sum Rules and Crewther Relation of Order $\mathcal{O}(\alpha_s^4)$: the Singlet Case*, *in preparation* (2012) , [hep-ph/12].

[16] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Massless propagators: applications in QCD and QED*, *PoS* **RADCOR2007** (2007) 023, [arXiv:0810.4048].

[17] K. G. Chetyrkin, J. H. Kühn, and A. Kwiatkowski, *QCD corrections to the e^+e^- cross-section and the Z boson decay rate: Concepts and results*, *Phys. Rept.* **277** (1996) 189–281.

[18] S. G. Gorishny, A. L. Kataev, S. A. Larin, and L. R. Surguladze, *The Analytical four loop corrections to the QED Beta function in the MS scheme and to the QED psi function: Total reevaluation*, *Phys. Lett.* **B256** (1991) 81–86.

[19] J. Erler, *Calculation of the QED coupling alpha (M(Z)) in the modified minimal subtraction scheme*, *Phys.Rev.* **D59** (1999) 054008, [hep-ph/9803453].

[20] O. V. Tarasov, A. A. Vladimirov, and A. Y. Zharkov, *The gell-mann-low function of qcd in the three loop approximation*, *Phys. Lett.* **B93** (1980) 429–432.

[21] S. A. Larin and J. A. M. Vermaseren, *The Three loop QCD Beta function and anomalous dimensions*, *Phys. Lett.* **B303** (1993) 334–336, [hep-ph/9302208].

[22] K. G. Chetyrkin and V. A. Smirnov, *R* OPERATION CORRECTED*, *Phys. Lett.* **B144** (1984) 419–424.

[23] K. G. Chetyrkin, *Corrections of order alpha(s)**3 to R(had) in pQCD with light gluinos*, *Phys. Lett.* **B391** (1997) 402–412, [hep-ph/9608480].

[24] P. A. Baikov, *A practical criterion of irreducibility of multi-loop feynman integrals*, *Phys. Lett.* **B634** (2006) 325–329, [hep-ph/0507053].

[25] P. A. Baikov, *Explicit solutions of the 3–loop vacuum integral recurrence relations*, *Phys. Lett.* **B385** (1996) 404–410, [hep-ph/9603267].

[26] P. A. Baikov and K. G. Chetyrkin, *Four Loop Massless Propagators: an Algebraic Evaluation of All Master Integrals*, *Nucl. Phys.* **B837** (2010) 186–220, [arXiv:1004.1153].

[27] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve*, *Nucl. Phys.* **B856** (2012) 95–110, [arXiv:1108.0732].

[28] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, *The 4-loop quark mass anomalous dimension and the invariant quark mass*, *Phys. Lett.* **B405** (1997) 327–333, [hep-ph/9703284].

[29] M. Tentyukov *et. al.*, *ParFORM: Parallel Version of the Symbolic Manipulation Program FORM*, cs/0407066.

[30] M. Tentyukov and J. A. M. Vermaseren, *The multithreaded version of FORM*, hep-ph/0702279.

[31] J. A. M. Vermaseren, *New features of form*, math-ph/0010025.

[32] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren, *Group theory factors for feynman diagrams*, *Int. J. Mod. Phys.* **A14** (1999) 41–96, [hep-ph/9802376].

[33] P. Nogueira, *Automatic feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279–289.

[34] J. A. M. Vermaseren, *Axodraw*, *Comput. Phys. Commun.* **83** (1994) 45–58.

[35] D. Binosi and L. Theussl, *JaxoDraw: A graphical user interface for drawing Feynman diagrams*, *Comput. Phys. Commun.* **161** (2004) 76–86, [hep-ph/0309015].