Adler Function, Sum Rules and Crewther Relation of Order $\mathcal{O}(\alpha_s^4)$: the Singlet Case

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Abstract

The analytic result for the singlet part of the Adler function of the vector current in a general gauge theory is presented in five-loop approximation. Comparing this result with the corresponding singlet part of the Gross-Llewellyn Smith sum rule [1], we successfully demonstrate the validity of the generalized Crewther relation for the singlet part. This provides a non-trivial test of both our calculations and the generalized Crewther relation. Combining the result with the already available non-singlet part of the Adler function [2, 3] we arrive at the complete $\mathcal{O}(\alpha_s^4)$ expression for the Adler function and, as a direct consequence, at the complete $\mathcal{O}(\alpha_s^4)$ correction to the e^+e^- annihilation into hadrons in a general gauge theory.

Keywords: QCD, Adler function, Gross-Llewellyn Smith sum rule, Crewther relation

1. Introduction

The Crewther relation (CR) [4, 5] connects in a non-trivial way two seemingly unrelated quantities, namely the Adler function [6] and perturbative corrections arising in the sum rules relevant for deep inelastic scattering (DIS). Originally the CR had been formulated for the case of a conformal-invariant gauge theory. Subsequently, observing a close relation between the $\mathcal{O}(\alpha_s^3)$ terms in the Adler function and the corrections to the Bjorken sum rule, its generalization for the case of QCD was suggested in [5], introducing as modification additional terms proportional to the *beta*-function. A more formal argument for the validity of this "generalized Crewther relation" (GCR) was given in [7, 8]. During the past years the perturbative corrections both for Adler function and Bjorken sum rule were extended from $\mathcal{O}(\alpha_s^3)$ [9, 10, 11, 12] to $\mathcal{O}(\alpha_s^4)$ [2, 3]. However, these results were restricted to the respective non-singlet parts. Nevertheless, they could be used to demonstrate the validity of the GCR between non-singlet Adler function and Bjorken sum rule [13, 14], thus providing at the same time an important cross check of the underlying, demanding calculations.

The $\mathcal{O}(\alpha_s^4)$ singlet piece of the sum rule was published in [1], thus completing the prediction for the Gross-Llewellyn Smith (GLS) sum rule [15]. Below we give the corresponding result for the Adler function. On the one hand this leads to a prediction of the familiar R-ratio measured in electron-positron annihilation, including the (small) up to now missing singlet pieces of $\mathcal{O}(\alpha_s^4)$, on the other hand this result allows to test the GCR also for the singlet case.

2. Singlet $\mathcal{O}(\alpha_s^4)$ contributions to the Adler function and R(s)

For the definition of the Adler function it is convenient to start with the polarization function of the flavor singlet vector current:

$$3\,Q^2\,\Pi(Q^2) = i\int \mathrm{d}^4x\, e^{iq\cdot x}\langle 0|\mathrm{T}\, j_\mu(x) j^\mu(0)|0\rangle\,, \qquad (1)$$

with $j_\mu = \sum_i \overline{\psi}_i\gamma_\mu\psi_i$ and $Q^2 = -q^2$. The corresponding Adler function

$$D(Q^2) = -12\,\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{dQ}^2}\Pi(Q^2) \qquad (2)$$

is naturally decomposed into a sum of the non-singlet (NS) and singlet (SI) components (see Fig. 1):

$$D(Q^2) = n_f\, D^{NS}(Q^2) + n_f^2\, D^{SI}(Q^2)\,. \qquad (3)$$

Here n_f stands for the total number of quark flavours; all quarks are considered as massless. The Adler function D^{EM} corresponding to the electromagnetic vector current $j_\mu^{EM} = \sum_i q_i\,\overline{\psi}_i\gamma_\mu\psi_i$ (q_i stands for the electric charge of the quark field ψ_i) is thus given by the following combination:

$$D^{EM} = \Big(\sum_i q_i^2\Big) D^{NS} + \Big(\sum_i q_i\Big)^2 D^{SI}\,. \qquad (4)$$

Similar decompositions hold for the corresponding polarization functions Π and Π^{EM}.

The physical observable $R(s) = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}$ is related to $\Pi^{EM}(Q^2)$ by the optical theorem

$$R(s) = 12\pi\,\Im\,\Pi^{EM}(-s-i\epsilon)\,. \qquad (5)$$

The result for the perturbative expansions of the non-singlet part ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$D^{NS}(Q^2) = d_R\Big(1 + \sum_{i=1}^{\infty} d_i^{NS}\, a_s^i(Q^2)\Big) \qquad (6)$$

has been presented in [3]. For the singlet part it reads:

$$D^{SI}(Q^2) = d_R\Big(\sum_{i=3}^{\infty} d_i^{SI}\, a_s^i(Q^2)\Big)\,, \qquad (7)$$

where the parameter d_R (the dimension of the quark color representation, $d_R = 3$ in QCD) is factorized in *both* non-singlet and singlet components.

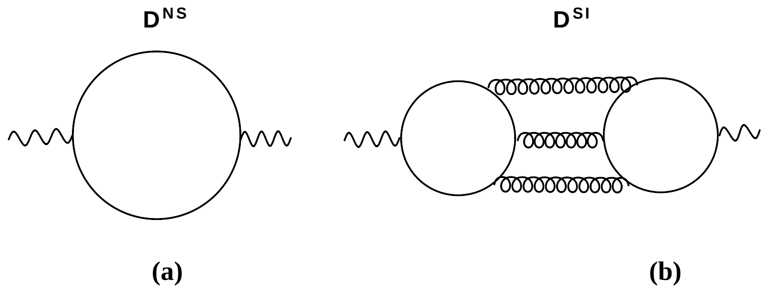


Figure 1: Lowest order non-singlet (a) and singlet (b) diagrams contributing to the Adler function.

The singlet component has the following structure at orders α_s^3 and α_s^4:

$$d_3^{SI} = d^{abc}\, d^{abc}/d_R \left(\tfrac{11}{192} - \tfrac{1}{8}\,\zeta_3\right), \tag{8}$$

$$d_4^{SI} = d^{abc}\, d^{abc}/d_R \left(C_F\, d_{4,1}^{SI} + C_A\, d_{4,2}^{SI} + T\, n_f\, d_{4,3}^{SI}\right). \tag{9}$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, $d^{abc} = 2\,\mathrm{Tr}(\{\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\}, \frac{\lambda^c}{2}\})$, T is the trace normalization of the fundamental representation. For QCD (colour gauge group SU(3)):

$$C_F = 4/3\,, \quad C_A = 3\,, \quad T = 1/2\,, \quad d^{abc}\, d^{abc} = 40/3\,.$$

Using the methods described in [16, 17, 18, 2, 19, 3] we obtain

$$d_{4,1}^{SI} = -\tfrac{13}{64} - \tfrac{1}{4}\,\zeta_3 + \tfrac{5}{8}\,\zeta_5\,, \tag{10}$$

$$d_{4,2}^{SI} = \tfrac{3893}{4608} - \tfrac{169}{128}\,\zeta_3 + \tfrac{45}{64}\,\zeta_5 - \tfrac{11}{32}\,\zeta_3^2\,, \tag{11}$$

$$d_{4,3}^{SI} = -\tfrac{149}{576} + \tfrac{13}{32}\,\zeta_3 - \tfrac{5}{16}\,\zeta_5 + \tfrac{1}{8}\,\zeta_3^2\,. \tag{12}$$

With the use of eqs. (10)-(12) and the result for D^{NS} from [3] we arrive at the complete result for the ratio $R(s)$ at order α_s^4 in (massless) QCD:

$$\begin{aligned}
R(s) = 3\sum_f q_f^2 \Big\{ & 1 + a_s + a_s^2\big(\tfrac{365}{24} - 11\,\zeta_3 - \tfrac{11}{12}\,n_f + \tfrac{2}{3}\,\zeta_3\,n_f\big) \\
& + a_s^3 \Big[n_f^2 \left(\tfrac{151}{162} - \tfrac{1}{108}\pi^2 - \tfrac{19}{27}\,\zeta_3\right) \\
& \quad + n_f \left(- \tfrac{7847}{216} + \tfrac{11}{36}\,\pi^2 + \tfrac{262}{9}\,\zeta_3 - \tfrac{25}{9}\,\zeta_5\right) \\
& \quad + \tfrac{87029}{288} - \tfrac{121}{48}\,\pi^2 - \tfrac{1103}{4}\,\zeta_3 + \tfrac{275}{6}\,\zeta_5 \Big] \\
& + a_s^4 \Big[n_f^3 \left(- \tfrac{6131}{5832} + \tfrac{11}{432}\,\pi^2 + \tfrac{203}{324}\,\zeta_3 - \tfrac{1}{54}\,\pi^2\,\zeta_3 + \tfrac{5}{18}\,\zeta_5\right) \\
& \quad + n_f^2 \Big(\tfrac{1045381}{15552} - \tfrac{593}{432}\,\pi^2 - \tfrac{40655}{864}\,\zeta_3 \\
& \qquad\qquad + \tfrac{11}{12}\,\pi^2\,\zeta_3 + \tfrac{5}{6}\,\zeta_3^2 - \tfrac{260}{27}\,\zeta_5 \Big) \\
& \quad + n_f \Big(- \tfrac{13044007}{10368} + \tfrac{2263}{96}\,\pi^2 + \tfrac{12205}{12}\,\zeta_3 - \tfrac{121}{8}\,\pi^2\,\zeta_3 \\
& \qquad\qquad - 55\,\zeta_3^2 + \tfrac{29675}{432}\,\zeta_5 + \tfrac{665}{72}\,\zeta_7 \Big) \\
& \quad + \tfrac{144939499}{20736} - \tfrac{49775}{384}\,\pi^2 - \tfrac{5693495}{864}\,\zeta_3 + \tfrac{1331}{16}\,\pi^2\,\zeta_3 \\
& \qquad\qquad + \tfrac{5445}{8}\,\zeta_3^2 + \tfrac{65945}{288}\,\zeta_5 - \tfrac{7315}{48}\,\zeta_7 \Big] \Big\} \\
+ \Big(\sum_f q_f\Big)^2 \Big\{ & a_s^3 \big(\tfrac{55}{72} - \tfrac{5}{3}\,\zeta_3\big) \\
& + a_s^4 \Big[n_f \left(- \tfrac{745}{432} + \tfrac{65}{24}\,\zeta_3 + \tfrac{5}{6}\,\zeta_3^2 - \tfrac{25}{12}\,\zeta_5\right) \\
& \quad + \left(\tfrac{5795}{192} - \tfrac{8245}{144}\,\zeta_3 - \tfrac{55}{4}\,\zeta_3^2 + \tfrac{2825}{72}\,\zeta_5\right) \Big] \Big\}\,,
\end{aligned} \tag{13}$$

where we set $\mu = Q$. The full results for Adler function and $R(s)$ for generic color factors and generic value of μ are rather lenghty and can be found (in computer-readable form) in `http://www-ttp.physik.uni-karlsruhe.de/Progdata /ttp12/ttp12-017`. Numerically, it reads:

$$\begin{aligned}
R(s) = 3\sum_f q_f^2 \Big\{ & 1 + a_s + a_s^2\left(1.986 - 0.1153\,n_f\right) \\
& + a_s^3\left(- 6.637 - 1.200\,n_f - 0.00518\,n_f^2\right) \\
& + a_s^4\big(- 156.608 + 18.7748\,n_f - 0.797434\,n_f^2 \\
& \qquad + 0.0215161\,n_f^3\big) \Big\} \\
- \Big(\sum_f q_f\Big)^2 & \left(1.2395\,a_s^3 + \left(17.8277 - 0.57489\,n_f\right) a_s^4\right).
\end{aligned}$$

Specifically, for the particular values of $n_f = 4$ and 5 one obtains (for the terms of order α_s^3 and α_s^4 we have explicitly decomposed the coefficient into non-singlet and singlet contributions):

$$\begin{aligned}
R^{n_f=4}(s) = \frac{10}{3} \Big[& 1 + a_s + 1.5245 a_s^2 \\
& + a_s^3\left(- 11.686 = -11.52 - 0.16527^{SI}\right) \\
& + a_s^4\left(- 94.961 = -92.891 - 2.0703^{SI}\right) \Big]\,,
\end{aligned} \tag{14}$$

$$\begin{aligned}
R^{n_f=5}(s) = \frac{11}{3} \Big[& 1 + a_s + 1.40902 a_s^2 \\
& + a_s^3\left(- 12.80 = -12.767 - 0.037562^{SI}\right) \\
& + a_s^4\left(- 80.434 = -79.981 - 0.4531^{SI}\right) \Big].
\end{aligned} \tag{15}$$

Note that for $n_f = 3$ the singlet contributions vanish in every order in α_s as the corresponding global coefficient $(\sum_i q_i)^2$ happens to be zero. Implications of this result for the determination of α_s in electron-positron annihilation and in Z-boson decays are discussed in [20].

3. GLS sum rule at order $O(\alpha_s^4)$ and the Crewther relation

The second quantity of interest, the GLS sum rule,

$$\frac{1}{2}\int_0^1 F_3(x,Q^2)dx = 3\,C^{CLS}(a_s), \quad (16)$$

relates the lowest moment of the isospin singlet structure function $F_3^{\nu p+\bar{\nu}p}(x,Q^2)$ to a coefficient $C^{CLS}(a_s)$, which appears in the operator product expansion of the axial and vector non-singlet currents

$$i\int TA_\mu^a(x)V_\nu^b(0)e^{iqx}dx|_{q^2\to-\infty} \approx C_{\mu\nu\alpha}^{V,ab}\,V_\alpha(0)+\dots \quad (17)$$

where

$$C_{\mu\nu\alpha}^{V,ab} = \delta^{ab}\epsilon_{\mu\nu\alpha\beta}\frac{q^\beta}{Q^2}C^{GLS}(a_s)$$

and $V_\alpha = \overline{\psi}\gamma_\alpha\psi$ is a flavour singlet quark current. At last $A_\mu^a = \overline{\psi}\gamma_\mu\gamma_5 t^a\psi$, $V_\nu^b = \overline{\psi}\gamma_\nu t^b\psi$ are axial vector and vector non-singlet quark currents, with t^a, t^b being the generators of the flavour group $SU(n_f)$.

Again diagrams contributing to $C^{GLS}(a_s)$ can be separated in two groups: non-singlet and singlet ones (see Fig. 2):

$$C^{GLS} = C^{NS} + C^{SI}\,, \quad (18)$$

$$C^{NS}(Q^2) = 1 + \sum_{i=1}^{\infty} c_i^{NS}\,a_s^i(Q^2)\,, \quad (19)$$

$$C^{SI}(Q^2) = \sum_{i=3}^{\infty} c_i^{SI}\,a_s^i(Q^2)\,. \quad (20)$$

The results for both functions C^{NS} and C^{SI} at order α_s^3 are known since early 90-ties [12]. Note that as a consequence of chiral invariance the closely related Bjorken sum rule receives contributions from the non-singlet piece only [12]:

$$C^{Bjp} \equiv C^{NS}. \quad (21)$$

The $O(\alpha_s^4)$ contribution to C^{Bjp} has been computed some time ago [3]. The calculation of the $O(\alpha_s^4)$ contribution to C^{SI} has been published in [1] for a generic gauge group and is repeated below:

$$c_3^{SI} = n_f\frac{d^{abc}d^{abc}}{d_R}\left(c_{3,1}^{SI} \equiv -\frac{11}{192}+\frac{1}{8}\zeta_3\right), \quad (22)$$

$$c_4^{SI} = n_f\frac{d^{abc}d^{abc}}{d_R}\left(C_F\,c_{4,1}^{SI} + C_A\,c_{4,2}^{SI} + T\,n_f\,c_{4,3}^{SI}\right), \quad (23)$$

$$c_{4,1}^{SI} = \frac{37}{128}+\frac{1}{16}\zeta_3-\frac{5}{8}\zeta_5\,, \quad (24)$$

$$c_{4,2}^{SI} = -\frac{481}{1152}+\frac{971}{1152}\zeta_3-\frac{295}{576}\zeta_5+\frac{11}{32}\zeta_3^2\,, \quad (25)$$

$$c_{4,3}^{SI} = \frac{119}{1152}-\frac{67}{288}\zeta_3+\frac{35}{144}\zeta_5-\frac{1}{8}\zeta_3^2\,. \quad (26)$$

Using the input from eqs. (10-12) and (22-26), the validity of the GCR can now be investigated. In fact, there exist two of them [5], one involving the non-singlet parts only and one involving also a singlet piece:

$$\begin{aligned} D^{NS}(a_s)\,C^{Bjp}(a_s) &= d_R\left[1+\frac{\beta(a_s)}{a_s}K^{NS}(a_S)\right], \quad (27)\\ K^{NS}(a_s) &= a_s K_1^{NS} + a_s^2 K_2^{NS} + a_s^3 K_3^{NS} + \dots \end{aligned}$$

and

$$\begin{aligned} D(a_s)\,C^{GLS}(a_s) &= d_R\,n_f\left[1+\frac{\beta(a_s)}{a_s}K(a_s)\right], \quad (28)\\ K(a_s) &= a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \dots \end{aligned}$$

Here $\beta(a_s) = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}a_s(\mu) = -\sum_{i\geq0}\beta_i a_s^{i+2}$ is the QCD β-function with its first term $\beta_0 = \frac{11}{12}C_A - \frac{T}{3}n_f$. The term proportional to the β-function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2.

Relation (27) has been studied in detail in [3], where its validity at order α_s^4 was demonstrated (a detailed discussion at orders α_s^2 and α_s^3 can be found in [5]).

Let us consider now eq. (28). Combining eqs. (3,18,21) and (27) leads to the following relations between coefficients K_i^{NS} and K_i:

$$\begin{aligned} K_1 &= K_1^{NS},\;\; K_2 = K_2^{NS}\,, \quad (29)\\ K_3 &= K_3^{NS} + K_3^{SI}\,, \quad (30)\\ K_3^{SI} &= k_{3,1}^{SI}\,n_f\,\frac{d^{abc}d^{abc}}{d_R}\,, \quad (31) \end{aligned}$$

with $k_{3,1}^{SI}$ being a numerical parameter.

Thus, we conclude that eq. (28) puts $3-1=2$ constraints between two triplets of (purely numerical) parameters $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$ and $\{c_{4,1}^{SI}, c_{4,2}^{SI}, c_{4,3}^{SI}\}$ appearing in eqs. (9) and (23) and completely describing the order α_s^4 singlet contributions to the Adler function and the Gross-Llewellyn Smith sum rule respectively.

The solution of the constraints and eqs. (24-26) produces the following relations for d_4^{SI}:

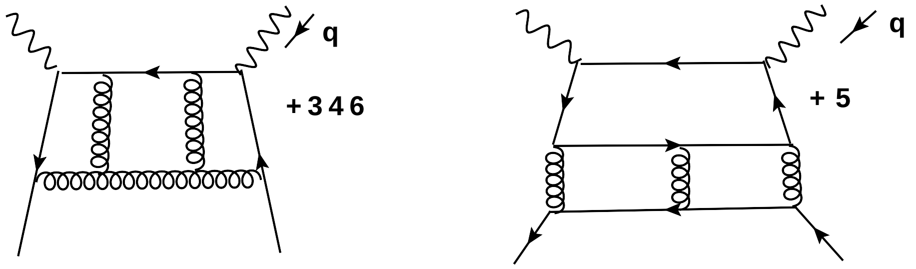
$$\begin{aligned} d_{4,1}^{SI} &= -\frac{3}{2}c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64}-\frac{\zeta_3}{4}+\frac{5\zeta_5}{8}\,, \quad (32)\\ d_{4,2}^{SI} &= -c_{4,2}^{SI} - \frac{11}{12}k_{3,1}^{SI}\,, \quad (33)\\ d_{4,3}^{SI} &= -c_{4,3}^{SI} + \frac{1}{3}k_{3,1}^{SI}\,, \quad (34) \end{aligned}$$

whose validity is indeed confirmed by the explicit calculations. As a result the remaining unknown $k_{3,1}^{SI}$ is fixed as:

$$k_{3,1}^{SI} = -\frac{179}{384}+\frac{25}{48}\zeta_3-\frac{5}{24}\zeta_5\,. \quad (35)$$

4. Conclusion

We have analytically computed coeffcients of all three colour structures contributing to the singlet part of the Adler function in massless QCD at $O(\alpha_s^4)$. We have checked that all constraints on these coefficients derived previously in [1] on the base of the



(a) **(b)**

Figure 2: (a),(b): $O(\alpha_s^3)$ non-singlet and singlet diagrams contributing to the Gross-Llewellyn Smith sum rule; note that the coefficient function C^{Bjp} is contributed by only non-singlet diagrams.

GCR are really fulfilled. This is an important cross-check of our calculations of D^{SI}, C^{SI} and the very GCR.

The calculations has been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using parallel MPI-based [21] as well as thread-based [22] versions of FORM [23]. For the evaluation of color factors we have used the FORM program *COLOR* [24]. The diagrams have been generated with QGRAF [25]. The figures have been drawn with the the help of Axodraw [26] and JaxoDraw [27].

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