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Expansion by regions: revealing potential and Glauber regions automatically

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Abstract

When performing asymptotic expansions using the strategy of expansion by regions, it is a non-trivial task to find the relevant regions. The recently published Mathematica code `asy.m` automates this task, but it has not been able to detect potential regions in threshold expansions or Glauber regions. In this work we present an algorithm and its implementation in the update `asy2.m` which also reveals potential and Glauber regions automatically.

1 Introduction

If a given Feynman integral depends on kinematic invariants and masses which essentially differ in scale, a natural idea is to expand it in ratios of small and large parameters. As a result, the integral is written as a series of simpler quantities than the original integral itself and it can be substituted by a sufficiently large number of terms of such an expansion. For limits typical of Euclidean space (for example, the

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off-shell large-momentum limit or the large-mass limit), one can write down the corresponding asymptotic expansion in terms of a sum over certain subgraphs of a given graph [1–6]. This prescription of expansion by subgraphs has been mathematically proven (see [4] and Appendix B.2 of [6]). Moreover, there is an automated tool [7, 8] where such an expansion by subgraphs is implemented.

For limits typical of Minkowski space (i.e. which cannot be formulated in Euclidean space) the universal strategy of expansion by regions [9–11, 6] is available. It consists of the following prescriptions:

- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a Taylor series with respect to the parameters that are considered small there.
- Integrate the integrand, expanded in the appropriate way in every region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

As shown in [11], this prescription can also be applied to parametric representations of Feynman integrals, i.e. alpha parameters (or generalized Feynman parameters) integrated from 0 to ∞, eventually restricted by a delta function. Then the regions are specified by scaling relations between the parameters.

There is no mathematical proof that this prescription is correct in all situations. But also no examples are known where a proper application of the expansion by regions leads to wrong results. An indirect proof exists for limits typical of Euclidean space because here the strategy of regions is equivalent to the mathematically proven expansion by subgraphs. A systematic study of the expansion by regions was presented recently in [12]. There it was shown explicitly and illustrated using various one-loop examples that one can start from a decomposition of a given integral into non-intersecting domains and arrive at an expansion by regions in the above sense. This requires certain conditions on the choice and completeness of the considered regions which were derived in [12]. As pointed out there, the appearance of additional overlap contributions can be avoided by adequate choices of the regions and regularization parameters.

While these findings provide some hints on the proper choice of the regions, it remains a non-trivial task to actually reveal the typical regions for a given limit. Usually, one starts from considering one-loop examples, checks the results against known analytical results, then proceeds in two loops etc. One can also use the second version [13] of the code `FIESTA` [14] to obtain numerically several first terms of a given asymptotic expansion.

Recently an algorithm for an automatic search of regions was suggested and implemented on a computer as the open source `Mathematica` code `asy.m` [15]. The algorithm uses a geometric approach based on finding the convex hull of a set of points determined from a parametric representation of the Feynman integral. In this way all possible sets of scalings for the (Feynman) parameters are found which lead to non-vanishing (because non-scaleless) integrals. These regions may then be used to expand the parametric integral, or they can be translated into regions for expanding the integral in loop-momentum space.

This code works successfully for a large variety of limits, at least in cases where the function $\mathcal{F}$ in the corresponding parametric representation which depends on kinematic invariants and masses is only composed of terms with the same sign. Moreover, it was shown in [15] that in this case there are no regions except for the ones produced by the code (all others result in scaleless integrals). In particular, the code `asy.m` works for Sudakov-type limits which are typical of Minkowski space. As it was pointed out in [15] the code does not reveal potential regions in threshold expansions, and similarly it fails to detect the so-called Glauber regions.

Our first goal in the present paper is to provide an algorithm and the corresponding update `asy2.m` which automatically reveals potential regions. This is done in Section 2. In Section 3 we consider the problem of revealing Glauber regions. Here the situation is more complicated than for the potential regions, but the updated code `asy2.m` is now also able to reveal Glauber regions automatically. In both Sections 2 and 3, we first formulate simple changes of variables and decompositions of a given Feynman integral, using instructive one-loop examples, which lead to integrals where `asy2.m` is able to detect the relevant regions corresponding to scalings of the parameters. Then, for both cases, we explain how to use the new features of `asy2.m` to perform these algorithmic steps automatically.

In the case of Section 3 with Glauber regions, the structure of the regions differs depending on whether the expansion is performed in loop-momentum space or at the level of the parametric integral. We show in Section 4 how to disentangle and match the various regions arising in this problem using generic propagator powers, and how `asy2.m` can be used to automate such an analysis.

A summary of the new features and the syntax of `asy2.m` (together with a download link) is provided in Section 5. In Section 6 we conclude by discussing the mathematical problem of proving the expansion by regions for a simple example which is not related to Feynman integrals, but where `asy2.m` works successfully.

2 Revealing potential contributions

Let us consider the one-loop propagator diagram with two massive lines in the threshold limit, i.e. when $y = m^2 - q^2/4 \to 0$ with q being the external momentum:

$$F(q^2, m^2) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)\left((k-q)^2 - m^2\right)}\,, \tag{1}$$

where $d = 4 - 2\varepsilon$, and the usual $+i0$ is implied in all the propagators. Within the strategy of expansion by regions, the hard and the potential regions give contributions to the expansion [9, 6]. The previous version of the code `asy.m` [15] reported only about the hard region. The reason for this can be seen in the corresponding parametric representation,

$$F(q^2, y) = i\pi^{d/2}\,\Gamma(\varepsilon) \iint \frac{(x_1+x_2)^{2\varepsilon-2}\,\delta\left(x_1+x_2-1\right)\,\mathrm{d}x_1\mathrm{d}x_2}{\left[\frac{q^2}{4}(x_1-x_2)^2 + y(x_1+x_2)^2 - i0\right]^{\varepsilon}}\,, \tag{2}$$

where the parameters x_i are integrated from 0 to ∞ (restricted by the delta function). As it was pointed out in [15], it is the region where $x_1 \approx x_2$ (more precisely $x_1 - x_2 \sim y^{1/2}$) which causes problems. In other words, the polynomial in the square brackets in (2) (considered at positive q^2 and y) has terms of different sign, such that cancellations occur because of the presence of the negative term $-q^2 x_1 x_2/2$.

To reveal the missing potential contribution, let us perform a simple trick. We decompose the integration domain into two subdomains, $x_1 \leq x_2$ and $x_2 \leq x_1$. The two resulting integrals are equal to each other, but such an equality will not generally take place for any integral. In the first domain we turn to new variables by $x_1 = x_1'/2,\ x_2 = x_2' + x_1'/2$, remove the primes at x_i and obtain the integral (again from 0 to ∞ with the usual restrictions via the delta function)

$$i\pi^{d/2}\,\frac{\Gamma(\varepsilon)}{2} \iint \frac{(x_1+x_2)^{2\varepsilon-2}\,\delta\left(x_1+x_2-1\right)\,\mathrm{d}x_1\mathrm{d}x_2}{\left[\frac{q^2}{4}x_2^2 + y(x_1+x_2)^2 - i0\right]^{\varepsilon}}\,. \tag{3}$$

The goal of this trick was to make the line $x_1 = x_2$ (in the old variables) the border of an integration domain which turned out to be (in the new variables) $x_2 = 0$. Now we can run the code `asy2.m`. Since this is a parametrical integral rather than a Feynman integral we use the newly introduced command[1] `WilsonExpand[]` for integrals where all parameters are integrated from 0 to ∞:

```
WilsonExpand[q^2/4*x2^2 + y*(x1 + x2)^2, x1 + x2,
  {x1, x2}, {q -> 1, y -> x}, Delta -> True]
```

The first two arguments of `WilsonExpand[]` are the polynomials $\mathcal{F}$ and $\mathcal{U}$ from the parametric integral (3) as defined in [15]. The third argument is the list of integration parameters, and the fourth argument specifies the scaling of the kinematic quantities with respect to the small parameter which is labelled by the global symbol `x`. The option `Delta -> True` tells `WilsonExpand[]` that, under the integration, the sum over an arbitrary non-empty subset of the integration parameters is restricted to 1 by a delta function.

Note that `WilsonExpand[]` can only take into account such a delta function if the specific choice of the sum over parameters in the argument of the delta function

[1] The name of the command refers to its application to parametric integrals contributing to Wilson loops.

is irrelevant. This is the case for the generalized Feynman-parameter integrals (2) and (3), where we could e.g. replace $\delta(x_1+x_2-1)$ by $\delta(x_1-1)$ without changing the value of the integral. But if a specific form of the delta function is assumed, e.g. by replacing $x_1+x_2 \to 1$ under the integral, then the option `Delta` of `WilsonExpand[]` does not apply (see Section 5 for details). Alternatively, `WilsonExpand[]` can be used without the option `Delta` after eliminating one of the integrations, e.g. via $\delta(x_1-1)$.

The output of `WilsonExpand[]` is a list of regions specified by the scaling of the parameters x_i. Here the result `{{0, 0}, {0, 1/2}}` reveals not only the hard region with the scaling $x_1, x_2 \sim y^0$ but also the region $x_1 \sim y^0, x_2 \sim y^{1/2}$ which we qualify as the potential region.

The contribution of the potential region can then be evaluated in terms of gamma functions at general ε. Taking into account that we have two identical integrals after our decomposition, we arrive at the following result for the potential contribution:

$$i\pi^{d/2}\,\Gamma(\varepsilon-1/2)\sqrt{\frac{\pi y}{q^2}}\,y^{-\varepsilon} \tag{4}$$

in agreement with [6].

In fact, such a trick of making manifest squares of some linear combination of the integration parameters was already used in the code `FIESTA` [14] in order to evaluate numerically Feynman integrals at a threshold. Using the implementation of this procedure in `FIESTA` it turned out to be possible to automate the above trick for a general Feynman integral. In the present version `asy2.m`, the user may call the command `AlphaRepExpand[]` with the additional option `PreResolve` enabled which automatically looks for the change of variables described above:

```
AlphaRepExpand[{k}, {k^2 - m^2, (k - q)^2 - m^2},
  {q^2 -> qq, m^2 -> qq/4 + y}, {qq -> 1, y -> x},
  PreResolve -> True]
```

As in the previous version of `asy.m`, the arguments of `AlphaRepExpand[]` are the list of loop momenta, the list of denominators of the loop integral, a list of replacements for the kinematic quantities, and the list of scalings with respect to the small parameter `x`. The output is a list of entries which indicate for each region the changes of variables, the Jacobian of the integral transformation, and the scalings of the new variables:

```
{{{x[1] -> y[1] + y[2]/2, x[2] -> y[2]/2}, 2, {0, -1/2}},
 {{x[1] -> y[1] + y[2]/2, x[2] -> y[2]/2}, 2, {0, 0}},
 {{x[1] -> y[1]/2, x[2] -> y[1]/2 + y[2]}, 2, {0, 0}},
 {{x[1] -> y[1]/2, x[2] -> y[1]/2 + y[2]}, 2, {0, 1/2}}}
```

Remember that a region remains invariant if all parameters are rescaled simultaneously, due to the homogeneity of the polynomials $\mathcal{F}$ and $\mathcal{U}$. So the scaling $\{0, -1/2\}$ shown above is equivalent to $\{1/2, 0\}$, i.e. the first new parameter is suppressed by

$y^{1/2}$ with respect to the second one. For both parts of the integral decomposition the hard and potential regions are found.

The preresolution algorithm implemented in `asy2.m` (and switched on with the option `PreResolve`) tries to eliminate factorized combinations of terms in the function $\mathcal{F}$ which potentially cancel each other, like $(x_1 - x_2)^2$ in the example above. It checks all pairs of variables (say, x and y) which are part of monomials with opposite sign. For all those pairs the code tries to build a linear combination z of x and y such that in the variables x and z or y and z this monomial disappears. The code checks whether in the new variables the number of monomials with opposite sign decreases. For all such pairs the code recursively repeats the initial procedure in the new variables. As a result it creates a tree of possible bisections and corresponding replacements of variables. A leaf of this tree is a set of sectors and functions such that one cannot decrease the number of monomials with opposite sign any longer. Ideally it means that all monomials now have the same sign. The code analyzes all leafs and chooses one of those with the minimal number of opposite-sign monomials (or the minimal number of sectors if the numbers of monomials with opposite sign coincide). After finishing with the preresolution, the code performs the replacements and looks for regions in all those sectors.

Note that the algorithm can only find the necessary variable transformations if it is able to determine the relative signs of all terms in the polynomial $\mathcal{F}$. As the signs of symbols are unknown to `asy2.m`, the substitution rules of the third and fourth arguments of `AlphaRepExpand[]` must replace all kinematic quantities by numbers (integers or fractions of integers) or powers of the small parameter `x`.

We have checked that the updated version `asy2.m` works in various examples of the threshold expansion (considered in [9,6]): a triangle, a box, the two-loop propagator diagram (with the masses $m, m, m, m, 0$), a two-loop vertex diagram. Because of the decomposition of a given integration domain into subdomains, the number of resulting integrals for various regions increases a little bit. For example, the (hard-hard) region for the two-loop propagator diagram is described by six integrals, the (potential-ultrasoft) region is also described by six integrals, etc. However, the (potential-hard) region is described by four integrals with some regions (with scalings composed of 1, 1/2 and 0), and four more integrals with a set of regions of a different type (composed of 1 and 0).

Let us finally mention that the preresolution algorithm also works for threshold expansions with unequal masses. Returning to the one-loop example (1), but now with different masses m_1 and m_2 in the propagators and the expansion parameter defined as $y = (m_1 + m_2)^2/4 - q^2/4 \to 0$, we call:

```
AlphaRepExpand[{k}, {k^2 - m1^2, (k - q)^2 - m2^2},
  {q^2 -> (m1 + m2)^2 - 4*y},
  {y -> x, m1 -> 2, m2 -> 3/2},
  PreResolve -> True]
```

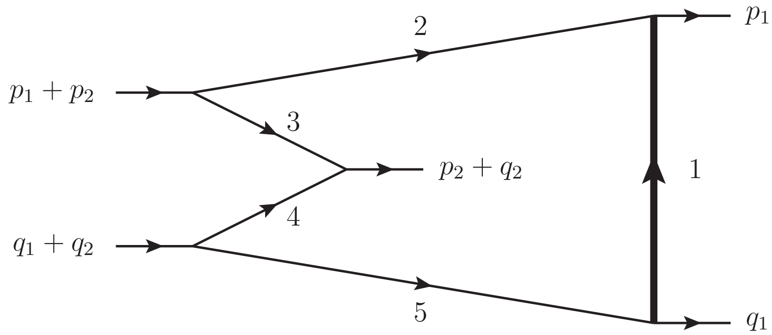

Figure 1: One-loop five-point integral exhibiting a Glauber contribution.

We have set the values of the masses to different rational numbers, $m_1 \to 2$ and $m_2 \to 3/2$ (the actual values are irrelevant), thus permitting the preresolution algorithm to know about their sign without assuming any equality or other relation between them. The polynomial $\mathcal{F}$ in the parametric representation of the Feynman integral reads $\mathcal{F} = (m_1 x_1 - m_2 x_2)^2 + 4 y x_1 x_2$, and cancellations occur where $m_1 x_1 = m_2 x_2$. These cancellations are automatically made explicit by adequate changes of variables. The output

```
{{{x[1] -> y[1] + 3*y[2]/7, x[2] -> 4*y[2]/7}, 7/4, {0, -1/2}},
 {{x[1] -> y[1] + 3*y[2]/7, x[2] -> 4*y[2]/7}, 7/4, {0, 0}},
 {{x[1] -> 3*y[1]/7, x[2] -> 4*y[1]/7 + y[2]}, 7/3, {0, 0}},
 {{x[1] -> 3*y[1]/7, x[2] -> 4*y[1]/7 + y[2]}, 7/3, {0, 1/2}}}
```

shows that `asy2.m` detects both regions. The variable transformations are always normalized such that sums of the parameters (here $x_1 + x_2$) remain invariant.

So, we now have a manifestly Lorentz-invariant treatment of threshold expansion and a code that automatically provides the set of relevant regions.

3 Revealing Glauber contributions

Let us consider the one-loop five-point integral in Fig. 1, where two initial-state partons both perform a collinear splitting into two partons each with momenta p_1, p_2 and q_1, q_2, respectively. While two partons, one of each pair, collide with a large centre-of-mass energy $Q = \sqrt{(p_2 + q_2)^2}$, the two remaining partons exchange a particle with the small mass m. We will use the simplified kinematics $p_1 = p_2 = p$ and $q_1 = q_2 = q$ with $p^2 = q^2 = 0$ and $(p+q)^2 = 2p \cdot q = Q^2$ in the limit $m^2/Q^2 \to 0$:

$$F(Q^2, m^2) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)(k^2 - 2p \cdot k)(k^2 + 2p \cdot k)(k^2 - 2q \cdot k)(k^2 + 2q \cdot k)} \,. \tag{5}$$

Before we search for regions using `asy.m`, we notice that this five-point integral is similar to the Sudakov form factor example treated in Section 6 of [12]. From the viewpoint of the convergence of the expansions, the second and third propagators of (5) are equivalent, and so are the fourth and fifth propagators. Effectively, the five-point integral has only three different types of propagators which are equivalent to the ones of the three-point integral in [12]. So the integral (5) can be expanded in loop-momentum space employing the regions and convergence domains known from [12] (and using generic propagator powers as analytic regulators where necessary):

- a hard region where $k \sim Q$,
- a 1-collinear region where $k^2 \sim p \cdot k \sim m^2$ and $q \cdot k \sim Q^2$,
- a 2-collinear region where $k^2 \sim q \cdot k \sim m^2$ and $p \cdot k \sim Q^2$,
- a Glauber region where $p \cdot k \sim q \cdot k \sim m^2$, and the components of k perpendicular to the plane spanned by p, q scale as $k_\perp \sim m$.

The collinear-plane region mentioned in [12] yields only scaleless contributions. But, in contrast to the three-point integral, the five-point integral has a non-vanishing Glauber contribution. The Glauber region even provides the leading contribution scaling as $(m^2)^{-2-\varepsilon}$, whereas the collinear contributions start with $(m^2)^{-1-\varepsilon}$ and the hard contribution starts with $(m^2)^0$.

The five-point integral (5) can be represented in terms of an integral over Feynman parameters,

$$F(Q^2, m^2) = -i\pi^{d/2}\,\Gamma(3+\varepsilon) \times \int \cdots \int \frac{(x_1 + \ldots + x_5)^{1+2\varepsilon}\,\delta\left(\sum_i x_i - 1\right)\, \mathrm{d}x_1 \cdots \mathrm{d}x_5}{\left[x_1(x_1 + \ldots + x_5)m^2 + (x_2 - x_3)(x_4 - x_5)Q^2 - i0\right]^{3+\varepsilon}}\,, \quad (6)$$

where one can choose the sum in the argument of the delta function in an appropriate way, i.e. restrict only the sum over a subset of the parameters to 1 and extend the integration over the rest of the parameters to the whole domain $[0, \infty)$.

Applying the strategy of expansion by regions in Feynman-parameter space and trying to reveal regions relevant to the given limit $m^2/Q^2 \to 0$ with the help of the code `asy.m` [15], we call:

```
AlphaRepExpand[{k},
  {k^2 - m^2, k^2 - 2*p*k, k^2 + 2*p*k, k^2 - 2*q*k, k^2 + 2*q*k},
  {p^2 -> 0, q^2 -> 0, p*q -> Q^2/2},
  {Q -> 1, m^2 -> x}]
```

The output states the following set of three regions:

```
{{0, 0, 0, 0, 0}, {0, 0, 0, 1, 1}, {0, 1, 1, 0, 0}}
```

As before, the regions are specified by the scaling of the Feynman parameters in terms of powers of the small parameter m^2. For example, for the second region we have $x_1 \sim x_2 \sim x_3 \sim (m^2)^0$, $x_4 \sim x_5 \sim (m^2)^1$. The first region is hard; its contribution starts with $(m^2)^0$. The second and third regions start with order $(m^2)^{-1-\varepsilon}$. They correspond to the two collinear regions stated for the momentum-space expansion above. But `asy.m` does not find anything corresponding to the Glauber region; in particular, none of the regions found by `asy.m` provides the leading $(m^2)^{-2-\varepsilon}$ contribution.

We notice that, as in the previous section about potential contributions, the polynomial in the square brackets of (6) has terms of different sign. The missing Glauber contribution stems from the parameter region where either $(x_2 - x_3) \sim (m^2)^1$ or $(x_4 - x_5) \sim (m^2)^1$. So let us decompose the parametric integral into four parts corresponding to the domains where the two factors $(x_2 - x_3)$ and $(x_4 - x_5)$ are either positive or negative and then introduce new variables in such a way that this product takes the form $\pm x_2' x_4'$. For example, in the domain $x_2 \leq x_3$, $x_5 \leq x_4$ we change the variables by $x_2 = x_3'/2,\ x_3 = x_2' + x_3'/2$ and by $x_4 = x_4' + x_5'/2,\ x_5 = x_5'/2$, similarly to our example in the previous section. However, in the threshold expansion the cancelling terms appeared in squared form such that a transformation between one pair of variables was sufficient. Here two separate factors involve cancellations, which requires a twofold change of variables.

Removing the primes from the variables x_i, the parametric integral reads

$$F(Q^2, m^2) = 2(I_+ + I_-) \tag{7}$$

with

$$\begin{aligned} I_\pm = &-i\pi^{d/2}\,\frac{\Gamma(3+\varepsilon)}{4} \\ &\times \int \cdots \int \frac{(x_1+x_2+x_3+x_4+x_5)^{1+2\varepsilon}\ \delta\left(x_1 - 1\right)\ \mathrm{d}x_1 \cdots \mathrm{d}x_5}{\left[x_1(x_1+x_2+x_3+x_4+x_5)m^2 \pm x_2 x_4 Q^2 - i0\right]^{3+\varepsilon}}\,, \end{aligned} \tag{8}$$

where we have chosen the argument of the delta function as $x_1 - 1$, so that we may also write

$$\begin{aligned} I_\pm = &-i\pi^{d/2}\,\frac{\Gamma(3+\varepsilon)}{4} \\ &\times \int_0^\infty \cdots \int_0^\infty \frac{(1+x_2+x_3+x_4+x_5)^{1+2\varepsilon}\ \mathrm{d}x_2 \cdots \mathrm{d}x_5}{\left[(1+x_2+x_3+x_4+x_5)m^2 \pm x_2 x_4 Q^2 - i0\right]^{3+\varepsilon}}\,. \end{aligned} \tag{9}$$

It is sufficient to consider the expansion of I_+ and obtain a result for I_- by analytically continuing $Q^2 \to -Q^2 - i0$, taking into account that the dependence on Q^2 is power-like.

Now we can apply `asy2.m` to the integral I_+ using either the command

```
WilsonExpand[x1*(x1 + x2 + x3 + x4 + x5)*m^2 + x2*x4*Q^2,
```

```
    x1 + x2 + x3 + x4 + x5,
    {x1, x2, x3, x4, x5},
    {Q^2 -> 1, m^2 -> x}, Delta -> True]
```

for integrals (8) restricted by a delta function (see Section 5), yielding the output:

```
{{0, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 0, 1, 0}}
```

Or we use the command

```
WilsonExpand[(1 + x2 + x3 + x4 + x5)*m^2 + x2*x4*Q^2,
  1 + x2 + x3 + x4 + x5,
  {x2, x3, x4, x5},
  {Q^2 -> 1, m^2 -> x}]
```

for integrals (9) over variables from 0 to ∞ without any restriction, and obtain the output:

```
{{0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 1, 0}}
```

The two results are equivalent, as $x_1 \sim (m^2)^0$ is implied in the second case. So we obtain again three regions. We will see in a moment that this list of regions is indeed correct and complete.

But first, let us emphasize that the new preresolution algorithm in `asy2.m` is capable of performing the transformation of the integral from (6) to (8) automatically:

```
AlphaRepExpand[{k},
  {k^2 - m^2, k^2 - 2*p*k, k^2 + 2*p*k, k^2 - 2*q*k, k^2 + 2*q*k},
  {p^2 -> 0, q^2 -> 0, p*q -> Q^2/2},
  {Q -> 1, m^2 -> x},
  PreResolve -> True]
```

The output of this command lists four different variable transformations according to the twofold decomposition described above. For each of the integrals over new parameters, the regions $\{0,0,0,0,0\}$, $\{0,1,0,0,0\}$ and $\{0,0,0,1,0\}$ are found, up to permutations in the order of the parameters from different changes of variables.

The evaluation of the contributions to each region, as found by `WilsonExpand[]` or `AlphaRepExpand[]` (including the `PreResolve` option), is straightforward. The first region is the hard one. The contributions of the second and third regions are not individually regularized by dimensional regularization, as it often happens for Sudakov-type limits. We use an auxiliary analytic regularization by introducing additional powers $x_2^{\delta_2} x_3^{\delta_3} x_4^{\delta_4} x_5^{\delta_5}$ of the new variables into the integrand of (9), taking the limit $\delta_2, \delta_3, \delta_4, \delta_5 \to 0$ in the end. The leading-order (LO) contribution of the second and third regions to the integral $F(Q^2, m^2)$ reads

$$-i\pi^{d/2} \frac{i\pi\Gamma(\varepsilon)}{2Q^2(m^2)^{2+\varepsilon}} \,. \tag{10}$$

This agrees with the leading contribution of the Glauber region in the momentum-space expansion.

We have found the leading Glauber contribution of order $(m^2)^{-2-\varepsilon}$. But we seem to have lost the two collinear regions with the scalings $\{0,0,0,1,1\}$ and $\{0,1,1,0,0\}$ found before the change of variables. In fact, we can evaluate the contributions from these two regions by expanding the integral (9). The resulting integrals are scaleless and regularized by the parameters δ_3, δ_5, so they vanish, and `asy2.m` is right in omitting these two regions.

We are also able to solve the integral (9) including the auxiliary analytic regularization factor $x_2^{\delta_2} x_3^{\delta_3} x_4^{\delta_4} x_5^{\delta_5}$ in terms of a onefold Mellin–Barnes representation:

$$
\begin{aligned}
I_\pm = & -i\pi^{d/2} \frac{\Gamma(1+\delta_3)\Gamma(1+\delta_5)}{4} \frac{1}{2\pi i} \int \mathrm{d}z \, (m^2)^z (\pm Q^2 - i0)^{-3-\varepsilon-z} \\
& \times \Gamma(-z)\Gamma(-2-\varepsilon+\delta_2-z)\Gamma(-2-\varepsilon+\delta_4-z) \\
& \times \frac{\Gamma(1-\delta_2-\delta_3-\delta_4-\delta_5+z)\Gamma(3+\varepsilon+z)}{\Gamma(-1-2\varepsilon-z)} \, . \qquad (11)
\end{aligned}
$$

The asymptotic expansion of $I_\pm$ in the limit $m^2/Q^2 \to 0$ is obtained by taking the residues of the poles of the functions $\Gamma(\ldots - z)$. The poles of $\Gamma(-z)$ correspond to the hard region, while the poles of the two functions $\Gamma(-2-\varepsilon+\delta_{2,4}-z)$ provide the contributions of the second and third regions. So `asy2.m` has found all contributing regions.

In the Mellin–Barnes integral (11) we can safely take the limit $\delta_2, \delta_3, \delta_4, \delta_5 \to 0$, add up I_+ and I_-, and arrive at the Mellin–Barnes representation

$$
\begin{aligned}
F(Q^2, m^2) = & \, i\pi^{d/2} \frac{i}{2} \frac{1}{2\pi i} \int \mathrm{d}z \, (m^2)^z (Q^2)^{-3-\varepsilon-z} e^{i\pi(\varepsilon+z)/2} \\
& \times \Gamma(-z)\Gamma(-2-\varepsilon-z)\Gamma\left(\tfrac{-1-\varepsilon-z}{2}\right) \frac{\Gamma(1+z)\Gamma\left(\frac{3+\varepsilon+z}{2}\right)}{\Gamma(-1-2\varepsilon-z)} \, . \qquad (12)
\end{aligned}
$$

The LO contribution to $F(Q^2, m^2)$ is obtained from the residue of the single pole at $z = -2-\varepsilon$, in agreement with (10). The next-to-leading-order (NLO) contribution stems from the residue of the double pole at $z = -1-\varepsilon$ and reads

$$
i\pi^{d/2} \frac{\Gamma(1+\varepsilon)}{(Q^2)^2 (m^2)^{1+\varepsilon}} \left(i\frac{\pi}{2} + 2\psi(-\varepsilon) - \psi(1+\varepsilon) + \gamma_{\mathrm{E}} - \ln\frac{Q^2}{m^2} - 1 \right) \, . \qquad (13)
$$

This agrees with the NLO contributions of the second and third regions.

At next-to-next-to-leading order (NNLO) there is a contribution from the residue of the single pole at $z = -\varepsilon$ which reads

$$
- i\pi^{d/2} \frac{i\pi\Gamma(2+\varepsilon)}{4(Q^2)^3 (m^2)^{\varepsilon}} \qquad (14)
$$

and agrees with the NNLO contributions of the second and third regions. The second NNLO contribution comes from the residue of the single pole at $z = 0$. It is given by

$$- i\pi^{d/2}\, \frac{i\, e^{i\pi\varepsilon/2}\, \Gamma(-2-\varepsilon)\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1-\varepsilon}{2}\right)}{2(Q^2)^{3+\varepsilon}\, \Gamma(-1-2\varepsilon)} \tag{15}$$

and agrees with the LO contribution of the hard region. So indeed all contributions to the five-point integral up to NNLO are correctly reproduced by the contributions of the three regions found by `asy2.m` after the decomposition of the integral and the change of variables.

When revealing Glauber regions for a general diagram, the preresolution algorithm of `asy2.m` tries to eliminate monomials with opposite sign in the polynomial $\mathcal{F}$ by automatically separating the integration into domains and performing changes of variables. If the option `PreResolve` is enabled for `AlphaRepExpand[]`, the code warns the user if the elimination of monomials with opposite sign has not been successful, such that possibly not all regions are revealed. This is the case if some monomials of opposite sign remain in the polynomial $\mathcal{F}$ after tries to eliminate them, or if symbols with unknown signs are present in the polynomial. We are therefore convinced that `AlphaRepExpand[]`, with the preresolution enabled, either reveals all relevant regions or issues a warning.

As for the threshold expansion of the previous section, `asy2.m` can also treat more complicated kinematical situations, e.g. the five-point integral depicted in Fig. 1 with $p_1 \neq p_2$ and $q_1 \neq q_2$ (retaining $p_1 \parallel p_2$ and $q_1 \parallel q_2$ such that $p_i \cdot p_j = q_i \cdot q_j = 0$ and $2p_i \cdot q_j = P_i Q_j$ with $P_i, Q_j > 0$):

```
AlphaRepExpand[{k},
  {k^2 - m^2, k^2 - 2*p1*k, k^2 + 2*p2*k,
              k^2 - 2*q2*k, k^2 + 2*q1*k},
  {p1^2 -> 0, p2^2 -> 0, p1*p2 -> 0,
    q1^2 -> 0, q2^2 -> 0, q1*q2 -> 0,
    p1*q1 -> P1*Q1/2, p1*q2 -> P1*Q2/2,
    p2*q1 -> P2*Q1/2, p2*q2 -> P2*Q2/2},
  {m^2 -> x, P1 -> 1, P2 -> 3, Q1 -> 2, Q2 -> 3/2},
  PreResolve -> True]
```

All kinematic invariants are replaced by rational numbers in order to enable the preresolution algorithm to work. The code correctly decomposes the integral into four pieces and finds the three regions for each of them.

Let us finally discuss the reason why, besides the hard region which is always present, the expansion in loop-momentum space requires two collinear regions and one Glauber region, whereas the expansion of the parametric integrals (9) has two regions providing the leading Glauber contribution and no further collinear regions. In fact, the momentum-space expansion is also valid for loop integrals (5) where each propagator is raised to an arbitrary, even non-integer power. For the decomposition of the parametric integral and the change of variables, however, we have assumed the specific form (5) with each propagator present exactly once. In this case, the loop integrand can be expanded into partial fractions as follows:

$$\begin{aligned}
&\frac{1}{(k^2-m^2)(k^2-2p\cdot k)(k^2+2p\cdot k)(k^2-2q\cdot k)(k^2+2q\cdot k)} \\
&\quad = \frac{1}{4(m^2)^2}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2}-\frac{m^2}{(k^2)^2}\right)\left(\frac{1}{k^2-2p\cdot k}+\frac{1}{k^2+2p\cdot k}\right) \\
&\quad \times \left(\frac{1}{k^2-2q\cdot k}+\frac{1}{k^2+2q\cdot k}\right) . \qquad (16)
\end{aligned}$$

Expanding this product of terms, one obtains twelve three-point integrals, which are well known. Because they only depend on $Q^2 = 2p \cdot q$, we recognize from the last two factors the structure $F(Q^2, m^2) = 2(I_+ + I_-)$, where I_+ and I_- are related by $Q^2 \to -Q^2$ as before. The three-point integrals with the massless propagators $1/k^2$ or $1/(k^2)^2$ only have a hard region. The massive three-point integral with propagator $1/(k^2 - m^2)$ is known to possess a hard and two collinear regions. Its LO and NLO hard contributions are cancelled by the massless three-point integrals, such that the uncancelled hard contributions start with $(m^2)^2/(m^2)^2 = (m^2)^0$, as for the five-point integral. The LO collinear contributions of the three-point integrals, enhanced by the $1/(m^2)^2$ prefactor, scale as $(m^2)^{-2-\varepsilon}$.

So, in the special case when all propagator powers are equal to 1, the five-point integral reduces to a linear combination of three-point integrals revealing the same structure of regions as found from the expansion of the parametric integral (9). This picture changes when generic propagator powers are introduced as analytic regulators, which is done in the next section.

4 Disentangling regions via propagator powers

In the previous section we have seen different patterns of regions arising when expanding either in loop-momentum space or in parametric space. The individual contributions can be disentangled more easily when the dependence on the propagator powers is retained. Instead of (5), let us consider the integral

$$\begin{aligned}
F(Q^2, m^2) = \int &\frac{\mathrm{d}^d k}{(k^2-m^2)^{1+\lambda_1}(k^2-2p\cdot k)^{1+\lambda_2}(k^2+2p\cdot k)^{1+\lambda_3}} \\
&\times \frac{1}{(k^2-2q\cdot k)^{1+\lambda_4}(k^2+2q\cdot k)^{1+\lambda_5}} , \qquad (17)
\end{aligned}$$

where the analytic regularization parameters λ_i make the propagator powers different from the previous case. The asymptotic expansion in loop-momentum space yields contributions from the four regions listed in the beginning of Section 3. The LO hard contribution still scales as $(m^2)^0$, but now the LO 1-collinear contribution scales as $(m^2)^{-1-\varepsilon-\lambda_1-\lambda_2-\lambda_3}$, the LO 2-collinear contribution as $(m^2)^{-1-\varepsilon-\lambda_1-\lambda_4-\lambda_5}$ and the LO Glauber contribution as $(m^2)^{-2-\varepsilon-\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5}$. So all regions are characterized by a distinct scaling. When performing the expansion in parametric space for generic λ_i, we are able to disentangle the contributions from each region and match them with the regions obtained in loop-momentum space.

For generic $\lambda_1, \ldots, \lambda_5$, the parametric integral corresponding to (6) reads

$$
\begin{aligned}
F(Q^2, m^2) &= -i\pi^{d/2}\, e^{-i\pi(\lambda_1+\ldots+\lambda_5)}\, \frac{\Gamma(3+\varepsilon+\lambda_1+\ldots+\lambda_5)}{\Gamma(1+\lambda_1)\cdots\Gamma(1+\lambda_5)} \\
&\quad\times \int\cdots\int \frac{x_1^{\lambda_1}\cdots x_5^{\lambda_5}\ (x_1+\ldots+x_5)^{1+2\varepsilon+\lambda_1+\ldots+\lambda_5}\ \delta\left(\sum_i x_i - 1\right)\ \mathrm{d}x_1\cdots\mathrm{d}x_5}{\left[x_1(x_1+\ldots+x_5)m^2 + (x_2-x_3)(x_4-x_5)Q^2 - i0\right]^{3+\varepsilon+\lambda_1+\ldots+\lambda_5}}\ .
\end{aligned}
\tag{18}
$$

The change of variables performed in Section 3, e.g. $x_2 = x_2' + x_3'/2$, $x_3 = x_3'/2$ for $x_2 \geq x_3$, is complicated by the presence of the factors $x_2^{\lambda_2} x_3^{\lambda_3} x_4^{\lambda_4} x_5^{\lambda_5}$, where parts of the monomials will change into polynomials. If we want to keep the simple form $F(Q^2, m^2) = 2(I_+ + I_-)$ from the previous section, then we have to require $\lambda_3 = \lambda_2$ and $\lambda_5 = \lambda_4$. Under this restriction of the parameters λ_i, the parametric integrals can be written as

$$
\begin{aligned}
I_\pm &= -i\pi^{d/2}\, \frac{e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)}}{4^{1+\lambda_2+\lambda_4}}\, \frac{\Gamma(3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4)}{\Gamma(1+\lambda_1)\Gamma^2(1+\lambda_2)\Gamma^2(1+\lambda_4)} \\
&\quad\times \int\cdots\int \frac{x_1^{\lambda_1}(2x_2+x_3)^{\lambda_2} x_3^{\lambda_2} (2x_4+x_5)^{\lambda_4} x_5^{\lambda_4}\ (x_1+\ldots+x_5)^{1+2\varepsilon+\lambda_1+2\lambda_2+2\lambda_4}}{\left[x_1(x_1+\ldots+x_5)m^2 \pm x_2 x_4 Q^2 - i0\right]^{3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4}} \\
&\qquad\times \delta\left(x_1 - 1\right)\ \mathrm{d}x_1\cdots\mathrm{d}x_5\ .
\end{aligned}
\tag{19}
$$

In order to find the regions for the asymptotic expansion of (19), we have to provide the additional polynomial factors $(2x_2 + x_3)$ and $(2x_4 + x_5)$ to `WilsonExpand[]`. We can do this by multiplying the new polynomials to the second argument of the command:

```
WilsonExpand[x1*(x1 + x2 + x3 + x4 + x5)*m^2 + x2*x4*Q^2,
  (x1 + x2 + x3 + x4 + x5)*(2*x2 + x3)*(2*x4 + x5),
  {x1, x2, x3, x4, x5},
  {Q^2 -> 1, m^2 -> x}, Delta -> True]
```

The output of this command reads:

```
{{0, 0, 0, 1, 0}, {0, 0, 0, 1, 1},
 {0, 1, 0, 0, 0}, {0, 1, 1, 0, 0},
 {0, 0, 0, 0, 0}}
```

In addition to the regions present in the analysis of Section 3, we retrieve the two collinear regions with scalings $\{0,0,0,1,1\}$ and $\{0,1,1,0,0\}$. The updated code `asy2.m` is capable of taking into account generic propagator powers automatically when told through the additional option `GenericPowers`:

```
AlphaRepExpand[{k},
  {k^2 - m^2, k^2 - 2*p*k, k^2 + 2*p*k, k^2 - 2*q*k, k^2 + 2*q*k},
  {p^2 -> 0, q^2 -> 0, p*q -> Q^2/2},
  {Q -> 1, m^2 -> x},
  PreResolve -> True, GenericPowers -> True]
```

When the option `GenericPowers` is enabled, the polynomial $\mathcal{U}$ obtained from the loop integral is multiplied by the product of all Feynman parameters, $x_1 x_2 \cdots$, before the changes of variables are performed. Some of these additional factors then turn into polynomials through replacements in the preresolution algorithm, while others remain monomials and are therefore irrelevant for the analysis of `asy2.m`. The call of `AlphaRepExpand[]` stated above correctly yields all five regions for each of the variable transformations.

For the evaluation of the Glauber contributions with scalings $\{0,0,0,1,0\}$ and $\{0,1,0,0,0\}$, an additional analytic regularization is needed, and we choose to multiply the integrand of (19) by $x_2^{\delta_2} x_4^{\delta_4}$. (The parameters δ_3, δ_5 from Section 3 are not needed here due to the presence of λ_2, λ_4.) The two Glauber contributions are individually singular in the limit $\delta_2, \delta_4 \to 0$, but this singularity cancels in the sum of the two contributions.

The leading contribution to the integral $F(Q^2, m^2)$ originates from the sum of the LO Glauber contributions. It reads

$$- i\pi^{d/2} \frac{i\, e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)}\, \Gamma\left(\frac{1}{2}+\lambda_2\right) \Gamma\left(\frac{1}{2}+\lambda_4\right) \Gamma(2+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4)}{2Q^2\,(m^2)^{2+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4}\, \Gamma(1+\lambda_1)\Gamma(1+\lambda_2)\Gamma(1+\lambda_4)\Gamma(1-\varepsilon)} \times \Gamma(-1-\varepsilon-2\lambda_2-2\lambda_4)\,. \quad (20)$$

This agrees with the LO contribution from the one Glauber region in the momentum-space expansion of (17) for $\lambda_3 = \lambda_2$ and $\lambda_5 = \lambda_4$. Here, in the expansion of the parametric integrals, we have two regions producing the Glauber contribution. This is possible because the contributions from both regions have the same scaling (for $\delta_2 = \delta_4 = 0$), starting with $(m^2)^{-2-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4}$. For $\lambda_i = 0$, the result (20) reproduces (10).

Among the NLO contributions to $F(Q^2, m^2)$, we expect NLO Glauber contributions and LO collinear contributions. However, the NLO Glauber contributions vanish exactly for general λ_1, $\lambda_2 = \lambda_3$ and $\lambda_4 = \lambda_5$, due to non-trivial cancellations between the pieces which contribute to the NLO expansion of (19) for either of the scalings $\{0,0,0,1,0\}$ and $\{0,1,0,0,0\}$. The same happens in loop-momentum space, where the NLO Glauber contribution is proportional to $(\lambda_3-\lambda_2)(\lambda_5-\lambda_4)$, thus vanishing in the case considered here. So the NLO contribution to $F(Q^2, m^2)$ is made up entirely from the LO collinear contributions. The 1-collinear region provides

$$- i\pi^{d/2} \frac{e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)} \, e^{i\pi\lambda_4} \, \Gamma(\lambda_2-\lambda_4)\Gamma(1-2\lambda_4)\Gamma(1+\varepsilon+\lambda_1+2\lambda_2)}{2(Q^2)^{2+2\lambda_4} \, (m^2)^{1+\varepsilon+\lambda_1+2\lambda_2} \, \Gamma(1+\lambda_1)\Gamma(1+\lambda_2)\Gamma(1-\lambda_4)} \times \frac{\Gamma(-\varepsilon-2\lambda_2)}{\Gamma(-\varepsilon-2\lambda_4)} \, \frac{1}{1+2\lambda_4} \,, \quad (21)$$

in agreement with the momentum-space expansion. The 2-collinear contribution is obtained from this by exchanging $\lambda_2 \leftrightarrow \lambda_4$. Adding the two collinear contributions together and performing the limit $\lambda_1, \lambda_2, \lambda_4 \to 0$, the result (13) is reproduced.

Considering finally the NNLO contributions to $F(Q^2, m^2)$, we expect NNLO Glauber contributions, NLO collinear contributions and a LO hard contribution. But here the NLO collinear contributions vanish exactly for $\lambda_3 = \lambda_2$ and $\lambda_5 = \lambda_4$, both in loop-momentum space and when expanding the parametric integrals. So we are left with the NNLO Glauber contributions yielding

$$- i\pi^{d/2} \frac{i \, e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)} \, \Gamma\left(\lambda_2-\frac{1}{2}\right) \Gamma\left(\lambda_4-\frac{1}{2}\right) \Gamma(\varepsilon+\lambda_1+2\lambda_2+2\lambda_4)}{16(Q^2)^3 \, (m^2)^{\varepsilon+\lambda_1+2\lambda_2+2\lambda_4} \, \Gamma(1+\lambda_1)\Gamma(1+\lambda_2)\Gamma(1+\lambda_4)\Gamma(-1-\varepsilon)} \times \Gamma(1-\varepsilon-2\lambda_2-2\lambda_4) \quad (22)$$

and the LO hard contribution,

$$i\pi^{d/2} \frac{i \, e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)} \, e^{i\pi(\varepsilon+\lambda_1+2\lambda_2+2\lambda_4)/2} \, \Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_2}{2}\right) \Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_4}{2}\right)}{2\sqrt{\pi} \, (2Q^2)^{3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4} \, \Gamma(1+\lambda_2)\Gamma(1+\lambda_4)\Gamma(-1-2\varepsilon-\lambda_1-2\lambda_2-2\lambda_4)} \times \Gamma\left(\frac{-2-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4}{2}\right) \Gamma\left(\frac{3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4}{2}\right) \,, \quad (23)$$

both consistent between the expansions of the loop integral and of the parametric integrals. Setting all $\lambda_i = 0$ in (22) and (23), we recover the results from (14) and (15), respectively.

We may also evaluate the parametric integral (19) in terms of a onefold Mellin–Barnes representation:

$$\begin{aligned} I_\pm = -i\pi^{d/2} & \frac{e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)}}{4\pi \, \Gamma(1+\lambda_1)\Gamma(1+\lambda_2)\Gamma(1+\lambda_4)} \\ & \times \frac{1}{2\pi i} \int \mathrm{d}z \, (m^2)^z (\pm 4Q^2 - i0)^{-3-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z} \\ & \times \Gamma(1+\lambda_1+z)\Gamma(3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4+z) \\ & \times \frac{\Gamma(-z)\Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_2-z}{2}\right) \Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_4-z}{2}\right) \Gamma^2\left(\frac{-2-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z}{2}\right)}{\Gamma(-1-2\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z)} \,. \end{aligned} \quad (24)$$

The relevant regions can easily be determined from the gamma functions in the numerator of the last line. In particular, the squared gamma function indicates that the expansion of the parametric integrals $I_\pm$ requires *two* regions for the Glauber contribution, both scaling as $(m^2)^{-2-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4}$ at leading order. When combining $F(Q^2, m^2) = 2(I_+ + I_-)$, one of these gamma functions is cancelled, and we obtain

$$
\begin{aligned}
F(Q^2, m^2) = i\pi^{d/2} & \frac{i\, e^{-i\pi(\lambda_1+2\lambda_2+2\lambda_4)}}{2\sqrt{\pi}\,\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)\Gamma(1+\lambda_4)} \\
& \times \frac{1}{2\pi i}\int \mathrm{d}z\,(m^2)^z (2Q^2)^{-3-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z}\, e^{i\pi(\varepsilon+\lambda_1+2\lambda_2+2\lambda_4+z)/2} \\
& \times \Gamma(1+\lambda_1+z)\Gamma\left(\tfrac{3+\varepsilon+\lambda_1+2\lambda_2+2\lambda_4+z}{2}\right) \\
& \times \frac{\Gamma(-z)\Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_2-z}{2}\right)\Gamma\left(\frac{-1-\varepsilon-\lambda_1-2\lambda_4-z}{2}\right)\Gamma\left(\frac{-2-\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z}{2}\right)}{\Gamma(-1-2\varepsilon-\lambda_1-2\lambda_2-2\lambda_4-z)}\,.
\end{aligned}
\tag{25}
$$

From this representation the contributions to the asymptotic expansion in the limit $m^2/Q^2 \to 0$ can be extracted: The hard contributions stem from the residues of the poles at $z = n$, the 1- and 2-collinear contributions from $z = -1 + 2n - \varepsilon - \lambda_1 - 2\lambda_2$ and $z = -1 + 2n - \varepsilon - \lambda_1 - 2\lambda_4$, respectively, the Glauber contributions from $z = -2 + 2n - \varepsilon - \lambda_1 - 2\lambda_2 - 2\lambda_4$ (with $n = 0, 1, 2, \ldots$). All LO, NLO and NNLO contributions reported in (20)–(23) are confirmed by the corresponding residue contributions from (25). In particular, the structure of the poles in (25) clearly shows that the Glauber region does not contribute to the NLO result, and that the collinear contributions are absent at NNLO, as obtained before.

The results reported in this section show that the use of generic propagator powers helps disentangling the Glauber and collinear regions from each other, making all regions contribute in the same way to the asymptotic expansion in loop-momentum space and to the expansion of the parametric integrals. Keeping the dependence of the contributions and their scalings on the propagator powers also facilitates the identification of regions found by `asy2.m` in parametric space for a subsequent expansion at the level of the loop integration.

5 Summary of `asy2.m`

The updated version of the code, `asy2.m`, can be downloaded from the known web site [16], where further installation instructions are found. The `Mathematica` code is loaded using `<<asy2.m`.

The main function `AlphaRepExpand[]` reveals regions for loop integrals:

```
AlphaRepExpand[{k1, k2, ...},
  {(k1 + p1)^2 - m1^2, (k2 + p2)^2 - m2^2, ...},
  {p1^2 -> Q1, p2^2 -> Q2, p1*p2 -> Q3, ...},
  {m1^2 -> x, m2^2 -> x^2, Q1 -> 1, Q2 -> 3/2, ...},
  options]
```

The first argument is the list of loop momenta. The second argument lists the denominators of the propagators. The third argument contains replacement rules for all kinematic invariants. In particular, all external momenta appearing in the denominators must be replaced here, otherwise they are not identified correctly as vectors. The fourth argument sets the scaling of the parameters by replacing all symbols with powers of the expansion parameter, labelled by the global symbol `x`, and rational numbers (integers or explicit fractions of integers).

The output is a list of regions, specified by the scaling (in powers of the expansion parameter) of the Feynman parameters $x_1, x_2, \ldots$ corresponding to the propagators in the order stated in the second argument. Possible options are:

- `PreResolve -> True`: Try to eliminate cancellations between terms in the parametric representation by decomposing the integral and performing changes of variables. The code warns if this fails. The output contains, for each region, the changes of variables, the Jacobian of the integral transformation and the scalings of the new variables. Without this option, regions occurring at such cancellations will not be revealed.

- `GenericPowers -> True`: Take into account generic (in particular non-integer) powers of the propagators, e.g. when these powers are used as analytic regulators. Without this option, the preresolution algorithm triggered by the option `PreResolve` only finds all regions if the propagator powers are positive integers.

- `Verbose -> True`: Print verbose internal information.

- `Scalar -> True`: Permit more complex structures of the denominators by specifying scalar products of momenta via the function `Scalar[k,p]` instead of simple products `k*p` or `Scalar2[k]` instead of `k^2`, e.g.:

```
AlphaRepExpand[{k},
  {Scalar[k, k] - m^2, Scalar2[k - q] - m^2},
  {Scalar2[q] -> qq, m^2 -> qq/4 + y}, {qq -> 1, y -> x},
  PreResolve -> True, Scalar -> True]
```

For expanding more general integrals, which need not originate from Feynman diagrams, the command `WilsonExpand[]` may be used:

```
WilsonExpand[F, U, {x1, x2, ...}, {... -> x, ...}, options]
```

Traditionally, the first two arguments are the polynomials $\mathcal{F}$ and $\mathcal{U}$, respectively, from the parametric representation of the Feynman integral. But, more generally, `WilsonExpand[]` will reveal regions for integrals over parameters $x_1, x_2, \ldots$, integrated from 0 to ∞ each, where all non-trivial polynomials of the parameters x_i occurring in the integrand are specified either in the first or the second argument. The third argument of `WilsonExpand[]` is the list of integration parameters, and the fourth argument specifies the scaling of all quantities with the expansion parameter `x`.

The output is a list of regions, specified by the scaling of the integration parameters in powers of the expansion parameters. Possible options are:

- `Delta -> True`: Under the integral, the sum over an arbitrary non-empty subset of the integration parameters is restricted to 1 via a delta function. The specific choice of the sum in this delta function must be irrelevant for the integral, which is the case for the generalized Feynman representations used in this paper.

- `Verbose -> True`: Print verbose internal information.

More generally, the option `Delta` of `WilsonExpand[]` works correctly for all integrals of the form

$$\int_0^\infty \cdots \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \delta\left(\sum_{i=1}^n a_i x_i - 1\right) f(x_1, \ldots, x_n) \; , \tag{26}$$

where the linear combination in the argument of the delta function has no negative and at least one positive coefficient ($a_i \geq 0 \, \forall i$ and $\exists \, a_i > 0$), and where the function f scales homogeneously with the parameters x_i as

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^{-n} f(x_1, \ldots, x_n) \; \forall \lambda > 0 \; , \tag{27}$$

the degree of homogeneity being equal to minus the number of integration parameters. It can be shown that for such integrals (26) the specific choice of the coefficients a_i is irrelevant.[2] All integrands of Feynman-parameter representations used in this paper (without the additional analytic regularization factors $x_i^{\delta_i}$) fulfill the homogeneity condition (27).

6 Conclusion

We have presented an algorithm for finding potential regions and Glauber regions when expanding loop integrals at the level of their parametric representations. The necessary decompositions and variable transformations of the integral are automated by the updated `Mathematica` code `asy2.m`. When the command `AlphaRepExpand[]`

[2] To see this, multiply the integrand of (26) by 1 in the form $x_j \int_0^\infty \mathrm{d}t \, e^{-t x_j}$, where x_j is any of the integration parameters. Then, inside the t-integration, transform the other integration variables as $x_i \to x_i/t \, \forall i$ and use the homogeneity relation (27) with $\lambda = 1/t$. Finally evaluate the t-integration first, yielding $\int_0^\infty \mathrm{d}t \, \delta(\sum_{i=1}^n a_i x_i - t) = 1$, independent of the coefficients a_i, as long as the linear combination is positive. The integral (26) is given by $\int_0^\infty \cdots \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n \, f(x_1, \ldots, x_n) \, x_j \, e^{-x_j}$.

is used with its option `PreResolve` enabled, we are convinced that it will either reveal all relevant regions or issue a warning. In particular, regions corresponding to cancellations between large positive and negative terms in the parametric representation of the loop integrals (such as potential and Glauber regions) will now be found.

Let us emphasize that to prove expansion by regions at least for some specific limit typical of Minkowski space is a natural mathematical problem. Perhaps, this problem is not specifically related to Feynman integrals. Let us present an example of a one-dimensional parametric integral, without any relevance to Feynman integrals, and show that expansion by regions works successfully. To do this, we will use `asy2.m`.

Let us consider the integral

$$F(t) = \int_0^\infty (t+u+u^2)^\lambda \mathrm{d}u \;, \tag{28}$$

with λ a complex parameter, in the limit $t \to 0$. We assume that λ is in the domain $\mathrm{Re}\lambda < -1/2$ in order to have an absolute convergence of the integral which then can be continued analytically to the whole complex plane as an analytic function of λ. Running

```
WilsonExpand[t + u + u^2, 1, {u}, {t -> x}]
```

we obtain the two regions `{{1}, {0}}`. The leading-order terms from each region can be evaluated analytically in terms of gamma functions at general λ, with the results

$$\frac{t^{\lambda+1}\Gamma(-\lambda-1)}{\Gamma(-\lambda)} \tag{29}$$

and

$$\frac{\Gamma(-2\lambda-1)\Gamma(\lambda+1)}{\Gamma(-\lambda)} \;. \tag{30}$$

They can be checked easily by deriving the onefold Mellin–Barnes representation

$$F(t) = \frac{1}{2\pi i}\frac{1}{\Gamma(-\lambda)}\int \Gamma(-z)\Gamma(\lambda-z+1)\Gamma(-2\lambda+2z-1)t^z \mathrm{d}z \tag{31}$$

and evaluating the first terms of the asymptotic expansion in the limit $t \to 0$ by shifting the contour to the right and taking residues at the poles of the two gamma functions in the integrand.

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