TTP12-012
SFB/CPP-12-23

Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model

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Abstract

We analytically compute the dominant contributions to the β-functions for the top-Yukawa coupling, the strong coupling and the Higgs self-coupling as well as the anomalous dimensions of the scalar, gluon and quark fields in the unbroken phase of the Standard Model at three-loop level. These are mainly the QCD and top-Yukawa corrections. The contributions from the Higgs self-interaction which are negligible for the running of the top-Yukawa and the strong coupling but important for the running of the Higgs self-coupling are also evaluated.

1 Introduction

Using perturbation theory in any renormalizable quantum field theory comes with the price that the parameters of this theory, e.g. the couplings and masses, in general depend on the renormalization scale μ. The precise description of the evolution of these parameters with the energy scale is an important task in any model. This is done by means of the Renormalization Group functions, that is the β-functions and anomalous dimensions. The knowledge of the three-loop contributions to the β-functions for the Standard Model (SM) and its extensions is important for physics at the very high energy frontier and for cosmology. Here are some examples: The running of the gauge couplings plays an important role for the construction of Grand Unified theories of the strong and electroweak interactions. The β-functions for the scalar self-interaction and for the top quark Yukawa coupling constant are important for the analysis of Higgs-inflation in the SM [1, 2, 3, 4]. The current investigations of these issues are based on the two-loop approximation. The inclusion of the next order could be essential. Further, in a recent work [5], the possibility has been discussed that the SM, supplemented by the asymptotically safe gravity could play the role of a fundamental, rather than effective field theory. Within this framework the mass of the Higgs boson has been predicted to be approximately 126 GeV. The theoretical uncertainty of the prediction is about 2 GeV.

Recent exciting evidence from several SM-like Higgs search channels at both the CERN Large Hadron Collider and the Fermilab Tevatron [6, 7, 8] point to the possibility of a SM Higgs boson with a mass in the vicinity of 125 GeV which is in truly remarkable agreement with the aforementioned prediction. This calls for more precise calculations, in particular of β-functions, in the SM.

In the present paper we are particularly interested in the evolution of the Higgs self-coupling

as well as the top-Yukawa coupling in the SM.

The underlying gauge group of the SM is an $\mathrm{SU}_C(3) \times \mathrm{SU}(2) \times \mathrm{U}_Y(1)$ which is spontaneously broken to $\mathrm{SU}_C(3) \times \mathrm{U}_Q(1)$ at the electroweak scale. As the renormalization constants for fields and vertices do not depend on masses and external momenta in the $\overline{\mathrm{MS}}$-scheme, we will perform our calculations in the unbroken phase of the SM.

The most important contributions to the running of the Higgs self-coupling λ arise from the top-Yukawa coupling and the strong sector. All other Yukawa couplings are significantly smaller due to to the smallness of the respective quark masses. From the top mass $M_t \approx 172.9$ GeV we get the Yukawa coupling at this scale $y_t(M_t) = \sqrt{2}\frac{M_t}{v} \approx 1$

where $v \approx 246.2$ GeV is proportional to the vacuum expectation value of the scalar field Φ from which results the Higgs field after the spontaneous symmetry breaking: $|\langle\Phi\rangle| = \frac{v}{\sqrt{2}}$.

The next Yukawa coupling to be considered would be $y_b = \sqrt{2}\frac{M_b}{v} \approx 0.02$. The strong coupling at the scale of the Z boson mass is $g_s(M_Z) \approx 1.22$ whereas the electroweak couplings $g_1(M_Z) = \frac{\sqrt{4\pi\alpha}}{\cos\theta_W} \approx 0.36$ and $g_2(M_Z) = \frac{\sqrt{4\pi\alpha}}{\sin\theta_W} \approx 0.65$ give much smaller contributions which are further suppressed by the isospin and hypercharge factors. For this reason we will consider a simplified version of the SM or - as one could also see it - a minimal extension of QCD by setting $g_1 = g_2 = 0$ in our calculation. For a Higgs mass of 125 GeV the value of the Higgs self-interaction would be $\lambda(M_H) =\approx 0.13$ at the scale of the Higgs mass. The relevance of this parameter will be examined in section 2.

The outline of the work is as follows. In the next section we discuss the main definitions and the general setup of our work. Section 3 deals with the technical details, including the treatment of γ_5. In sections 4 and 5 we present our results for the β-functions of the top-Yukawa, the strong and the Higgs self-couplings and the relevant field anomalous dimensions. The numerical influence of the computed three-loop corrections on the evolution of the quartic Higgs coupling is discussed in section 6. For this analysis we will include the already known contributions with g_1 and g_2 at one-loop and two-loop level. Finally, section 7 contains our conclusions and acknowledgements.

All our results for β-functions and anomalous dimensions can be retrieved from
`http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp12/ttp12-012/`

2 General Setup

The Lagrangian of our model consists of three pieces:

$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{y_t} + \mathcal{L}_{\Phi}. \tag{1}$$

The QCD part is defined by

$$\begin{aligned}\mathcal{L}_{QCD} = &-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{2(1-\xi)}\left(\partial_\mu A^{a\,\mu}\right)^2 + \partial_\mu\bar{c}^a\partial^\mu c^a + g_s f^{abc}\,\partial_\mu\bar{c}^a A^{b\,\mu}c^c \\ &+ \sum_q \left\{\frac{i}{2}\bar{q}\overleftrightarrow{\not\partial}q + g_s\bar{q}\not{A}^a T^a q\right\}\end{aligned}, \tag{2}$$

where q runs over all quark flavours, the gluon field strength tensor is given by

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{3}$$

and f^{abc} are the structure constants of the colour gauge group with the generators T^a:

$$\left[T^a, T^b\right] = i f^{abc} T^c. \tag{4}$$

The complex scalar field Φ and the left-handed parts of the top and bottom quarks t_L and b_L are doublets under SU(2):

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \qquad Q_L = \begin{pmatrix} t \\ b \end{pmatrix}_L. \tag{5}$$

Setting all Yukawa couplings to zero except for the top coupling y_t the Lagrangian for the Yukawa sector is given by

$$\begin{aligned} \mathcal{L}_{y_t} &= -y_t \left\{ \bar{t}_R \Phi^{\dagger\, c} Q_L + \bar{Q}_L \Phi^c t_R \right\} \\ &= -y_t \left\{ \bar{t}_R \left(\Phi_2, -\Phi_1\right) \cdot \begin{pmatrix} t \\ b \end{pmatrix}_L + \left(\bar{t}, \bar{b}\right)_L \cdot \begin{pmatrix} \Phi_2^* \\ -\Phi_1^* \end{pmatrix} t_R \right\} \\ &= -y_t \left\{ \left(\bar{t} P_R t\right) \Phi_2^* + \left(\bar{t} P_L t\right) \Phi_2 - \left(\bar{b} P_R t\right) \Phi_1^* - \left(\bar{t} P_L b\right) \Phi_1 \right\}. \end{aligned} \tag{6}$$

Finally, we have the scalar sector of the model

$$\mathcal{L}_\Phi = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2. \tag{7}$$

The indices L and R indicate the left- and right-handed part of the fields as obtained by the projectors

$$P_L = \frac{1}{2}\left(1 - \gamma_5\right) \qquad P_R = \frac{1}{2}\left(1 + \gamma_5\right). \tag{8}$$

This model is renormalized with the counterterm Lagrangians

$$\begin{aligned} \delta\mathcal{L}_{QCD} = &-\frac{1}{4}\delta Z_3^{(2g)} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2 - \frac{1}{2}\delta Z_1^{(3g)} g_s f^{abc} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A_\mu^b A_\nu^c \\ &- \frac{1}{4}\delta Z_1^{(4g)} g_s^2 \left(f^{abc} A_\mu^b A_\nu^c\right)^2 + \delta Z_3^{(2c)} \partial_\mu \bar{c}^a \partial^\mu c^a + \delta Z_1^{(ccg)} g_s f^{abc}\, \partial_\mu \bar{c}^a A^{b\,\mu} c^c \\ &+ \sum_q \left\{ \frac{i}{2} \bar{q} \overleftrightarrow{\not\partial} \left[\delta Z_{2,L}^{(2q)} P_L + \delta Z_{2,R}^{(2q)} P_R\right] q + g_s \bar{q} \not{A}^a T^a \left[\delta Z_{1,L}^{(qqg)} P_L + \delta Z_{1,R}^{(qqg)} P_R\right] q \right\} \end{aligned} \tag{9}$$

for the QCD part,

$$\delta\mathcal{L}_{Yukawa} = -\delta Z_1^{(tb\Phi)} y_t \left\{ \left(\bar{t} P_R t\right) \Phi_2^* + \left(\bar{t} P_L t\right) \Phi_2 - \left(\bar{b} P_R t\right) \Phi_1^* - \left(\bar{t} P_L b\right) \Phi_1 \right\} \tag{10}$$

for the Yukawa part and

$$\delta\mathcal{L}_\Phi = \delta Z_2^{(2\Phi)} \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2\, \delta Z_{\Phi^2} \Phi^\dagger \Phi + \delta Z_1^{(4\Phi)} \left(\Phi^\dagger \Phi\right)^2 \tag{11}$$

for the Φ-sector. Note that in general the left- and right-handed parts of quark fields and quark-vertices are renormalized differently.[1] So the renormalization constant for the strong gauge coupling g_s can be obtained for example from

$$Z_{g_s} = \frac{Z_{1,L}^{(ttg)}}{Z_{2,L}^{(2t)} \sqrt{Z_3^{(2g)}}} = \frac{Z_{1,R}^{(ttg)}}{Z_{2,R}^{(2t)} \sqrt{Z_3^{(2g)}}} \tag{12}$$

[1] In our case this is true for the quark fields participating in the Yukawa sector. As we do not consider the electroweak interaction here and neglect all Yukawa couplings except for y_t the light quark fields u,d,s and c have the same renormalization constant for the left- and right-handed part.

or the renormalization constant for y_t from

$$Z_{y_t} = \frac{Z_1^{(tb\Phi)}}{\sqrt{Z_{2,L}^{(2t)} Z_{2,R}^{(2t)} Z_2^{(2\Phi)}}}. \tag{13}$$

Here the renormalization constants have been defined in a minimal way as

$$Z = 1 + \delta Z, \tag{14}$$

with δZ containing only poles in the regulating parameter $\varepsilon = (4 - D)/2$ of the dimensional regularization and D being the engineering space-time dimension. The Higgs self-coupling λ is related to the Higgs mass at tree level M_H via

$$\lambda = \frac{M_H^2}{2v^2}, \tag{15}$$

which for $M_H = 125$ GeV yields $\lambda(M_H = 125\text{GeV}) \approx 0.13$. For the running of the top-Yukawa coupling the contribution from λ is negligible compared to the top-Yukawa and strong coupling. The corresponding four-Φ vertex is nevertheless needed for the renormalization at three-loop level, namely to kill the subdivergence from the fermion loop in diagrams like Fig. 1 (d). The β-function for a coupling X is defined as

$$\beta_X = \mu^2 \frac{dX}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \beta_X^{(n)} \tag{16}$$

and is given as a power series in all couplings of the model, namely g_s, y_t and λ. Note that the β-functions β_{g_s} and β_{y_t} are proportional to g_s and y_t respectively whereas β_λ has one part proportional to λ and one part proportional to y_t^4 with no λ-dependence at all. The anomalous dimension of a field f is defined as

$$\gamma_2^f = -\mu^2 \frac{d\ln Z_f^{-1}}{d\mu^2} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \gamma_2^{f\,(n)}, \tag{17}$$

where Z_f is the field strength renormalization constant for the respective field.[2] The β-functions for all couplings are independent of the gauge parameter ξ whereas the anomalous dimensions of the fields are not.

3 Calculation

As we are interested in β-functions it is enough to compute the UV divergent part of all diagrams in order to determine the necessary renormalization constants. In the $\overline{\text{MS}}$-scheme the latter depends only polynomially on external momenta and masses. Therefore most of our renormalization constants could be computed from massless propagator-like diagrams using the FORM 3 [9] version[3] of the package MINCER [10]. In some diagrams, e.g. with four

[2] For an n-point vertex V or a mass m the anomalous dimension is defined as $\gamma_n^V = -\mu^2 \frac{d\ln Z_V}{d\mu^2}$ or $\gamma_m = -\mu^2 \frac{d\ln Z_m}{d\mu^2}$. For a field we take the inverse renormalization constant.

[3] The program can be downloaded from
http://www.nikhef.nl/~ form/maindir/packages/mincer/mincer.html

external Φ-fields where two external momenta are set to zero, this leads to IR divergences which mix with the UV ones in dimensional regularization. Another convenient method to compute renormalization constants has been suggested in [11] and elaborated in the context of three-loop calculations in [12]. The idea is an exact decomposition of all propagators using an auxiliary mass parameter M^2:

$$\frac{1}{(q+p)^2} = \frac{1}{q^2-M^2} + \frac{-p^2-2q\cdot p-M^2}{q^2-M^2}\frac{1}{(q+p)^2}, \tag{18}$$

where p is a combination of external and q of internal momenta. This can be done recursively until the power in the denominator of the last term is high enough for this contribution to be finite, e.g.

$$\begin{aligned}\frac{1}{(q+p)^2} =& \frac{1}{q^2-M^2} + \frac{-p^2-2q\cdot p}{(q^2-M^2)^2} + \frac{(-p^2-2q\cdot p)^2}{(q^2-M^2)^3} - \frac{M^2}{(q^2-M^2)^2} \\ &+ \frac{M^2(M^2+2p^2+4q\cdot p)}{(q^2-M^2)^3} + \frac{(-p^2-2q\cdot p-M^2)^3}{(q^2-M^2)^3}\frac{1}{(q+p)^2}.\end{aligned} \tag{19}$$

As the result is independent of M^2 we can omit the contributions $\sim M^2$ in the above decomposition as long as we introduce counterterms into the Lagrangian to cancel M^2-dependent subdivergences. In our case only a term $\frac{M^2}{2}\delta Z^{(2g)}_{M^2}\, A^a_\mu A^{a\,\mu}$ and $\frac{M^2}{2}\delta Z^{(2\Phi)}_{M^2}\, \Phi^\dagger\Phi$ are possible.[4] The first one is not gauge invariant but this does not matter as it is only used for the cancellation of subdivergences which works nevertheless. This method effectively amounts to introducing the same auxiliary mass parameter M^2 in every denominator of propagators and all possible M^2-counterterms. We expand in the external momenta[5] and arrive at massive tadpole diagrams with one scale M. Due to the auxiliary mass no IR divergences can appear while the UV counterterms which we are interested in (that is the ones without any dependence on the auxiliary mass M) will stay untouched.

For the calculation of massive tadpoles we have used the FORM-based program MATAD [13]. Where possible, i.e. for the propagators and three point functions, we have employed both the MINCER and the MATAD setups which served as an extra check. To generate the diagrams we used QGRAF [14] and to compute the colour factors the FORM package COLOR [15].

An important aspect of calculations such as these is the proper treatment of γ_5. As is well known, a naive treatment of γ_5 can be applied if it only appears in an external fermion line. In fermion loops we have to be more careful. In four dimensions we define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \text{ with } \varepsilon_{0123} = 1 = -\varepsilon^{0123}. \tag{20}$$

In Fig. 1 to 5 we show a few diagrams that had to be calculated for the various ingredients of our final result. In order to have a contribution from a fermion loop with one γ_5 in it at least four free Lorentz indices or momenta on the external lines of the minimal subgraph containing this fermion loop are required. These can be indices from the gluon vertices or the internal momenta from other loops which act as external momenta to the minimal subgraph containing the fermion loop in question. External momenta of the whole diagram can be set to zero as the renormalization constants in the $\overline{\text{MS}}$-scheme do not depend on those.

[4] Counterterms $\sim M$ that would arise for fermions cannot appear because we have no M in the numerators of propagators. The ghost mass term $\frac{M^2}{2}\delta Z^{(2c)}_{M^2}\, \bar{c}^a c^a$ does not appear because of the momentum dependence of the ghost-gluon-vertex.

[5] This method also works in massive theories. In this case we expand in the physical masses as well.

Consider for example one of the fermion loops in Fig.2 (c). The momenta on the two external Φ-legs can be set to zero. Then we have two indices from the gluon lines attached to our fermion loop and one loop momentum going through the two gluons and acting as an external momentum to the subgraph containing only our fermion loop. This is not enough to have a non-naive γ_5 contribution from this graph.

For this reason diagrams like Fig.1 (a,b,c,d), Fig.2 (a,b,c), Fig.3 (a,b), Fig.4 (a,b,c), Fig.5 (a,b) can be treated naively. Fig.3 (c) has enough indices and momenta but no γ_5 in it. Diagrams like Fig.1 (c,e), Fig.2 (b), Fig.3 (b), Fig.4 (a,c), Fig.5 (a,c) are zero because of an odd number of γ-matrices in at least one fermion loop. And diagrams like Fig.1 (b), Fig.2 (b), Fig.3 (a), Fig.4 (b), Fig.5 (b) are zero because of their colour structure. The only problematic type is Fig.1 (f) which fortunately only contributes a $\frac{1}{\varepsilon}$ pole and can therefore be treated as described in [16]. We use the fact that γ_5 anticommutes with every other γ-matrix in four dimensions and that $\gamma_5^2 = 1$. Then we apply relation (20) for the case when one γ_5 remains on each fermion line. The two $\varepsilon_{\mu\nu\rho\sigma}$ can be rewritten as a combination of metric tensors which can be handled in dimensional regularization. The error we make with this treatment is of order ε and does therefore not affect the pole part of our result.

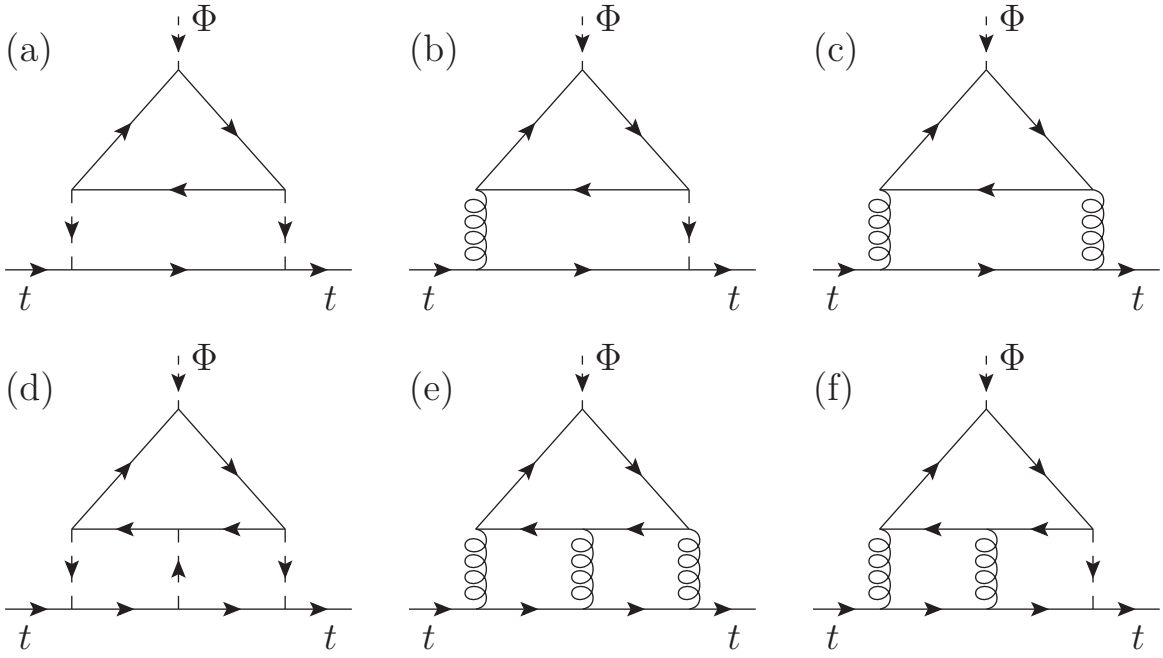

Figure 1: Some diagrams contributing to $Z_1^{(tb\Phi)}$

4 Results for the β-functions

First we give the results for the three-loop β-functions of couplings λ, y_t and g_s with the general gauge group factors for the strong interacting sector. Below C_F and C_A are the quadratic Casimir operators of the quark and the adjoint representation of the corresponding Lie algebra, d_R is the dimension of the quark representation, T_F is defined so that $T_F\delta^{ab} = \mathbf{Tr}\left(T^aT^b\right)$

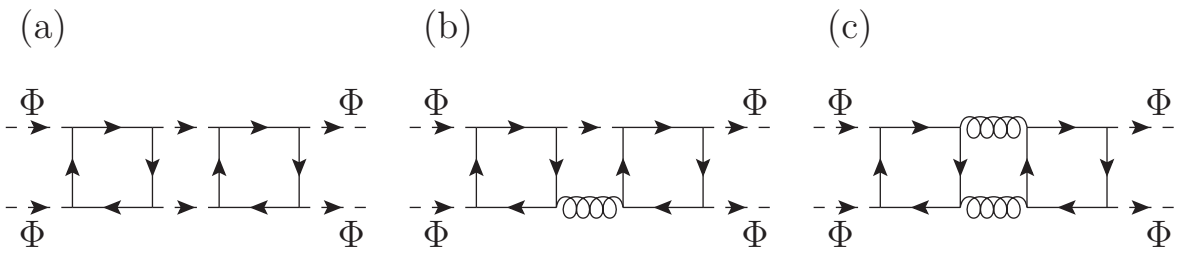

Figure 2: Some diagrams contributing to $Z_1^{(4\Phi)}$

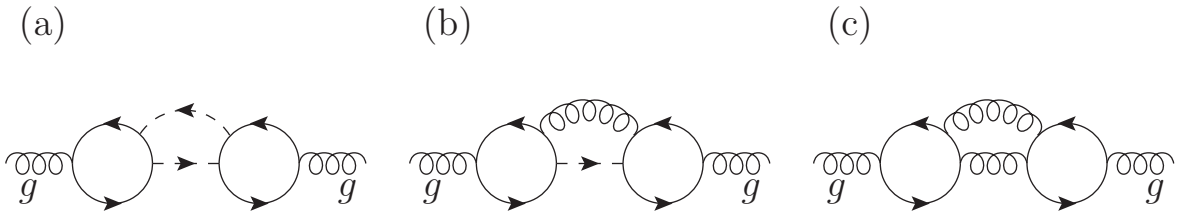

Figure 3: Some diagrams contributing to $Z_3^{(2g)}$

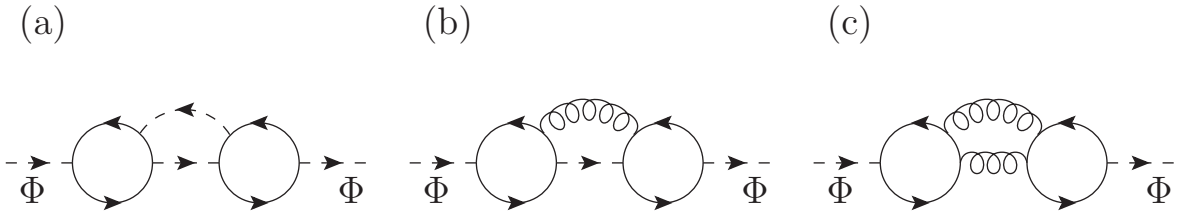

Figure 4: Some diagrams contributing to $Z_2^{(2\Phi)}$

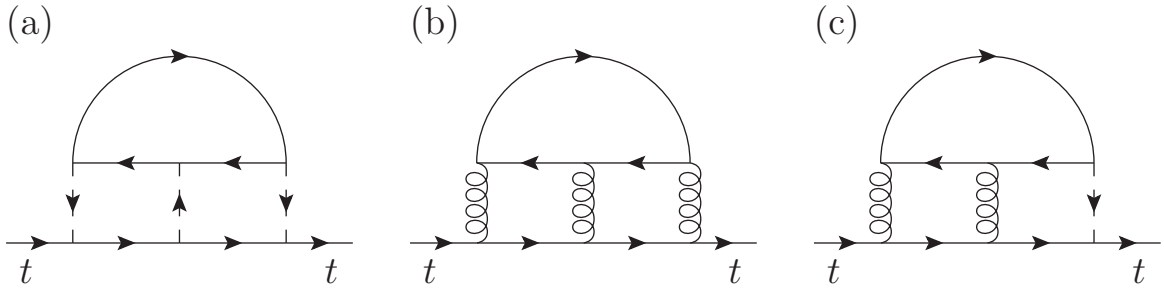

Figure 5: Some diagrams contributing to $Z_{2,L/R}^{(2t)}$

is the trace of two group generators of the quark representation.[6] For QCD (colour gauge group SU(3)) we have $C_F = 4/3$, $C_A = 3$, $T_F = 1/2$ and $d_R = 3$. Furthermore we denote the

[6]For an SU(N) gauge group these are $d_R = N$, $C_A = 2T_F N$ and $C_F = T_F\left(N - \frac{1}{N}\right)$.

number of fermions by $n_f = n_l + 1$.

$$
\begin{aligned}
\beta_\lambda^{(1)} =& 12\,\lambda^2 + 2d_R\,y_t^2\lambda - d_R\,y_t^4,\\
\beta_\lambda^{(2)} =& -156\,\lambda^3 - 24d_R\,y_t^2\lambda^2 - \frac{1}{2}d_R\,y_t^4\lambda + 5d_R y_t^6\\
&+ 10C_F d_R\,g_s^2 y_t^2\lambda - 4C_F d_R\,g_s^2 y_t^4,\\
\beta_\lambda^{(3)} =& \lambda^4\,(3588 + 2016\zeta_3) + 291 d_R\,y_t^2\lambda^3 + y_t^4\lambda^2\left(\frac{789}{2}d_R + 252\zeta_3 d_R - 36 d_R^2\right)\\
&+ y_t^6\lambda\left(-\frac{1881}{8}d_R - 66\zeta_3 d_R + 80 d_R^2\right) + y_t^8\left(\frac{13}{2}d_R - 12\zeta_3 d_R - \frac{195}{8}d_R^2\right)\\
&+ g_s^2 y_t^2\lambda^2\,(-306 C_F d_R + 288\zeta_3 C_F d_R) + g_s^2 y_t^4\lambda\left(\frac{895}{4}C_F d_R - 324\zeta_3 C_F d_R\right)\\
&+ g_s^2 y_t^6\left(-\frac{19}{2}C_F d_R + 60\zeta_3 C_F d_R\right) + g_s^4 y_t^2\lambda\left(-\frac{119}{2}C_F^2 d_R + 77 C_A C_F d_R\right.\\
&\left.-16 n_f T_F C_F d_R + 72\zeta_3 C_F^2 d_R - 36\zeta_3 C_A C_F d_R\right) + g_s^4 y_t^4\left(\frac{131}{2}C_F^2 d_R\right.\\
&\left.+48 T_F C_F d_R - \frac{109}{2}C_A C_F d_R + 10 n_f T_F C_F d_R - 48\zeta_3 C_F^2 d_R + 24\zeta_3 C_A C_F d_R\right)
\end{aligned}
\qquad . \tag{21}
$$

The purely λ-dependent parts of eq. (21) have been known for a while [17, 18], the full one-loop and two-loop result are in agreement with [19, 20] (for an SU(3) colour gauge group). It has also been a useful check for our setup to see that the same result can be derived from the four-Φ_1 vertex, the four-Φ_2 vertex and the $(\Phi_1^*\Phi_2^*\Phi_1\Phi_2)$ vertex.

$$
\begin{aligned}
\frac{\beta_{y_t}^{(1)}}{y_t} =& y_t^2\left(\frac{3}{4} + \frac{1}{2}d_R\right) - 3C_F\,g_s^2,\\
\frac{\beta_{y_t}^{(2)}}{y_t} =& 3\,\lambda^2 - 6\,y_t^2\lambda + y_t^4\left(\frac{3}{4} - \frac{9}{4}d_R\right)\\
&+ g_s^2 y_t^2\left(6C_F + \frac{5}{2}C_F d_R\right) + g_s^4\left(-\frac{3}{2}C_F^2 - \frac{97}{6}C_A C_F + \frac{10}{3}n_f T_F C_F\right),\\
\frac{\beta_{y_t}^{(3)}}{y_t} =& -18\,\lambda^3 + y_t^2\lambda^2\left(\frac{285}{8} - \frac{45}{4}d_R\right) + y_t^4\lambda\left(\frac{63}{2} + \frac{45}{2}d_R\right)\\
&+ y_t^6\left(-\frac{345}{32} + \frac{9}{4}\zeta_3 + \frac{107}{32}d_R + \frac{3}{2}\zeta_3 d_R + \frac{39}{16}d_R^2\right)\\
&+ 6C_F\,g_s^2 y_t^2\lambda - g_s^2 y_t^4\left(\frac{57}{2}C_F + \frac{81}{8}C_F d_R\right)\\
&+ g_s^4 y_t^2\left(\frac{471}{16}C_F^2 - \frac{119}{8}C_F^2 d_R + 25 T_F C_F + \frac{717}{16}C_A C_F + \frac{77}{4}C_A C_F d_R\right.\\
&-\frac{33}{4}n_f T_F C_F - 4 n_f T_F C_F d_R - 27\zeta_3 C_F^2 + 18\zeta_3 C_F^2 d_R - \frac{27}{2}\zeta_3 C_A C_F\\
&\left.-9\zeta_3 C_A C_F d_R) + g_s^6\left(-\frac{129}{2}C_F^3 + \frac{129}{4}C_A C_F^2 - \frac{11413}{108}C_A^2 C_F + 46 n_f T_F C_F^2\right.\right.\\
&\left.\left.+\frac{556}{27}n_f C_A T_F C_F + \frac{140}{27}n_f^2 T_F^2 C_F - 48\zeta_3 n_f T_F C_F^2 + 48\zeta_3 n_f C_A T_F C_F\right).
\end{aligned}
\tag{22}
$$

The one-loop and two-loop part of this result have been found before in [19, 21] (for $d_R = 3$, $T_F = \frac{1}{2}$) and the contributions of order g_s^2, g_s^4, $y_t^2 g_s^2$ and $g_s^4 y_t$ to Z_{y_t} have been successfully checked against [22]. In this reference the calculation of Z_{M_t} has been performed in the

broken phase of the SM with a massive top quark. When comparing these two results one has to take into account that in the broken SM the top quark mass is to be renormalized as a product $y_t\,(\Phi_2 + \Phi_2^\dagger)$ so the corresponding top quark mass renormalization constant is

$$Z_{M_t} = Z_{y_t}\, Z_2^{(2\Phi)} = \frac{Z_1^{(tb\Phi)}}{\sqrt{Z_{2,L}^{(2t)}\, Z_{2,R}^{(2t)}}}.$$

Again the setup could be checked for consistency by using the renormalization of the four different vertices t-t-Φ_2, t-t-Φ_2^*, t-b-Φ_1, t-b-Φ_1^* for the calculation.

$$\begin{aligned}
\frac{\beta_{g_s}^{(1)}}{g_s} =& g_s^2 \left(-\frac{11}{6} C_A + \frac{2}{3} n_f T_F\right), \\
\frac{\beta_{g_s}^{(2)}}{g_s} =& -2 T_F\, g_s^2 y_t^2 + g_s^4 \left(-\frac{17}{3} C_A^2 + 2 n_f T_F C_F + \frac{10}{3} n_f C_A T_F\right), \\
\frac{\beta_{g_s}^{(3)}}{g_s} =& + g_s^2 y_t^4 \left(\frac{9}{2} T_F + \frac{7}{2} T_F d_R\right) - g_s^4 y_t^2 \left(3 T_F C_F + 12 C_A T_F\right) \\
& + g_s^6 \left(-\frac{2857}{108} C_A^3 - n_f T_F C_F^2 + \frac{205}{18} n_f C_A T_F C_F + \frac{1415}{54} n_f C_A^2 T_F \right. \\
& \left. - \frac{22}{9} n_f^2 T_F^2 C_F - \frac{79}{27} n_f^2 C_A T_F^2\right).
\end{aligned} \tag{23}$$

In the case of $y_t = 0$ this is in agreement with the well-known result [23, 24]. The one-loop and two-loop parts of eq. (23) are known from [25, 26, 27, 28, 29, 30], the term $\propto g_s^4 y_t^2$ can be found in [22] and the full three-loop result has been computed in [31] (for $d_R = 3$, $T_F = \frac{1}{2}$). The β-function describing the running of the "mass" parameter m^2 in eq. (7) can be computed from the renormalization constant of the local operator $O_{2\Phi} := \Phi^\dagger \Phi$. An insertion of $O_{2\Phi}$ into a Green's function, e.g. with two external Φ-fields, is renormalized as $[O_{2\Phi}] = Z_{\Phi^2} O_{2\Phi}$ where $[O_{2\Phi}]$ is the corresponding finite operator. From $[O_{2\Phi}] = Z_{m^2} O_{2\Phi}^{\text{bare}}$ and $O_{2\Phi}^{\text{bare}} = Z_2^{(2\Phi)} O_{2\Phi}$ it follows that

$$Z_{m^2} = \left(Z_2^{(2\Phi)}\right)^{-1} Z_{\Phi^2}. \tag{24}$$

This yields the following contributions to β_{m^2}:

$$\begin{aligned}
\frac{\beta_{m^2}^{(1)}}{m^2} =& 6\,\lambda + d_R\, y_t^2, \\
\frac{\beta_{m^2}^{(2)}}{m^2} =& -30\,\lambda^2 - 12 d_R\, y_t^2 \lambda - \frac{9}{4} d_R\, y_t^4 + 5 C_F d_R\, g_s^2 y_t^2, \\
\frac{\beta_{m^2}^{(3)}}{m^2} =& 1026\,\lambda^3 + \frac{99}{2} d_R\, y_t^2 \lambda^2 + y_t^4 \lambda \left(\frac{333}{4} d_R - 18 d_R^2 + 72 \zeta_3 d_R\right) \\
& + y_t^6 \left(-\frac{617}{16} d_R + 24 d_R^2 + 15 \zeta_3 d_R\right) \\
& + g_s^2 y_t^2 \lambda \left(-153 C_F d_R + 144 \zeta_3 C_F d_R\right) \\
& + g_s^2 y_t^4 \left(\frac{447}{8} C_F d_R - 90 \zeta_3 C_F d_R\right) \\
& + g_s^4 y_t^2 \left(-\frac{119}{4} C_F^2 d_R + \frac{77}{2} C_A C_F d_R - 8 n_f T_F C_F d_R + 36 \zeta_3 C_F^2 d_R - 18 \zeta_3 C_A C_F d_R\right)
\end{aligned} \tag{25}$$

.

The one-loop and two-loop parts of this result are in agreement with [19] where they have been computed before. The purely λ-dependent part can be found in [17, 18]. For $d_R = 3$ and $T_F = \frac{1}{2}$ (QCD) we get the following results:

$$\begin{aligned}
\beta_\lambda^{(1)} =& 12\,\lambda^2 + 6\,y_t^2\lambda - 3\,y_t^4, \\
\beta_\lambda^{(2)} =& -156\,\lambda^3 - 72\,y_t^2\lambda^2 - \frac{3}{2}\,y_t^4\lambda + 15\,y_t^6 + 40\,g_s^2 y_t^2\lambda - 16\,g_s^2 y_t^4, \\
\beta_\lambda^{(3)} =& \lambda^4\,(3588 + 2016\zeta_3) + 873\,y_t^2\lambda^3 + y_t^4\lambda^2\left(\frac{1719}{2} + 756\zeta_3\right) \\
&+ y_t^6\lambda\left(\frac{117}{8} - 198\zeta_3\right) - y_t^8\left(\frac{1599}{8} + 36\zeta_3\right) + g_s^2 y_t^2\lambda^2\,(-1224 + 1152\zeta_3) \\
&+ g_s^2 y_t^4\lambda\,(895 - 1296\zeta_3) + g_s^2 y_t^6\,(-38 + 240\zeta_3) \\
&+ g_s^4 y_t^2\lambda\left(\frac{1820}{3} - 32n_f - 48\zeta_3\right) + g_s^4 y_t^4\left(-\frac{626}{3} + 20n_f + 32\zeta_3\right).
\end{aligned} \tag{26}$$

To get an idea of the size of these contributions and therefore the significance of our calculation we evaluate β_λ at the scale $\mu = M_Z$ (with an assumed Higgs mass of 125 GeV and $n_f = 6$) which yields a value of $\beta_\lambda \sim (-0.01)$ at one-loop level. The two and three-loop contributions change this result by $\sim 1\%$ and $\sim (-0.04)\%$ respectively. To estimate the importance of the individual terms we introduce the labels

$$G = \frac{g_s}{g_s(\mu = M_Z)}, \quad Y = \frac{y_t}{y_t(\mu = M_Z)}, \quad L = \frac{\lambda}{\lambda(\mu = M_Z)} \tag{27}$$

and get

$$\begin{aligned}
\beta_\lambda|_{\mu=M_Z} =& \left(\underbrace{-1.7\,Y^4}_{1\text{ loop}}\right)10^{-2} + \left(\underbrace{5.0\,LY^2}_{1\text{ loop}}\underbrace{+1.5\,L^2}_{1\text{ loop}}\right)10^{-3} \\
&+ \left(\underbrace{-8.5\,G^2Y^4}_{2\text{ loop}}\underbrace{+5.0\,Y^6}_{2\text{ loop}}\underbrace{+3.1\,G^2LY^2}_{2\text{ loop}}\right)10^{-4} \\
&+ \left(\underbrace{7.9\,G^2Y^6}_{3\text{ loop}}\underbrace{-4.8\,Y^8}_{3\text{ loop}}\underbrace{-5.3\,L^2Y^2}_{2\text{ loop}}\underbrace{-3.1\,G^2LY^4}_{3\text{ loop}}\right. \\
&\left.\underbrace{-2.5\,G^4Y^4}_{3\text{ loop}}\underbrace{+2.6\,G^4LY^2}_{3\text{ loop}}\underbrace{-1.7\,L^3}_{2\text{ loop}}\right)10^{-5} \\
&+ \left(\underbrace{-7.5\,LY^4}_{2\text{ loop}}\underbrace{+7.8\,L^2Y^4}_{3\text{ loop}}\underbrace{-6.6\,LY^6}_{3\text{ loop}}\underbrace{+1.1\,G^2L^2Y^2}_{3\text{ loop}}\right)10^{-6} \\
&+ \left(\underbrace{5.7\,L^3Y^2}_{3\text{ loop}}\underbrace{+5.9\,L^4}_{3\text{ loop}}\right)10^{-7}.
\end{aligned} \tag{28}$$

We see that the decrease of the effective four-Φ coupling with increasing energy is induced by top quark loops. Without quarks there would be an increase. It is also worth noting that the individual contributions at three-loop level are much larger than the overall effect due to huge cancellations. Consider for example the five numerically largest three-loop terms at $\mu = M_Z$:

$$\left(7.9\,G^2Y^6 - 4.8\,Y^8 - 3.1\,G^2LY^4 - 2.5\,G^4Y^4 + 2.6\,G^4LY^2\right)10^{-5}.$$

The total contribution from these terms is by almost two orders of magnitude smaller than the size of the largest one.
For the top-Yukawa β-functions we find

$$\begin{aligned}
\frac{\beta_{y_t}^{(1)}}{y_t} =& \frac{9}{4}\,y_t^2 - 4\,g_s^2,\\
\frac{\beta_{y_t}^{(2)}}{y_t} =& 3\,\lambda^2 - 6\,y_t^2\lambda - 6\,y_t^4 + 18\,g_s^2 y_t^2 + g_s^4\left(-\frac{202}{3} + \frac{20}{9}n_f\right),\\
\frac{\beta_{y_t}^{(3)}}{y_t} =& -18\,\lambda^3 + \frac{15}{8}\,y_t^2\lambda^2 + 99\,y_t^4\lambda + y_t^6\left(\frac{339}{16} + \frac{27}{4}\zeta_3\right)\\
&+ 8\,g_s^2 y_t^2\lambda - \frac{157}{2}\,g_s^2 y_t^4 + g_s^4 y_t^2\left(\frac{4799}{12} - \frac{27}{2}n_f - 114\zeta_3\right)\\
&+ g_s^6\left(-1249 + \frac{2216}{27}n_f + \frac{140}{81}n_f^2 + \frac{160}{3}\zeta_3 n_f\right).
\end{aligned} \tag{29}$$

As has already been mentioned above the λ-corrections are negligible here. Evaluating β_{y_t} at the scale $\mu = M_Z$ (with an assumed Higgs mass of 125 GeV and $n_f = 6$) we get a value of $\sim (-0.023)$ at one-loop level which means a decrease of y_t and therefore the top mass with increasing energy. This is due to the QCD corrections. In the absence of QCD the opposite would be the case as we can see from the term $\propto y_t^2$ in $\frac{\beta_{y_t}}{y_t}$. The two and three-loop corrections are $\sim 16.6\%$ and $\sim 0.7\%$ with respect to the one-loop result and so quite high compared e.g. to the case of β_{g_s} discussed below. We use again the labels (27) to get an impression of the individual terms:

$$\begin{aligned}
\beta_{y_t}|_{\mu=M_Z} =& \left(\underbrace{-3.7\,YG^2}_{1\text{ loop}}\underbrace{+1.3\,Y^3}_{1\text{ loop}}\right)10^{-2} + \left(\underbrace{-4.7\,YG^4}_{2\text{ loop}}\right)10^{-3}\\
&+ \left(\underbrace{9.8\,Y^3G^2}_{2\text{ loop}}\underbrace{-2.5\,YG^6}_{3\text{ loop}}\underbrace{-2.1\,Y^5}_{2\text{ loop}}\right)10^{-4}\\
&+ \left(\underbrace{9.3\,Y^3G^4}_{3\text{ loop}}\underbrace{-3.1\,LY^3}_{2\text{ loop}}\underbrace{-2.5\,Y^5G^2}_{3\text{ loop}}\right)10^{-5}\\
&+ \left(\underbrace{6.0\,Y^7}_{3\text{ loop}}\underbrace{+3.0\,LY^5}_{3\text{ loop}}\underbrace{+2.3\,L^2Y}_{2\text{ loop}}\right)10^{-6}\\
&+ \left(\underbrace{3.9\,LY^3G^2}_{3\text{ loop}}\right)10^{-7} + \left(\underbrace{-1.2\,L^3Y}_{3\text{ loop}}\right)10^{-8} + \left(\underbrace{8.5\,L^2Y^3}_{3\text{ loop}}\right)10^{-9}.
\end{aligned} \tag{30}$$

For the strong coupling we get

$$\begin{aligned}
\frac{\beta_{g_s}^{(1)}}{g_s} =& g_s^2\left(-\frac{11}{2} + \frac{1}{3}n_f\right),\\
\frac{\beta_{g_s}^{(2)}}{g_s} =& -g_s^2 y_t^2 + g_s^4\left(-51 + \frac{19}{3}n_f\right),\\
\frac{\beta_{g_s}^{(3)}}{g_s} =& \frac{15}{2}\,g_s^2 y_t^4 - 20\,g_s^4 y_t^2 + g_s^6\left(-\frac{2857}{4} + \frac{5033}{36}n_f - \frac{325}{108}n_f^2\right).
\end{aligned} \tag{31}$$

In order to numerically compare the higher order corrections to the above β-functions we also give the evaluation of β_{g_s} at the scale $\mu = M_Z$ and with $n_f = 6$. The one-loop contribution is $\sim (-0.04)$ to which the two and three-loop calculations give corrections of $\sim 3.7\%$ and $\sim (-0.02)\%$ respectively. Very small λ-corrections to β_{g_s} do not appear until four loops. With eq. (27) and the above assumptions we get

$$\begin{aligned}\beta_{g_s}|_{\mu=M_Z} = &\left(\underbrace{-4.0\,G^3}_{\text{1 loop}}\right)10^{-2} + \left(\underbrace{-1.4\,G^5}_{\text{2 loop}}\right)10^{-3} \\ &+ \left(\underbrace{-6.9\,Y^2G^3}_{\text{2 loop}}\underbrace{+1.7\,G^7}_{\text{3 loop}}\underbrace{-1.3\,Y^2G^5}_{\text{3 loop}}\right)10^{-5} + \left(\underbrace{3.1\,Y^4G^3}_{\text{3 loop}}\right)10^{-6}.\end{aligned} \tag{32}$$

The running of the m^2 parameter is given by

$$\begin{aligned}\frac{\beta^{(1)}_{m^2}}{m^2} =& 6\,\lambda + 3\,y_t^2, \\ \frac{\beta^{(2)}_{m^2}}{m^2} =& -30\,\lambda^2 - 36\,y_t^2\lambda - \frac{27}{4}\,y_t^4 + 20\,g_s^2y_t^2, \\ \frac{\beta^{(3)}_{m^2}}{m^2} =& 1026\,\lambda^3 + \frac{297}{2}\,y_t^2\lambda^2 + y_t^4\lambda\left(\frac{351}{4} + 216\zeta_3\right) \\ &+ y_t^6\left(\frac{1605}{16} + 45\zeta_3\right) + g_s^2y_t^2\lambda\,(-612 + 576\zeta_3) \\ &+ g_s^2y_t^4\left(\frac{447}{2} - 360\zeta_3\right) + g_s^4y_t^2\left(\frac{910}{3} - 16n_f - 24\zeta_3\right).\end{aligned} \tag{33}$$

Again we evaluate β_{m^2} at the scale $\mu = M_Z$ (with an assumed Higgs mass of 125 GeV and $n_f = 6$) and get a value of $\sim (+0.023)$ at one-loop level which means an increase of m^2 at higher energy scales. The two and three-loop corrections are $\sim 2.9\%$ and $\sim 0.32\%$. With the labels (27) we can estimate the contributions of the individual terms:

$$\begin{aligned}\frac{\beta_{m^2}}{m^2}|_{\mu=M_Z} = &\left(\underbrace{1.8\,Y^2}_{\text{1 loop}}\right)10^{-2} + \left(\underbrace{5.3\,L}_{\text{1 loop}}\underbrace{+1.1\,G^2Y^2}_{\text{2 loop}}\right)10^{-3} + \left(\underbrace{-2.4\,Y^4}_{\text{2 loop}}\underbrace{-1.9\,LY^2}_{\text{2 loop}}\right)10^{-4} \\ &+ \left(\underbrace{9.4\,G^4Y^2}_{\text{3 loop}}\underbrace{-7.0\,G^2Y^4}_{\text{3 loop}}\underbrace{+3.3\,Y^6}_{\text{3 loop}}\underbrace{-2.4\,L^2}_{\text{2 loop}}\underbrace{+1.1\,LY^4}_{\text{3 loop}}\right)10^{-5} \\ &+ \left(\underbrace{4.0\,G^2LY^2}_{\text{3 loop}}\right)10^{-6} + \left(\underbrace{7.1\,L^3}_{\text{3 loop}}\underbrace{+7.0\,L^2Y^2}_{\text{3 loop}}\right)10^{-7}.\end{aligned} \tag{34}$$

5 Results for the anomalous dimensions

In this section we give the anomalous dimensions of the physical fields in this setup. Note that because of the SU(2) symmetry Φ_1 and Φ_2 must have the same anomalous dimension γ_2^Φ. The same holds for the left-handed part of top and bottom quarks: $\gamma_{2,L}^t = \gamma_{2,L}^b$. For the quark flavours q which do not participate in the Yukawa interaction there is no difference between the left- and right-handed part as they are renormalized by the same Z-factor:

$\gamma_2^q \equiv \gamma_{2,L}^q = \gamma_{2,R}^q$. This also applies to the right-handed part of the bottom quark: $\gamma_{2,R}^b = \gamma_2^q$. All these relations have been tested explicitly during our calculation which provides a nice additional check.

$$
\begin{aligned}
\gamma_2^{q\,(1)} =& g_s^2\, C_F\, (1-\xi)\,, \\
\gamma_2^{q\,(2)} =& g_s^4 \left(-\frac{3}{2} C_F^2 + \frac{17}{2} C_A C_F - 2 n_f T_F C_F - \frac{5}{2} \xi C_A C_F + \frac{1}{4} \xi^2 C_A C_F \right), \\
\gamma_2^{q\,(3)} =& 6 T_F C_F\, g_s^4 y_t^2 + g_s^6 \left(\frac{3}{2} C_F^3 - \frac{143}{4} C_A C_F^2 + \frac{10559}{144} C_A^2 C_F + 3 n_f T_F C_F^2 \right. \\
& - \frac{1301}{36} n_f C_A T_F C_F + \frac{20}{9} n_f^2 T_F^2 C_F + 12 \zeta_3 C_A C_F^2 - \frac{15}{2} \zeta_3 C_A^2 C_F - \frac{371}{32} \xi C_A^2 C_F \\
& \left. + \frac{17}{4} \xi n_f C_A T_F C_F - \frac{3}{2} \xi \zeta_3 C_A^2 C_F + \frac{69}{32} \xi^2 C_A^2 C_F + \frac{3}{8} \xi^2 \zeta_3 C_A^2 C_F - \frac{5}{16} \xi^3 C_A^2 C_F \right).
\end{aligned}
\tag{35}
$$

For $y_t = 0$ this is in agreement with the well-known QCD result [24]. The renormalization constants for t,b and q can also be found in [22] up to order g_s^6 and $g_s^4 y_t^2$ where the calculation has been performed in the broken phase of the SM.

$$
\begin{aligned}
\gamma_{2,L}^{t\,(1)} =& \gamma_2^{q\,(1)} + \frac{1}{2}\, y_t^2, \\
\gamma_{2,L}^{t\,(2)} =& \gamma_2^{q\,(2)} - y_t^4 \left(\frac{1}{4} + \frac{3}{4} d_R \right) - 2 C_F\, g_s^2 y_t^2, \\
\gamma_{2,L}^{t\,(3)} =& \gamma_2^{q\,(3)} - \frac{33}{4}\, y_t^2 \lambda^2 + 6\, y_t^4 \lambda + y_t^6 \left(-\frac{3}{2} + \frac{29}{8} d_R - \frac{3}{8} d_R^2 + \frac{3}{2} \zeta_3 \right) \\
& + g_s^2 y_t^4 \left(\frac{13}{2} C_F + \frac{5}{8} C_F d_R + 6 \zeta_3 C_F d_R \right) \\
& + g_s^4 y_t^2 \left(-\frac{51}{8} C_F^2 + \frac{31}{8} C_A C_F - \frac{3}{2} n_f T_F C_F + 6 \zeta_3 C_F^2 - 15 \zeta_3 C_A C_F \right).
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\gamma_{2,R}^{t\,(1)} =& \gamma_2^{q\,(1)} + y_t^2, \\
\gamma_{2,R}^{t\,(2)} =& \gamma_2^{q\,(2)} - y_t^4 \left(\frac{1}{4} + \frac{3}{2} d_R \right) - 4 C_F\, g_s^2 y_t^2, \\
\gamma_{2,R}^{t\,(3)} =& \gamma_2^{q\,(3)} - \frac{33}{2}\, y_t^2 \lambda^2 + 12\, y_t^4 \lambda + y_t^6 \left(-\frac{33}{16} + \frac{53}{8} d_R - \frac{3}{4} d_R^2 + 3 \zeta_3 \right) \\
& + g_s^2 y_t^4 \left(5 C_F + \frac{5}{4} C_F d_R + 12 \zeta_3 C_F d_R \right) \\
& + g_s^4 y_t^2 \left(-\frac{51}{4} C_F^2 + \frac{31}{4} C_A C_F - 3 n_f T_F C_F + 12 \zeta_3 C_F^2 - 30 \zeta_3 C_A C_F \right).
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\gamma_2^{\Phi\,(1)} =& d_R\, y_t^2, \\
\gamma_2^{\Phi\,(2)} =& 6\, \lambda^2 - \frac{9}{4} d_R\, y_t^4 + 5 C_F d_R\, g_s^2 y_t^2, \\
\gamma_2^{\Phi\,(3)} =& -36\, \lambda^3 - \frac{45}{2} d_R\, y_t^2 \lambda^2 + 15 d_R\, y_t^4 \lambda \\
& + y_t^6 \left(-\frac{25}{16} d_R + 6 d_R^2 + 3 \zeta_3 d_R \right) + g_s^2 y_t^4 \left(\frac{15}{8} C_F d_R - 18 \zeta_3 C_F d_R \right) \\
& + g_s^4 y_t^2 \left(-\frac{119}{4} C_F^2 d_R + \frac{77}{2} C_A C_F d_R - 8 n_f T_F C_F d_R + 36 \zeta_3 C_F^2 d_R - 18 \zeta_3 C_A C_F d_R \right),
\end{aligned}
\tag{38}
$$

The purely λ-dependent part of this has been computed before in [17, 18].

$$\begin{aligned}
\gamma_2^{g\,(1)} =& g_s^2 \left(-\frac{5}{3}C_A + \frac{4}{3}n_f T_F - \frac{1}{2}\xi C_A\right), \\
\gamma_2^{g\,(2)} =& -4T_F\, g_s^2 y_t^2 + g_s^4 \left(-\frac{23}{4}C_A^2 + 4n_f T_F C_F + 5n_f C_A T_F - \frac{15}{8}\xi C_A^2 + \frac{1}{4}\xi^2 C_A^2\right), \\
\gamma_2^{g\,(3)} =& g_s^2 y_t^4 \left(9T_F + 7T_F d_R\right) - g_s^4 y_t^2 \left(6T_F C_F + \frac{25}{2}C_A T_F\right) \\
&+ g_s^6 \left(-\frac{4051}{144}C_A^3 - 2n_f T_F C_F^2 + \frac{5}{18}n_f C_A T_F C_F + \frac{875}{18}n_f C_A^2 T_F - \frac{44}{9}n_f^2 T_F^2 C_F\right. \\
&-\frac{76}{9}n_f^2 C_A T_F^2 + \frac{3}{2}\zeta_3 C_A^3 + 24\zeta_3 n_f C_A T_F C_F - 18\zeta_3 n_f C_A^2 T_F - \frac{127}{16}\xi C_A^3 \\
&\left.+2\xi n_f C_A^2 T_F - \frac{9}{8}\xi\zeta_3 C_A^3 + \frac{27}{16}\xi^2 C_A^3 + \frac{3}{16}\xi^2\zeta_3 C_A^3 - \frac{7}{32}\xi^3 C_A^3\right).
\end{aligned} \tag{39}$$

This is also in agreement with [24] for $y_t = 0$ and with [23] for $y_t = 0$, $\xi = 0$. For $d_R = 3$ and $T_F = \frac{1}{2}$ (QCD) these results are as follows:

$$\begin{aligned}
\gamma_2^{q\,(1)} =& \frac{4}{3}\left(1-\xi\right) g_s^2, \\
\gamma_2^{q\,(2)} =& g_s^4 \left(\frac{94}{3} - \frac{4}{3}n_f - 10\xi + \xi^2\right), \\
\gamma_2^{q\,(3)} =& 4\, g_s^4 y_t^2 + g_s^6 \left(+\frac{24941}{36} - \frac{1253}{18}n_f + \frac{20}{27}n_f^2 - 26\zeta_3 - \frac{1113}{8}\xi\right. \\
&\left.+\frac{17}{2}\xi n_f - 18\xi\zeta_3 + \frac{207}{8}\xi^2 + \frac{9}{2}\xi^2\zeta_3 - \frac{15}{4}\xi^3\right),
\end{aligned} \tag{40}$$

$$\begin{aligned}
\gamma_{2,L}^{t\,(1)} =& \gamma_2^{q\,(1)} + \frac{1}{2}\, y_t^2, \\
\gamma_{2,L}^{t\,(2)} =& \gamma_2^{q\,(2)} - \frac{5}{2}\, y_t^4 - \frac{8}{3}\, g_s^2 y_t^2, \\
\gamma_{2,L}^{t\,(3)} =& \gamma_2^{q\,(3)} - \frac{33}{4}\, y_t^2\lambda^2 + 6\, y_t^4\lambda + y_t^6\left(6 + \frac{3}{2}\zeta_3\right) \\
&+ g_s^2 y_t^4 \left(\frac{67}{6} + 24\zeta_3\right) + g_s^4 y_t^2 \left(\frac{25}{6} - n_f - \frac{148}{3}\zeta_3\right).
\end{aligned} \tag{41}$$

$$\begin{aligned}
\gamma_{2,R}^{t\,(1)} =& \gamma_2^{q\,(1)} + y_t^2, \\
\gamma_{2,R}^{t\,(2)} =& \gamma_2^{q\,(2)} - \frac{19}{4}\, y_t^4 - \frac{16}{3}\, g_s^2 y_t^2, \\
\gamma_{2,R}^{t\,(3)} =& \gamma_2^{q\,(3)} - \frac{33}{2} y_t^2\lambda^2 + 12\, y_t^4\lambda + y_t^6\left(\frac{177}{16} + 3\zeta_3\right) \\
&+ g_s^2 y_t^4 \left(\frac{35}{3} + 48\zeta_3\right) + g_s^4 y_t^2 \left(\frac{25}{3} - 2n_f - \frac{296}{3}\zeta_3\right),
\end{aligned} \tag{42}$$

$$
\begin{aligned}
\gamma_2^{\Phi\,(1)} =& 3\,y_t^2, \\
\gamma_2^{\Phi\,(2)} =& 6\,\lambda^2 - \frac{27}{4}\,y_t^4 + 20\,g_s^2 y_t^2, \\
\gamma_2^{\Phi\,(3)} =& -36\,\lambda^3 - \frac{135}{2}\,y_t^2\lambda^2 + 45\,y_t^4\lambda + y_t^6\left(\frac{789}{16} + 9\zeta_3\right) \\
& + g_s^2 y_t^4\left(\frac{15}{2} - 72\zeta_3\right) + g_s^4 y_t^2\left(\frac{910}{3} - 16n_f - 24\zeta_3\right),
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\gamma_2^{g\,(1)} =& g_s^2\left(-5 + \frac{2}{3}n_f - \frac{3}{2}\xi\right), \\
\gamma_2^{g\,(2)} =& -2\,g_s^2 y_t^2 + g_s^4\left(-\frac{207}{4} + \frac{61}{6}n_f - \frac{135}{8}\xi + \frac{9}{4}\xi^2\right), \\
\gamma_2^{g\,(3)} =& 15\,g_s^2 y_t^4 - \frac{91}{4}\,g_s^4 y_t^2 \\
& + g_s^6\left(-\frac{12153}{16} + \frac{7831}{36}n_f - \frac{215}{27}n_f^2 + \frac{81}{2}\zeta_3 - 33\zeta_3 n_f - \frac{3429}{16}\xi\right. \\
& \left. + 9\xi n_f - \frac{243}{8}\xi\zeta_3 + \frac{729}{16}\xi^2 + \frac{81}{16}\xi^2\zeta_3 - \frac{189}{32}\xi^3\right).
\end{aligned}
\tag{44}
$$

6 The evolution of the quartic Higgs coupling

The quartic Higgs coupling λ is of special interest as it is directly related to the Higgs mass M_H. If we assume that the SM is valid up to some high energy scale Λ, then the value of M_H should meet the constraints

$$m_{min} < M_H < m_{max}.$$

Here the upper limit is related to the well-known fact that the running Higgs self-coupling develops a Landau pole[7] if M_H is large [32,33,34]. For $\Lambda = M_{Planck} = 10^{18}$ GeV the estimated value of m_{max} is around 175 GeV [32, 33, 34, 35], which is already excluded by experiments carried out at the LHC and the Tevatron.

The lower limit m_{min} follows from the requirement of the vacuum stability [36, 37, 38]. In order to find m_{min} one should construct the effective Higgs potential $V[\phi]$ including radiative corrections and sum possible large logarithms using the standard method of the Renormalization Group (for a review see, e.g. [39]). Once this has been done, the condition that the potential $V[\phi]$ does not develop a deeper minimum in addition to the standard one for all values of $\phi < \Lambda$ fixes m_{min}.

In our analysis we will use a simplified approach for finding m_{min}, namely the requirement that the running coupling constant $\lambda(\mu)$ stay non-negative for all μ less than Λ. It has been shown in [33,40] that the simplified approach is essentially equivalent to the one based on the use of the effective potential provided the instabilty of $V[\phi]$ can only happen at $\phi \gg M_Z$.

[7]This is true in the one-loop approximation. At two loops the Landau pole is replaced by an ultraviolet metastable fixed point with the resulting fixed point value of λ being outside the weak coupling region.

In this section we investigate the effect of the three-loop result $\beta_\lambda^{(3)}$ on the running of λ and therefore its effect on the stability of the electroweak vacuum in the SM. For this we also include the electroweak contributions up to the two-loop level. The two-loop β-functions for the SM gauge couplings have been derived in [27, 28, 29, 30]. The two-loop results for the Yukawa-couplings and λ can be found in [19, 21, 20]. Now we add the three-loop results derived in the previous section and investigate the effect this has on the evolution of our couplings (for a recent similar analysis, using the two-loop running, see, e.g. [41, 42]).

To find starting values for the running of the couplings we should account for the fact that the physical parameters (e.g. pole masses) are related to the ones in the $\overline{\rm MS}$-scheme in a non-trivial way (see e.g. [43, 44, 45]). For example eq. (15) is only valid at tree level. For the higher order corrections we take the electroweak ones at one-loop and the QCD ones at two-loop level from [43, 44]. These matching relations depend on the exact values of $\alpha_s(M_Z)$ and the pole mass M_t of the top quark and of course the mass of the Higgs boson M_H. For the latter we consider the cases $M_H = 124$ GeV and $M_H = 126$ GeV. For the other two we use the values

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007 \qquad \text{and} \qquad M_t = 172.9 \pm 0.6 \pm 0.9 \text{ GeV} \quad [46]. \tag{45}$$

One should also keep in mind that the matching relations themselves receive contributions from not yet known higher order corrections . The corresponding uncertainty in m_{min} has been estimated in [4, 41] and found to be about 2 GeV.

Fig. 6 shows the evolution of λ in this framework up to the Planck scale. To estimate the dependence of the λ-running on the parameters $\alpha_s(M_Z)$ and M_t we give the shifted curves for $\lambda(\mu)$ when we change these parameters by $\pm\sigma$ as given in eq. (45).[8] As the two- and three-loop curves are very close together we zoom in on the region where λ crosses over to negative values in Fig. 7. In this plot we give the α_s-uncertainty for the two- and three-loop curves to compare between this uncertainty and the shift from two to three loops. Note that there is a considerable difference between $M_H = 124$ GeV and $M_H = 126$ GeV which means that the evolution of λ is very sensitive to the value of the Higgs mass. Given a fixed value for M_H the largest uncertainty lies in the exact value of the top mass. The second largest uncertainty comes from α_s. The total effect due to the three-loop part of the β-functions is somewhat smaller than latter as can be seen best in Fig. 7. Still, it is worthy of note that the three-loop corrections to the β-functions presented here enhance the stability of the SM electroweak vacuum.

The smallness of the three-loop correction to β_λ seems to be somewhat coincidental as the aforementioned cancellations of individual terms in $\beta_\lambda^{(3)}$ depend strongly on the value of M_H. Finding a Higgs with a mass of 124 to 126 GeV would therefore mean an excellent convergence of the perturbation series for β_λ. Another intriguing consequence of a Higgs mass in that region is the uncertainty whether λ becomes indeed negative at high scales or not. If we take e.g. $M_H = 126$ GeV and $\alpha_s = 0.1184$ and decrease[9] the top mass from $M_t = 172.9$ GeV to $M_t = 171.25$ GeV ($M_t = 171.16$ GeV without the three-loop corrections), then λ stays positive up to the Planck scale M_{Planck} in our framework. The same effect can be achieved for $M_H = 126$ GeV and $M_t = 172.9$ GeV by increasing $\alpha_s = 0.1184$ by $6.5\,\sigma_{\alpha_s}$ ($7\,\sigma_{\alpha_s}$

[8] In order not to make the plot too crowded these shifted curves are only given for the two-loop result. The difference to the three-loop result is similar to the one between the two- and three-loop curves for $\alpha_s(M_Z) = 0.1184$ and $M_t = 172.9$ GeV.

[9] Smaller values for M_t or larger values for α_s increase the stability of the vacuum.

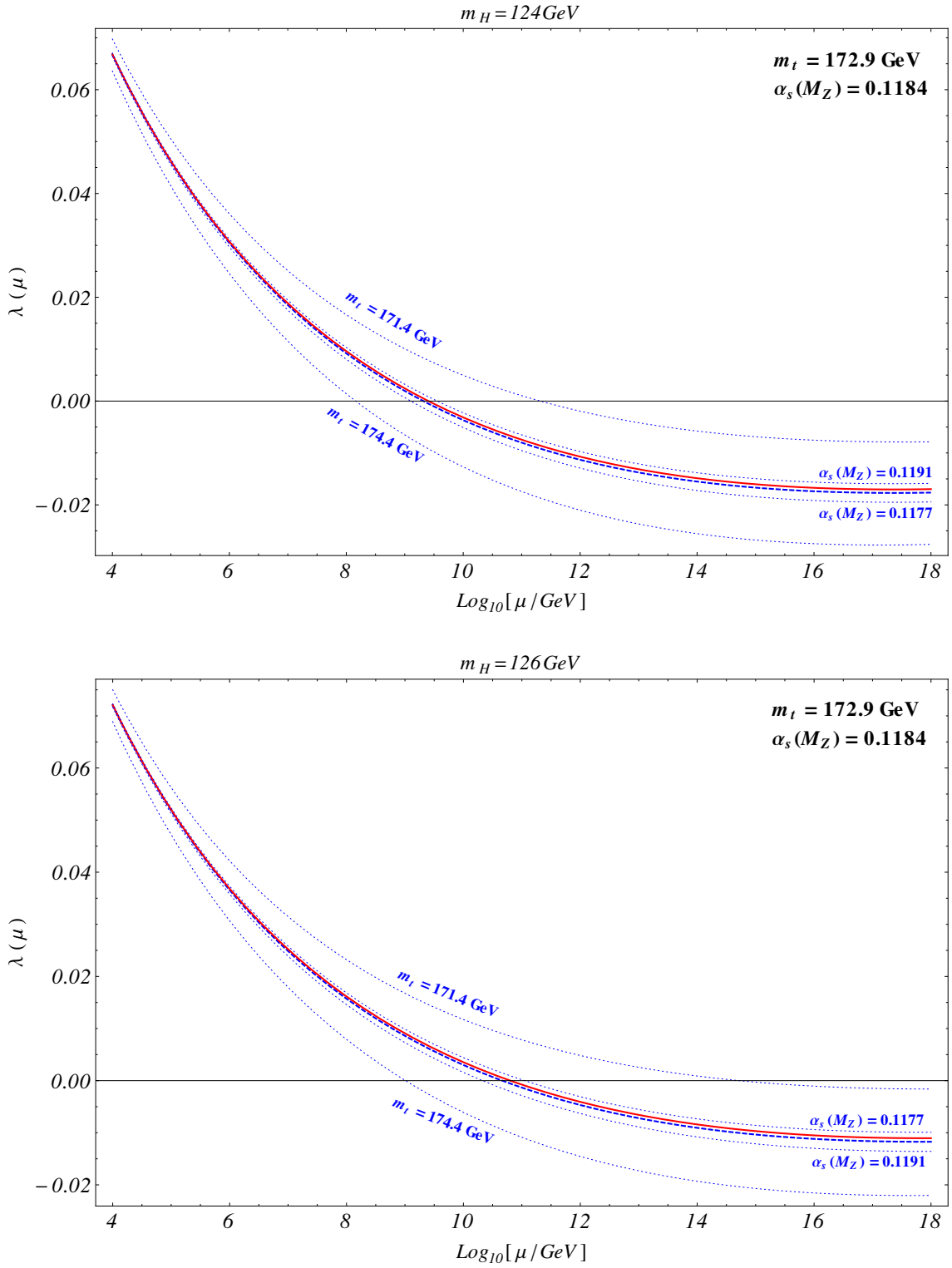

Figure 6: Evolution of λ with the scale μ: 2 loop (dashed, blue) and 3 loop (continuous, red) results; Uncertainties with respect to the two-loop result: $\pm 1\sigma_{\alpha_s}$, $\pm 1\sigma_{M_t}$ (dotted)

without the three-loop corretions). A combined scenario for $M_H = 126$ GeV would be a shift

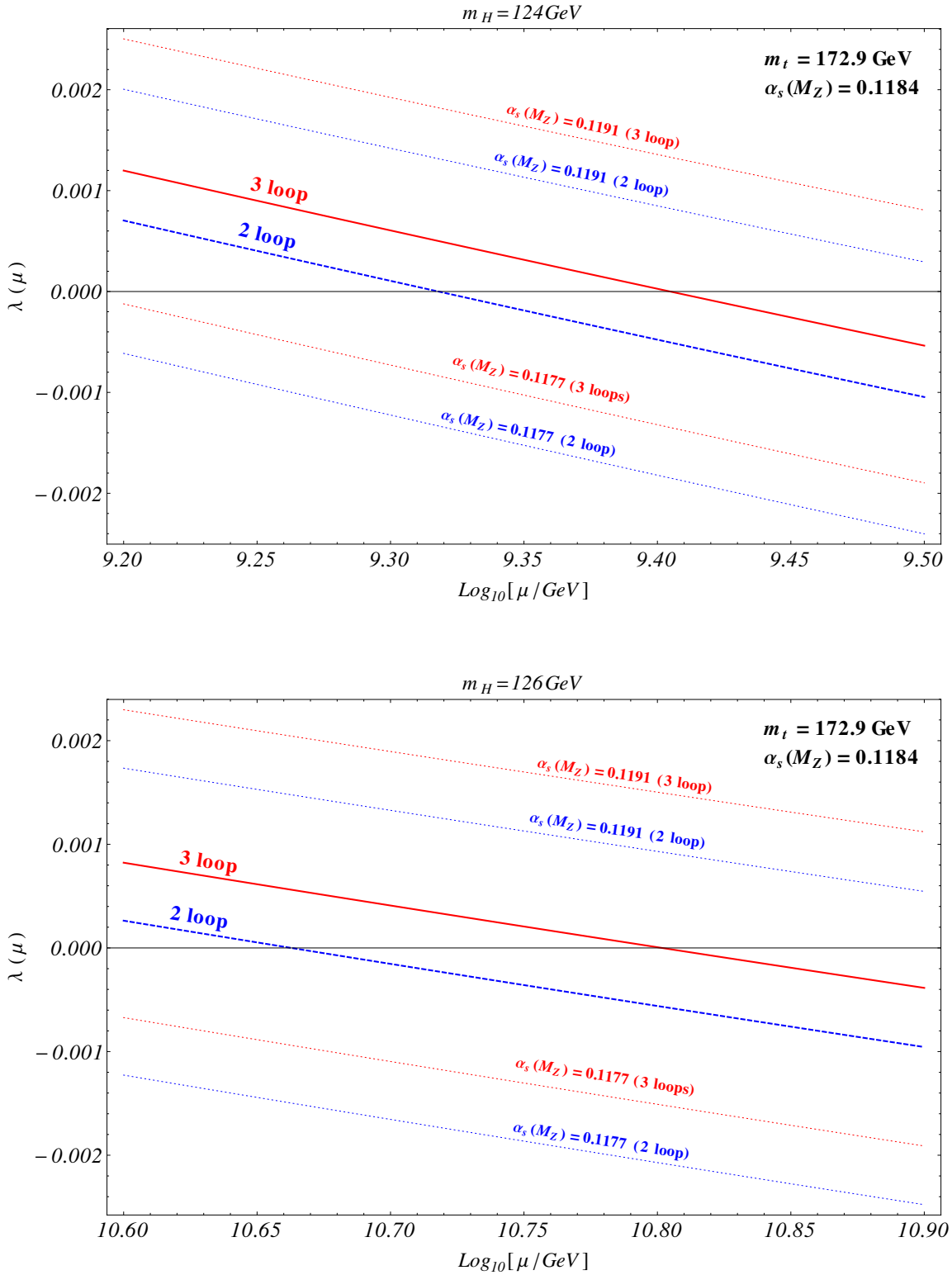

Figure 7: Evolution of λ with the scale μ: 2 loop (dashed, blue) and 3 loop (continuous, red) results; Uncertainties with respect to the two- and three-loop results: $\pm 1\sigma_{\alpha_s}$ (dotted)

of $M_t = 172.9$ GeV by $-1\,\sigma_{M_t} = -1.5$ GeV and of $\alpha_s = 0.1184$ by $+1\,\sigma_{\alpha_s} = +0.0007$ GeV which would also make λ positive up to the Planck scale.

Thus, we conclude that at present no definite answer can be given to the question whether the SM vacuum is stable all the way up to the Planck scale or not. If indeed a SM Higgs boson is found with a mass of 124 to 126 GeV, this is a good motivation for determining $\alpha_s(M_Z)$ and M_t as accurately as possible as well as calculating the SM β-functions to the highest achievable accuracy.

7 Conclusions

We have computed the three-loop corrections to the evolution the top-Yukawa coupling, the strong coupling and the quartic Higgs self-coupling in the unbroken SM with the numerically small gauge coupling constants g_1 and g_2 and all Yukawa couplings except for y_t set to zero.

The implications of our calculation on the stability of the electroweak vacuum in the SM can be summarized as follows:

- The total effect of the three-loop terms is relatively small which is not self-evident as the individual terms in $\beta_\lambda^{(3)}$ are much larger than the final value due to significant cancellations for a Higgs mass in the vicinity of 125 GeV.
- The evolution of λ is very sensitive to the values of the Higgs mass, the top mass and $\alpha_s(M_Z)$. If we take e.g. $M_H = 126$ GeV and decrease the top mass by about 1.7 GeV, then λ stays positive up to the Planck scale (a similar observation has been made in [41]). This is a very good motivation for high precision measurements of $\alpha_s(M_Z)$ and M_t. With the latter values known more precisely, the account of the the three-loop effects in the evolution of the quartic Higgs self-coupling would be essential in considering the problem of the stability of the electroweak vacuum in the SM.
- In this context it may also be useful to calculate both the electroweak contributions to $\beta_\lambda^{(3)}$ and $\beta_{y_t}^{(3)}$ at three-loop level as well as the matching of experimentally measurable on-shell parameters and $\overline{\rm MS}$-parameters to a higher accuracy.

We thank Luminita Mihaila, Jens Salomon and Matthias Steinhauser for useful discussions and informing us on the results of [31] before their publication. We thank Johann Kühn for valuable comments and support. Last but not least, we are grateful to Fedor Bezrukov and Mikhail Shaposhnikov for gently drawing our attention to the subject, numerous discussions and providing us with their version of a Mathematica package to perform one-loop matching in the SM.

In conclusion we want to mention that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using the thread-based [47] version of FORM [9]. The Feynman diagrams have been drawn with the Latex package Axodraw [48].

This work has been supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics".

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