

# Gauge Coupling Beta Functions in the Standard Model to Three Loops

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In this paper we compute the three-loop corrections to the beta functions of the three gauge couplings in the Standard Model of particle physics using the minimal subtraction scheme and taking into account Yukawa and Higgs self couplings.

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Renormalization group functions are fundamental quantities of each quantum field theory and play an important role in various aspects. Besides controlling the energy dependence of parameters and fields they are also crucial for the resummation of large logarithms. Furthermore, renormalization group functions are important for the development of grand unified theories and the extrapolation of low-energy precision data to high energies, not accessible by collider experiments.

As far as the strong interaction part of the Standard Model is concerned the corresponding gauge coupling beta function is known up to four-loop order [1–9]. At three-loop level also the corrections involving two strong and one top quark Yukawa coupling have been computed [10]. On the other hand, for the  $SU(2)_L \times U(1)_Y$  part only one- [1, 2, 11] and two-loop [12–15] corrections are available since about 30 years. (Two-loop corrections are also known for the Yukawa [15–17] and Higgs boson self coupling [15, 18, 19] beta function, see also Ref. [20].) In this Letter we provide results for the three-loop gauge coupling beta functions taking into account all sectors of the Standard model, i.e., the gauge, Yukawa and Higgs boson self couplings.

Let us in a first step define the beta functions. We denote the three gauge couplings by  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  and adopt a  $SU(5)$ -like normalization with

$$\begin{aligned}\alpha_1 &= \frac{5}{3} \frac{\alpha}{\cos^2 \theta_W}, \\ \alpha_2 &= \frac{\alpha}{\sin^2 \theta_W}, \\ \alpha_3 &= \alpha_s,\end{aligned}\tag{1}$$

where  $\alpha$  is the fine structure constant,  $\theta_W$  the weak mixing angle and  $\alpha_s$  the strong coupling. In our calculation we consider in addition to the gauge couplings also the third-generation Yukawa couplings [32]  $\alpha_4 = \alpha_t$ ,  $\alpha_5 = \alpha_b$  and  $\alpha_6 = \alpha_\tau$ , and the Higgs boson self coupling  $\alpha_7 = \lambda/(4\pi)$ .  $\alpha_x = \alpha m_x^2/(2 \sin^2 \theta_W M_W^2)$  ( $x = t, b, \tau$ ) where  $m_x$  and  $M_W$  are the fermion and  $W$  boson mass, respectively, and  $-\lambda(\Phi^\dagger \Phi)^2$  is the part of the Lagrange density describing the quartic Higgs self interaction.

In the modified minimal subtraction ( $\overline{\text{MS}}$ ) renormalization scheme the beta functions are then defined through

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_i}{\pi} = \beta_i(\{\alpha_j\}, \epsilon) = -\epsilon \frac{\alpha_i}{\pi}$$

$$- \left( \frac{\alpha_i}{\pi} \right)^2 \left[ a_i + \sum_{j=1}^7 \frac{\alpha_j}{\pi} b_{ij} + \sum_{j,k=1}^7 \frac{\alpha_j}{\pi} \frac{\alpha_k}{\pi} c_{ijk} + \dots \right],\tag{2}$$

where  $\epsilon = (4-d)/2$  is the regulator of Dimensional Regularization with  $d$  being the space-time dimension used for the evaluation of the momentum integrals and the dependence of  $\alpha_i$  on the renormalization scale  $\mu$  is suppressed. We assume the definition in Eq. (2) for  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ . In this Letter we evaluate the three-loop terms (coefficients  $c_{ijk}$ ) only for the gauge couplings (i.e.  $i = 1, 2, 3$ ). For our calculation the beta functions for the Yukawa couplings are needed to the one-loop order (as compared to Eq. (2)  $\alpha_i a_i$  has to be replaced by  $\sum_j \alpha_j a_j$ ) and the tree-level expression (first term in Eq. (2)) is sufficient for  $\beta_\lambda$ .

The functions  $\beta_i$  are obtained from the renormalization constants of the corresponding gauge couplings which are defined as  $g_i^{\text{bare}} = \mu^\epsilon Z_{g_i} g_i$  where  $\alpha_i = g_i^2/(4\pi)$ . Exploiting the fact that the  $g_i^{\text{bare}}$  are  $\mu$ -independent and taking into account that  $Z_{g_i}$  may depend on all seven couplings leads to the following formula

$$\beta_i = - \left[ \epsilon \frac{\alpha_i}{\pi} + 2 \frac{\alpha_i}{Z_{g_i}} \sum_{j \neq i} \frac{\partial Z_{g_i}}{\partial \alpha_j} \beta_j \right] \left( 1 + 2 \frac{\alpha_i}{Z_{g_i}} \frac{\partial Z_{g_i}}{\partial \alpha_i} \right)^{-1}.\tag{3}$$

From Eq. (3) it is clear that the renormalization constants  $Z_{g_i}$  ( $i = 1, 2, 3$ ) have to be computed up to three-loop order.

In the  $\overline{\text{MS}}$  scheme the beta functions are mass independent which allows us to use the Standard Model in the unbroken phase as a framework for our calculation. By construction less vertices have to be considered than after spontaneous symmetry breaking. The electroweak gauge bosons are denoted by  $W$  and  $B$  corresponding to the  $SU(2)_L \times U(1)_Y$  gauge groups, respectively.

In principle each vertex containing the gauge coupling  $g_i$  at tree level can be used in order to obtain  $Z_{g_i}$  via

$$Z_{g_i} = \frac{Z_{\text{vert}}}{\prod_k \sqrt{Z_{k,\text{wf}}}},\tag{4}$$

where  $Z_{\text{vert}}$  stands for the renormalization constant of the vertex and  $Z_{k,\text{wf}}$  for the wave function renormalization constant;  $k$  runs over all external particles. We have

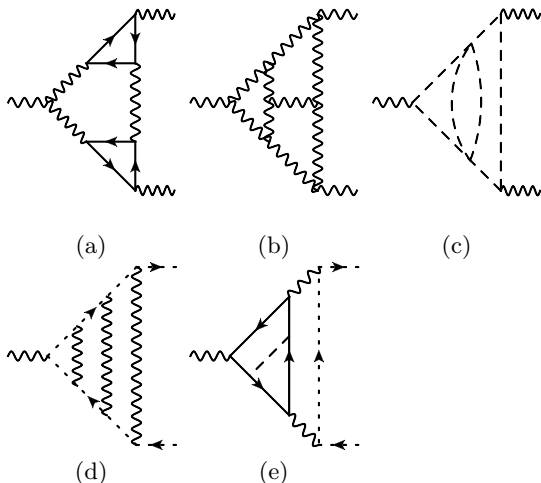


FIG. 1: Sample Feynman diagrams contributing to the  $W_1W_2W_3$  vertex [(a), (b) and (c)] and the  $W_3$  ghost vertex [(d) and (e)]. Dashed, dotted, solid and wavy lines correspond to Higgs boson, ghost, fermion and  $W$  boson lines, respectively. Sample diagrams for gluonic Green's functions are obtained for (a), (b), (d) and (e) by replacing the  $W$  lines by gluon lines.

computed  $Z_{g_3}$  using both the ghost-gluon and the three-gluon vertex and  $Z_{g_2}$  has been evaluated with the help of the ghost- $W_3$ , the  $W_1W_2W_3$  and the  $\phi^+\phi^-W_3$  vertex where  $\phi^\pm$  is the charged component of the Higgs doublet corresponding to the Goldstone boson in the broken phase and  $W_1$ ,  $W_2$  and  $W_3$  are the components of the  $W$  boson. Both for  $Z_{g_2}$  and  $Z_{g_3}$  the different ways lead to the same result which constitutes a strong check for the correctness of the final result.  $Z_{g_1}$  is obtained from vertices containing the  $B$  boson. Due to the Ward identity there is a cancellation of  $Z_{\text{vert}}$  and the factors  $\sqrt{Z_{k,wf}}$  other than the one corresponding to the  $B$  boson. Thus  $Z_{g_1}$  is solely computed from the wave function renormalization constant of the  $B$  boson. We have computed the three-loop corrections to the  $BBB$  vertex and checked that the result is zero. Several three-loop sample diagrams contributing to the considered three-point functions are shown in Fig. 1. Due to the fact that the  $\beta_i$  do not depend on any kinematical scale we evaluate the vertex functions in the limit where one external momentum is set to zero. In this way all loop-integrals are mapped to massless two-point functions which up to three-loops can be computed with the help of MINCER [21].

An important issue in the present calculation is the treatment of  $\gamma_5$  within dimensional regularization. Non-trivial contributions may arise if in the course of the calculation two fermion traces occur where both of them

contain an odd number of  $\gamma_5$  matrices and four or more  $\gamma$  matrices. It is straightforward to see that the Green's functions which we have chosen for calculating the beta functions contain at most one-loop triangle sub-diagrams (see, e.g., Fig. 1(a)) which could potentially lead to contributions where a careful treatment of  $\gamma_5$  is required. In our case, however, all these contributions vanish identically due to anomaly cancellations (see, e.g., Ref. [22]) since we always sum over all fermions of one generation. This has also been checked by an explicit calculation [23].

Our calculation is based on a high level of automation in order to avoid errors due to manual interaction. As a core of our set-up we use the well-tested chain of programs which work hand-in-hand: QGRAF [24] generates all contributing Feynman diagrams. The output is passed via q2e to exp [25, 26] which generates the FORM [27] code. The latter is processed by MINCER [21] which computes the Feynman integrals and outputs the  $\epsilon$  expansion of the result. A serious problem which had to be overcome in the course of the present calculation is the enormous number of diagrams which contribute to the individual renormalization factors. E.g., in the case of the  $W_1W_2W_3$  vertex one has about 380 000 diagrams for the vertex itself and 60 000 for the  $W_3$  self energy with similar numbers for the other Green's functions. Thus, in total more than one million diagrams have to be computed. In order to handle such an amount of diagrams we have decided to automatically split the original problems into blocks containing only of the order of 1000 Feynman diagrams. In total we end up with a CPU time of about 100 days on a single core. Since our calculation is highly parallelizable the final wall-clock time is about one day on 100 cores.

Note that at three-loop level all sectors of the Standard Model contribute to the  $\beta$  functions of the gauge couplings. Thus the huge number of diagrams mainly results from the numerous interaction vertices in the Standard Model. In the set-up described above the strong interaction part of the Standard Model has already been used for a variety of calculations. We have added the electroweak part by establishing an interface to a model file of FeynArts [28]. It generates all Standard Model Feynman rules in a format readable by QGRAF and q2e. The model file for FeynArts has been generated using FeynRules [29].

In the following we present results for  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  up to three loops. In order to keep the expressions compact we set the Yukawa couplings  $\alpha_b$  and  $\alpha_\tau$  to zero. Also, we use unit CKM matrix in our calculations. The complete result with  $\alpha_b$  and  $\alpha_\tau$  kept non-zero can be found elsewhere [23]. Omitting the  $\mu$  dependence in the arguments of  $\alpha_i$  (and taking the limit  $\epsilon \rightarrow 0$ ) we obtain

$$\beta_1 = \left(\frac{\alpha_1}{\pi}\right)^2 \left\{ \frac{1}{40} + \frac{n_G}{3} + \frac{\alpha_1}{\pi} \left( \frac{9}{800} + \frac{19n_G}{240} \right) + \frac{\alpha_2}{\pi} \left( \frac{9}{160} + \frac{3n_G}{80} \right) + \frac{\alpha_3}{\pi} \frac{11n_G}{60} + \left(\frac{\alpha_1}{\pi}\right)^2 \left( \frac{489}{512000} - \frac{29n_G}{2400} - \frac{209n_G^2}{8640} \right) \right\}$$

$$\begin{aligned}
& + \frac{\alpha_1 \alpha_2}{\pi \pi} \left( \frac{783}{51200} - \frac{7n_G}{6400} \right) - \frac{\alpha_1 \alpha_3}{\pi \pi} \frac{137n_G}{14400} + \left( \frac{\alpha_2}{\pi} \right)^2 \left( \frac{3401}{20480} + \frac{83n_G}{1920} - \frac{11n_G^2}{960} \right) - \frac{\alpha_2 \alpha_3}{\pi \pi} \frac{n_G}{320} \\
& + \left( \frac{\alpha_3}{\pi} \right)^2 \left( \frac{275n_G}{576} - \frac{121n_G^2}{2160} \right) + n_t \frac{\alpha_t}{\pi} \left[ -\frac{17}{160} - \frac{\alpha_1}{\pi} \frac{2827}{51200} - \frac{\alpha_2}{\pi} \frac{471}{2048} - \frac{\alpha_3}{\pi} \frac{29}{320} + \frac{\alpha_t}{\pi} \left( \frac{339}{5120} + \frac{303n_t}{2560} \right) \right] \\
& + \frac{\lambda}{4\pi^2} \left( \frac{\alpha_1}{\pi} \frac{27}{3200} + \frac{\alpha_2}{\pi} \frac{9}{640} - \frac{\lambda}{4\pi^2} \frac{9}{320} \right) \Bigg\}, \\
\beta_2 = & \left( \frac{\alpha_2}{\pi} \right)^2 \left\{ -\frac{43}{24} + \frac{n_G}{3} + \frac{\alpha_1}{\pi} \left( \frac{3}{160} + \frac{n_G}{80} \right) + \frac{\alpha_2}{\pi} \left( -\frac{259}{96} + \frac{49n_G}{48} \right) + \frac{\alpha_3}{\pi} \frac{n_G}{4} + \left( \frac{\alpha_1}{\pi} \right)^2 \left( \frac{163}{102400} - \frac{7n_G}{960} - \frac{11n_G^2}{2880} \right) \right. \\
& + \frac{\alpha_1 \alpha_2}{\pi \pi} \left( \frac{561}{10240} + \frac{13n_G}{1280} \right) - \frac{\alpha_1 \alpha_3}{\pi \pi} \frac{n_G}{960} + \left( \frac{\alpha_2}{\pi} \right)^2 \left( -\frac{667111}{110592} + \frac{1603n_G}{432} - \frac{415n_G^2}{1728} \right) + \frac{\alpha_2 \alpha_3}{\pi \pi} \frac{13n_G}{64} \\
& + \left( \frac{\alpha_3}{\pi} \right)^2 \left( \frac{125n_G}{192} - \frac{11n_G^2}{144} \right) + n_t \frac{\alpha_t}{\pi} \left[ -\frac{3}{32} - \frac{\alpha_1}{\pi} \frac{593}{10240} - \frac{\alpha_2}{\pi} \frac{729}{2048} - \frac{\alpha_3}{\pi} \frac{7}{64} + \frac{\alpha_t}{\pi} \left( \frac{57}{1024} + \frac{45n_t}{512} \right) \right] \\
& \left. + \frac{\lambda}{4\pi^2} \left( \frac{\alpha_1}{\pi} \frac{3}{640} + \frac{\alpha_2}{\pi} \frac{3}{128} - \frac{\lambda}{4\pi^2} \frac{3}{64} \right) \right\}, \\
\beta_3 = & \left( \frac{\alpha_3}{\pi} \right)^2 \left\{ -\frac{11}{4} + \frac{n_G}{3} + \frac{\alpha_1}{\pi} \frac{11n_G}{480} + \frac{\alpha_2}{\pi} \frac{3n_G}{32} + \frac{\alpha_3}{\pi} \left( -\frac{51}{8} + \frac{19n_G}{12} \right) + \left( \frac{\alpha_1}{\pi} \right)^2 \left( -\frac{13n_G}{7680} - \frac{121n_G^2}{17280} \right) \right. \\
& - \frac{\alpha_1 \alpha_2}{\pi \pi} \frac{n_G}{2560} + \frac{\alpha_1 \alpha_3}{\pi \pi} \frac{77n_G}{2880} + \left( \frac{\alpha_2}{\pi} \right)^2 \left( \frac{241n_G}{1536} - \frac{11n_G^2}{384} \right) + \frac{\alpha_2 \alpha_3}{\pi \pi} \frac{7n_G}{64} \\
& \left. + \left( \frac{\alpha_3}{\pi} \right)^2 \left( -\frac{2857}{128} + \frac{5033n_G}{576} - \frac{325n_G^2}{864} \right) + n_t \frac{\alpha_t}{\pi} \left[ -\frac{1}{8} - \frac{\alpha_1}{\pi} \frac{101}{2560} - \frac{\alpha_2}{\pi} \frac{93}{512} - \frac{\alpha_3}{\pi} \frac{5}{8} + \frac{\alpha_t}{\pi} \left( \frac{9}{128} + \frac{21n_t}{128} \right) \right] \right\}, \quad (5)
\end{aligned}$$

where  $n_G$  labels the number of generations and  $n_t$  the number of heavy up-type quarks. In the Standard Model we have  $n_G = 3$  and  $n_t = 1$ . The result including a fourth generation [33] (with Yukawa coupling  $\alpha_{t'}$ ) is obtained with the replacements  $(\alpha_t n_t)^n \rightarrow (\alpha_t + \alpha_{t'})^n$  ( $n = 1, 2$ ) and  $\alpha_t^2 n_t \rightarrow \alpha_t^2 + \alpha_{t'}^2$ . It is interesting to note that, although the two-loop expression is  $\lambda$ -independent, the three-loop term of  $\beta_1$  and  $\beta_2$  contains both linear and quadratic terms in  $\lambda$ . The latter arise from diagrams as those in Fig. 1(c).

There are several checks on the correctness of our result. The interface between `FeynArts` and `QGraf/q2e` has been checked by evaluating several one- and two-loop results which are known in the literature. We have even considered quantities within the Minimal Supersymmetric Standard Model, like the relation between the squark masses within one generation, which is quite involved in case electroweak interactions are kept non-zero. We have furthermore reproduced the two-loop results in the literature both for the gauge [12–15] and the Yukawa coupling beta functions [15–17] which constitutes a strong check on the correctness of the Feynman rules and the general set-up. Also the three-loop corrections of order  $\alpha_3^3 \alpha_t$  to  $\beta_3$  from Ref. [10] have been reproduced. All our calculations have been performed for three general gauge parameters, one for each gauge group. Whereas the renormalization constants for the vertices and wave functions still depend on the gauge parameters the  $Z_{g_i}$  are independent leading to gauge parameter independent results for

$\beta_i$ . In a further check we have ensured that in the vertex diagrams no infra-red divergence is introduced although one external momentum is set to zero [34]. This is done by assigning a non-vanishing mass  $m$  to internal lines (a common mass for all particles is sufficient) and performing an asymptotic expansion [30] in the limit  $q^2 \gg m^2$  where  $q$  is the non-vanishing external momentum of the vertex diagram. Asymptotic expansion for this limit is automated in the program `exp` and thus can be performed with the set-up described above. It is sufficient to restrict to the leading term in  $m^2/q^2$  and check that no  $\ln(m^2/\mu^2)$  terms appear in the final result. Nevertheless the calculation becomes significant more complex; some diagrams develop up to 35 sub-diagrams when applying the rules of asymptotic expansion. With this method we have explicitly checked that the  $W_1 W_2 W_3$  and three-gluon vertex are free from infra-red divergences. Since the results agree with the ones obtained from the other vertices also the latter are infra-red safe.

In order to estimate the numerical effect of the new terms we use the experimental measured values for  $\alpha_i(M_Z^2)$  ( $i = 1, 2, 3$ ) and run to a higher scale  $\mu$  using two or three loops for the gauge and two loops for top Yukawa beta functions. For  $\mu = 2$  TeV the relative difference between two and three loops amounts to 0.003%, 0.010% and 0.005% for  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ , respectively. In the case of  $\alpha_3$  the shift is significantly smaller than the one introduced by the experimental uncertainty at the  $Z$  boson mass scale. On the other hand, for  $\alpha_1$  and  $\alpha_2$

the three-loop effect is of the same order of magnitude compared to the experimental uncertainty. Similar conclusions hold for  $\mu = 10^{16}$  GeV where a relative shift between two- and three-loop running amounts to 0.012%, 0.027% and 0.004% for  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ , respectively.

To conclude, in this Letter we have computed the three-loop corrections to the gauge coupling beta functions in the Standard Model. Whereas Yukawa corrections are already present at two-loop order the Higgs boson self coupling appears for the first time at three loops. The numerical effect of the new terms is small, however, in the case of  $\alpha_1$  and  $\alpha_2$  it is comparable to the experimental uncertainties.

The method used for the current calculation can also

be applied to the calculation of the beta functions of the Yukawa and Higgs boson self coupling. However, in the case of the Yukawa couplings the issue of  $\gamma_5$  is likely to be more serious. For the Higgs boson self coupling, on the other hand, one has to consider four-point functions and thus a mapping to massless two-point functions introduces infra-red divergences already at one-loop order. A promising method would be to assign a common mass to all fields and set all external momenta to zero (see, e.g., Ref. [31]).

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