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# **CRunDec**: a C++ package for running and decoupling of the strong coupling and quark masses

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## **Abstract**

In this paper we present the C++ package **CRunDec** which implements all relevant formulae needed for the running and decoupling for the strong coupling constant and light quark masses. Furthermore, several formulae are implemented which can be used to transform the heavy quark masses among different renormalization schemes. **CRunDec** is the C++ version on the *Mathematica* package **RunDec** containing several updates and improvements.

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## Program summary

*Title of program:* CRunDec

*Available from:*

<http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp12/ttp12-02/>

*Computer for which the program is designed and others on which it is operable:* Any computer where a C++ compiler is running.

*Operating system or monitor under which the program has been tested:* Linux, Windows

*No. of bytes in distributed program including test data etc.:* 85 000

*Distribution format:* source code

*Keywords:* Quantum Chromodynamics, running coupling constant, running quark mass, on-shell mass,  $\overline{\text{MS}}$  mass, decoupling of heavy particles

*Nature of physical problem:* The value for the coupling constant of Quantum Chromodynamics,  $\alpha_s^{(n_f)}(\mu)$ , depends on the considered energy scale,  $\mu$ , and the number of active quark flavours,  $n_f$ . The same applies to light quark masses,  $m_q^{(n_f)}(\mu)$ , if they are, e.g., evaluated in the  $\overline{\text{MS}}$  scheme. In the program CRunDec all relevant formulae are collected and various procedures are provided which allow for a convenient evaluation of  $\alpha_s^{(n_f)}(\mu)$  and  $m_q^{(n_f)}(\mu)$  using the state-of-the-art correction terms.

*Method of solution:* CRunDec is implemented in C++. For the solution of the differential equations an adaptive Runge-Kutta procedure has been implemented.

*Restrictions on the complexity of the problem:* It could be that for an unphysical choice of the input parameters the results are nonsensical.

*Typical running time:* In general the run time for the individual operations is below a millisecond.

## 1. Introduction

Among the fundamental quantities of Quantum Chromodynamics there are certainly the anomalous dimensions which control the scale dependence of the parameters and fields. In this context a particular role is taken over by the beta function and anomalous mass dimension in the  $\overline{\text{MS}}$  scheme which govern the running of the strong coupling constant  $\alpha_s$  and the quark masses  $m_q$ . Every time a flavour threshold is crossed in the running process decoupling relations have to be applied which guarantee that the heavy quark is integrated out from the theory. More than ten years ago the `Mathematica` package `RunDec` has been written [1] which incorporates all relevant formulae. However, for many application it is more convenient to have the running and decoupling routines available in the framework of

a commonly used programming language and not within a computer algebra system. Thus we have decided to convert the most important routines to C++ adding at the same time new routines and improving others w.r.t. the *Mathematica* version. In what follows we describe the C++ file `CRunDec` which constitutes a C++ class and contains the functions known from `RunDec` as public components.

The use of `CRunDec` does not require any knowledge about object oriented programming. The following skeletons exemplify the usage and can easily be adapted to the problem at hand. It is possible to work with pointers to an object of the type `CRunDec` and access the member functions correspondingly:

```
#include <iostream>
#include "CRunDec.h"
using namespace std;
int main(){
    CRunDec * <pointer> = new CRunDec();
    double <result> = <pointer> -> <function>(<parameters>);
    return(0);
}
```

Alternatively also the following realization is possible:

```
#include <iostream>
#include "CRunDec.h"
using namespace std;
int main(){
    CRunDec <object>;
    double <result> = <object>.<function>(<parameters>);
    return(0);
}
```

Explicit examples will be given below.

The remainder of the paper is organized as follows: In the next Section all available functions are described, Section 3 contains typical examples which exemplify the usage of `CRunDec`.

## 2. Structure of `CRunDec`

All public components of the C++ class `CRunDec` consist of functions which have the same name as the corresponding function in the *Mathematica* version [1]. In the following we list the function header (including the variable names; see also Tab. 1) which — together with the description in the Appendix of Ref. [1] — specifies both the usage and the purpose of the function. There are some additions implemented in `CRunDec` which are also described.

Let us mention that all functions listed in Subsections 2.2–2.4 are overloaded w.r.t. the argument  $n_f$  (number of active flavours). I.e., this argument can be omitted in case  $n_f$  is

symbol in C++ code	mathematical symbol	meaning
asmu, asmu0, asmu1	$\alpha_s(\mu), \alpha_s(\mu_0), \alpha_s(\mu_1)$	strong coupling constant
mq, mq0	$m_q(\mu), m_q(\mu_0)$	$\overline{\text{MS}}$ quark mass
mOS, mMS	$M_q, m_q$	on-shell and $\overline{\text{MS}}$ quark mass
mRI	$m^{\text{RI}}$	regularization invariant mass
mRGI	$\hat{m}$	renormalization group invariant mass
Mth	$M_{\text{th}}$	heavy quark mass
mu, mu0, mu1	$\mu, \mu_0, \mu_1$	renormalization scale
muth	$\mu_{\text{th}}$	decoupling scale
nf	$n_f$	number of active flavours
nloops	—	number of loops

Table 1: Meaning of the variables used in the function headers.

specified in the declaration of the `CRunDec` object (see also examples in Section 3). In this context two auxiliary functions are quite useful: `GetNf()` returns the specified number of active flavours and `SetNf(int nf)` can be used to set the number of active flavours.

### 2.1. Input parameters

For convenience of the user some frequently used input parameters are pre-defined in the file `CRunDec.h` and can be used during the calculation. They read [2, 3, 4, 5]

```
#define asMz 0.1183
#define Mz 91.18
#define Mt 173.2
#define Mb 4.8
#define Mc 1.5
#define muc 1.279
#define mub 4.163
#define Mtau 1.777
```

If not stated otherwise these numbers are used in the examples presented in Section 3. In case other numerical values shall be used it is straightforward to redefine the preprocessor variables. The examples given in the appendix of Ref. [1] can be reproduced with

```
#define asMz 0.118
#define Mz 91.18
#define Mt 175.
#define Mb 4.7
#define Mc 1.6
#define muc 1.2
#define mub 3.97
#define Mtau 1.777
```

## 2.2. Functions related to the running of $\alpha_s$ and $m_q$

- `double LamExpl(double asmu, double mu, int nf, int nloops);`
- `double LamImpl(double asmu, double mu, int nf, int nloops);`
- `double AlphasLam(double Lambda, double mu, int nf, int nloops);`
- `double AlphasExact(double asmu0, double mu0, double mu1,  
int nf, int nloops);`
- `double mMS2mMS(double mq0, double asmu0, double asmu1,  
int nf, int nloops);`
- `AsmMS AsmMSrunexact(double mq0, double asmu0, double mu0,  
double mu, int nf, int nloops);`

The function `AsmMSrunexact` is new in `CRunDec`. It solves simultaneously the differential equations for  $\alpha_s$  and  $m_q$  with initial values  $m_q(\mu_0)$  and  $\alpha_s(\mu_0)$  and  $n_f$  active quark flavours. The return type of `AsmMSrunexact` is a struct with two double components,  $\alpha_s(\mu)$  and  $m_q(\mu)$ . The corresponding code in `CRunDec` looks as follows:

```
struct AsmMS {  
    double Asexact;  
    double mMSexact;  
};
```

For convenience of the user there is a pre-defined variable `AsmMS AM`. Both components of `AM` are initialized to zero when creating a `CRunDec` object.

## 2.3. Functions relating different mass definitions

- `double mOS2mMS(double mOS, double mq[], double asmu, double mu,  
int nf, int nloops);`
- `double mMS2mOS(double mMS, double mq[], double asmu, double mu,  
int nf, int nloops);`
- `double mOS2mMSrun(double mOS, double mq[], double asmu, double mu,  
int nf,int nloops);`
- `double mMS2mOSrun(double mMS, double mq[], double asmu, double mu,  
int nf,int nloops);`
- `double mOS2mMSit(double mOS, double mq[], double asmu, double mu,  
int nf,int nloops);`
- `double mOS2mSI(double mOS, double mq[], double asM,  
int nf, int nloops);`

- `double mMS2mSI(double mMS, double asmu, double mu, int nf, int nloops);`
- `double mMS2mRI(double mMS, double asmu, int nf, int nloops);`
- `double mRI2mMS(double mRI, double asmu, int nf, int nloops);`<sup>1</sup>
- `double mMS2mRGI(double mMS, double asmu, int nf, int nloops);`
- `double mRGI2mMS(double mRGI, double asmu, int nf, int nloops);`
- `double mMS2mRGImod(double mMS, double asmu, int nf, int nloops);`

Note that the light-quark-mass effects can be taken into account with the help of the array `mq[]` which is defined as

```
double mq[4];
```

By default all elements of `mq[]` are zero. In case non-zero values are needed the array has to be filled before the corresponding function is called.

In `CRunDec` the implementation of `mMS2mSI` has been modified as compared to the `Mathematica` version. It is now based on `AsmMSrunexact` and avoids the computation of  $\Lambda_{\text{QCD}}$  in intermediate steps which is perturbatively more stable, in particular for lower renormalization scales. Similar modifications have been performed in `mMS2mOSrun`.

The function `mMS2mRGImod` is new in `CRunDec`. It is defined in analogy to `mMS2mRGI`, however, the more commonly used convention has been adopted where the function  $c(x)$  in Eq. (11) of Ref. [1] is evaluated for  $x = 2\beta_0\alpha_s/\pi$  instead of  $x = \alpha_s/\pi$ .

#### 2.4. Functions related to the decoupling of heavy quarks

As compared to the `Mathematica` version `CRunDec` contains the decoupling relations only for the case of on-shell heavy quarks which are most relevant for the practical purposes. Furthermore, the functions `DecLambdaUp` and `DecLambdaDown` have not been implemented in the `C++` version since it is recommended to use `All2AlH` and `AlH2AlL` in case a flavour threshold is crossed during the running of  $\alpha_s$ .

- `double DecAsDownOS(double asmu, double Mth, double muth, int nf, int nloops);`
- `double DecAsUpOS(double asmu, double Mth, double muth, int nf, int nloops);`
- `double DecMqDownOS(double mq, double asmu, double Mth, double muth, int nf, int nloops);`

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<sup>1</sup>Note a typo in the example to `mRI2mMS` in the Appendix of Ref. [1]: `mRI2mMS[175,0.107,175,6,3]` should read `mRI2mMS[175,0.107,6,3]`.

- `double DecMqUpOS(double mq, double asmu, double Mth, double muth,  
int nf, int nloops);`

In `CRunDec` the functions `DecAsDownOS` and `DecAsUpOS` also contain the four-loop decoupling relations which have been computed in Refs. [6, 7]. Note that in the functions of this subsection the parameters  $n_f$  refers to the number of flavours in the effective theory [1].

### 2.5. Functions related to the combination of running and decoupling

- `double All2A1H(double asmu0, double mu0, TriplenfMmu decpar[],  
double mu1, int nloops);`
- `double A1H2A1L(double asmu0, double mu0, TriplenfMmu decpar[],  
double mu1, int nloops);`
- `double mL2mH(double mq0, double asmu0, double mu0,  
TriplenfMmu decpar[], double mu1, int nloops);`
- `double mH2mL(double mq0, double asmu0, double mu0,  
TriplenfMmu decpar[], double mu1, int nloops);`

The parameters governing the decoupling are contained in the array `decpar[]` where each element contains the triple  $\{n_f, M_{\text{th}}, \mu_{\text{th}}\}$  which is realized in the structure

```
struct TriplenfMmu {
    int nf;
    double Mth;
    double muth;
};
```

There is a pre-defined variable `TriplenfMmu nfMmu[4]`; which can be used when calling the above functions. Note that the components of `decpar` are set to zero at the end of the above functions.

In `CRunDec` we refrain to implement the function `AsRunDec` which automatically determines the number of active flavours for the initial and final energy scale and performs the corresponding running and decoupling steps. In practice it turns out that the decoupling of the heavy quark with mass  $M_{\text{th}}$  at the scale  $\mu_{\text{th}} = M_{\text{th}}$  is not convenient for all applications. Furthermore, there are ambiguities as far as the number of active flavours is concerned in case  $\alpha_s(M_{\text{th}})$  has to be evaluated using `AsRunDec`. Thus, it is recommended to use `All2A1H` and `A1H2A1L` instead.

## 3. Typical examples

In this section we present some typical examples which exemplify the usage of `CRunDec`. In the following we only display the part of the code related to `CRunDec`; the complete programs can be found in the file `example.cc` which comes together with `CRunDec`.

*Running of  $\alpha_s$  from  $M_t$  to  $M_b$  with five active flavours*

It is either possible to create a `CRunDec` object where five active flavours are already specified

```
CRunDec* pObjnf5 = new CRunDec(5);
```

or leave  $n_f$  unspecified

```
CRunDec* pObj = new CRunDec();
```

In the former case  $\alpha_s^{(5)}(M_b)$  is computed from  $\alpha_s^{(5)}(M_t) = 0.108$  via

```
pObjnf5 -> AlphasExact(0.108,Mz,Mb,4);
```

whereas in the latter case one has

```
pObj -> AlphasExact(0.108,Mz,Mb,5,4);
```

In both evaluations four-loop accuracy is assumed leading to the result  $\alpha_s^{(4)}(M_b) = 0.183$ .

The number of active flavours for the object `pObjnf5` can be obtained with the help of `pObjnf5 -> GetNf()`; and `pObj -> SetNf(5)`; sets  $n_f = 5$  for `pObj`.

*Compute  $\alpha_s^{(5)}(M_Z)$  from  $\alpha_s^{(3)}(M_\tau)$*

Assuming a value of the strong coupling as extracted from  $\tau$  decay as  $\alpha_s^{(3)}(M_\tau) = 0.332$  [8] the task is the computation of  $\alpha_s^{(5)}(M_Z)$ . If the decoupling of the charm and bottom quark is performed for  $\mu_{\text{th}} = 2M_c$  and  $\mu_{\text{th}} = M_b$ , respectively, one has to specify

```
CRunDec crundec;  
crundec.nfMmu[0].nf = 4;  
crundec.nfMmu[0].Mth = Mc;  
crundec.nfMmu[0].muth = 2*Mc;  
crundec.nfMmu[1].nf = 5;  
crundec.nfMmu[1].Mth = Mb;  
crundec.nfMmu[1].muth = Mb;
```

Afterwards  $\alpha_s^{(5)}(M_Z)$  is computed via

```
crundec.All2AlH(0.332, Mtau, crundec.nfMmu, Mz, 4);
```

where four-loop accuracy for the running (corresponding to three-loop decoupling relations) has been assumed. As a result one obtains  $\alpha_s^{(5)}(M_Z) = 0.1200$ .



*Compute  $m_b(m_b)$  from  $m_b(10 \text{ GeV})$*

In Ref. [2] the  $\overline{\text{MS}}$  bottom quark mass has been extracted for  $\mu = 10 \text{ GeV}$  as  $m_b(10 \text{ GeV}) = 3.610 \pm 0.016 \text{ GeV}$ . The scale-invariant mass  $m_b(m_b)$  can be computed with four-loop accuracy via

```
CRunDec cmpasmb(5);
double asmu = cmpasmb.AlphasExact(0.1189,Mz,10,4);
cmpasmb.mMS2mSI(3.610,asmu,10,5,4);
```

where  $\alpha_s^{(5)}(M_Z) = 0.1189$  has been used. The result reads  $m_b(m_b) = 4.163 \text{ GeV}$ . In the *Mathematica* version of *RunDec* it is not recommended to use `mMS2mSI` since in intermediate steps  $\Lambda_{\text{QCD}}$  is used and thus the final result does not have the required precision.

*Compute  $m_t(m_t)$  from the top quark on-shell mass*

The on-shell top quark mass has been measured at the *Tevatron* experiments *D0* and *CDF* to  $M_t = 173.2 \pm 0.9 \text{ GeV}$  [4]. If this value shall be transformed to  $m_t(m_t)$  one proceeds in the following way

```
CRunDec cmpmt(6);
double MtOS = 173.2;

cmpmt.nfMmu[0].nf = 6;
cmpmt.nfMmu[0].Mth = MtOS;
cmpmt.nfMmu[0].muth = MtOS;

double as6Mt = cmpmt.All2AlH(asMz,Mz,cmpmt.nfMmu,MtOS,4);
double mtMt = cmpmt.mOS2mMS(MtOS,cmpmt.mq,as6Mt,MtOS,3);
double mtmt = cmpmt.mMS2mSI(mtMt,as6Mt,MtOS,4);
```

The final result reads  $m_t(m_t) = 164.0 \text{ GeV}$ .

*Compute on-shell charm and bottom quark mass from  $m_c(m_c)$  and  $m_b(m_b)$*

After defining a pointer (`pObjnf5`) to a *CRunDec* object with five active flavours (as in the example above) one obtains  $\alpha_s^{(5)}(m_b(m_b))$  via

```
double alpha5mub = pObjnf5 -> AlphasExact(asMz, Mz, mub, 4);
```

and subsequently the on-shell mass  $M_b$  with two and three-loop accuracy with the help of

```
double mbOS2 = pObjnf5 -> mMS2mOS(mub, pObjnf5->mq, alpha5mub, mub, 2);
double mbOS3 = pObjnf5 -> mMS2mOS(mub, pObjnf5->mq, alpha5mub, mub, 3);
```

The results read  $M_b = 4.762 \text{ GeV}$  and  $M_b = 4.909 \text{ GeV}$ , respectively.

In the case of the charm quark one proceeds in analogy by evaluating in a first step  $\alpha_s^{(4)}(m_c(m_c))$

```

pObjnf5 -> nfMmu[0].nf    = 4;
pObjnf5 -> nfMmu[0].Mth  = Mc;
pObjnf5 -> nfMmu[0].muth = 2*Mc;
double alpha4muc = pObjnf5 -> AlH2AlL(asMz, Mz, pObjnf5 -> nfMmu, mub, 4);

```

and afterwards the on-shell mass  $M_c$  to two and three loops

```

double mcOS2 = pObjnf5 -> mMS2mOS(muc, pObjnf5->mq, alpha4muc, muc, 2);
double mcOS3 = pObjnf5 -> mMS2mOS(muc, pObjnf5->mq, alpha4muc, muc, 3);

```

leading to  $M_c = 1.494$  GeV and  $M_c = 1.573$  GeV.

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## References

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