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Moments of heavy-light current correlators up to three loops

Jens Hoff and Matthias Steinhauser

Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT)
76128 Karlsruhe, Germany

Abstract

We consider moments of the non-diagonal vector, axial-vector, scalar and pseudo-scalar current correlators involving two different massive quark flavours up to three-loop accuracy. Expansions around the limits where one mass is zero and the equal-mass case are computed. These results are used to construct approximations valid for arbitrary mass values.

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1 Introduction

Current correlators are building blocks for a number of physical quantities. Among them is the total cross section of hadrons in electron positron annihilation which is obtained from the imaginary part of the vector correlator. The axial-vector correlator leads in an analogue way to contributions to the Z boson decay rate. The scalar and pseudo-scalar correlators govern the decay rate of Higgs bosons with the respective CP property.

An important role in the context of current correlators is taken over by moments obtained from an expansion of the two-point functions for small external momenta. As far as the correlators are concerned where the current couples to the same quark flavour they have been used to extract precise values for the charm and bottom quark masses by comparing the results for the vector correlator up to four-loop order [1–8] with the moments obtained from the experimentally measured cross section (see, e.g., Refs. [9–11]). Also the moments for the other correlators [6–8] can be employed once the experimental data is replaced by precise lattice simulations as has been done in Refs. [12, 13]. Besides the mass values also the strong coupling constant can be extracted from the comparison of moments of the pseudo-scalar currents obtained both in perturbation theory and with the help of lattice gauge theory calculations [12, 13].

The focus of the paper lies on moments of heavy-light current correlators, i.e., two-point functions involving two quark flavours with different masses. Results for the case where one of the masses is zero have been obtained in Refs. [14, 15] and [16, 17] to two- and three-loop order, respectively. These results have been used in [16, 17] to reconstruct, in combination with high-energy expansions and information about the threshold behaviour, approximations valid for all external momenta and quark masses. Applications of these results are corrections for single-top production or decay rates of charged Higgs bosons in extensions of the Standard Model [16, 17]. The three-loop corrections have furthermore been used to obtain precise results for the B and D meson decay constants, f_B and f_D, based on sum rules [18,19]. Also the extraction of f_B and f_D from lattice gauge simulations requires perturbative input. As discussed in Ref. [20] it is particularly desirable to have moments of the heavy-light currents for general values of the two quark masses, not only for the physical masses of the involved quarks in order to perform simulations for a variety of different masses, which then can be extrapolated to the mass values of physical interest. We therefore extend the known three-loop results for the moments of the vector, axial-vector, scalar and pseudo-scalar current correlators where one of the quark masses is zero to the case of two different masses. This is done by computing analytic expansions around two limiting cases, the one for equal masses, the other for one of the masses equal to zero. Subsequently, approximations are constructed which cover the whole range.

In the next Section we provide details on the calculation, the results for the pseudo-scalar correlator are presented in Section 3. We conclude with a summary in Section 4. Long analytic formulae are relegated to the Appendix where also numerical results for the vector, axial-vector and scalar correlator can be found. All results for the moments can be downloaded from [21].

2 Calculation

Let us in a first step present our notation. We define the vector (v), axial-vector (a), scalar (s) and pseudo-scalar (p) currents via

$$\begin{aligned} j^v_\mu &= \bar{\psi}_1 \gamma_\mu \psi_2 \,, \\ j^a_\mu &= \bar{\psi}_1 \gamma_\mu \gamma^5 \psi_2 \,, \\ j^s &= \bar{\psi}_1 \psi_2 \,, \\ j^p &= \bar{\psi}_1 i \gamma^5 \psi_2 \,, \end{aligned} \tag{1}$$

where ψ_1 and ψ_2 denote the two quark flavours. Using these currents the vector and axial-vector correlator is defined through ($\delta = v, a$)

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi^\delta(q^2) + q_\mu q_\nu \, \Pi^\delta_L(q^2) = i \int \mathrm{d}x \, e^{iqx} \langle 0 | T j^\delta_\mu(x) j^{\delta\dagger}_\nu(0) | 0 \rangle \,, \tag{2}$$

where $\Pi^\delta(q^2)$ and $\Pi^\delta_L(q^2)$ are the transverse and longitudinal contributions, respectively. The scalar and pseudo-scalar polarization function reads ($\delta = s, p$)

$$q^2 \, \Pi^\delta(q^2) = i \int \mathrm{d}x \, e^{iqx} \langle 0 | T j^\delta(x) j^{\delta\dagger}(0) | 0 \rangle \,. \tag{3}$$

Throughout this paper we consider anti-commuting γ_5 which is justified as for $\psi_1 \neq \psi_2$ only non-singlet diagrams contribute. As a consequence the axial-vector (pseudo-scalar) correlator coincides with the vector (scalar) one if either m_1 or m_2 is zero. Sample Feynman diagrams occurring at one-, two- and three-loop order are shown in Fig. 1.

It is convenient to introduce the dimensionless variables

$$z = \frac{q^2}{m_1^2} \,, \tag{4}$$

$$x = \frac{m_2}{m_1} \,, \tag{5}$$

which enables us to cast $\Pi^\delta(q^2)$ for $q^2 \to 0$ in the form

$$\bar{\Pi}^\delta(q^2) = \frac{3}{16\pi^2} \sum_{n \geq -1} \bar{C}^\delta_n(x) z^n \,, \tag{6}$$

where the bar indicates that the overall renormalization of the polarization function is performed in the $\overline{\mathrm{MS}}$ scheme. Note that we have the symmetry relations $\bar{C}^s_n(x) = \bar{C}^p_n(-x)$ and $\bar{C}^v_n(x) = \bar{C}^a_n(-x)$ (see, e.g., Ref. [22]).

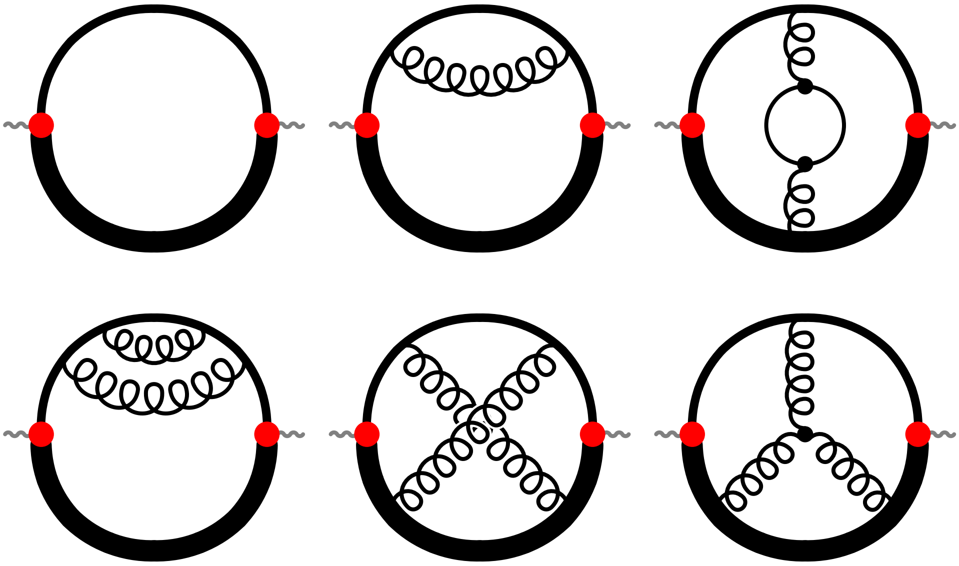

Figure 1: Sample diagrams contributing to Π^δ at one, two and three loops. The thick (lower) and thin (upper) lines correspond to quarks with mass m_1 and m_2, respectively, and the curly lines represent gluons.

We perform an explicit calculation only for the transverse part of the vector and axial-vector current and reconstruct the coefficients for the longitudinal part with the help of ($n = -1, \ldots, 3$)

$$\begin{aligned} \bar{C}^v_{L,n} &= (1-x)^2 \bar{C}^s_{n+1} \,, \\ \bar{C}^a_{L,n} &= (1+x)^2 \bar{C}^p_{n+1} \,. \end{aligned} \tag{7}$$

For $n = -1, 0, 1$ and 2 and six expansion terms in x and $(1-x)$ we have checked that these equations hold.

The expansion of the coefficients in α_s is given by

$$\bar{C}^\delta_n = \bar{C}^{(0),\delta}_n + \frac{\alpha_s}{\pi}\bar{C}^{(1),\delta}_n + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}^{(2),\delta}_n + \ldots \,. \tag{8}$$

where the arguments x and μ are suppressed. Often it is advantageous to require a QED-like normalization $\Pi^\delta(0) = 0$ where the relation to $\bar{\Pi}^\delta(q^2)$ is given by

$$\Pi^\delta(q^2) \;=\; \bar{\Pi}^\delta(q^2) - \frac{3}{16\pi^2}\left(\bar{C}^\delta_0 + \frac{\bar{C}^\delta_{-1}}{z}\right) \,. \tag{9}$$

For the practical calculation we use a well tested set-up which is highly automated in order to avoid errors. All Feynman diagrams are generated with `QGRAF` [23] and the various diagram topologies are identified and transformed to `FORM` [24] with the help of `q2e` and `exp` [25, 26]. `exp` also applies the asymptotic expansion for $x \to 0$ on a diagrammatic level. The expansion around $x = 1$ leads to a naive Taylor series. In a next step the expressions are passed to FORM where appropriate projectors are applied, the expansions in small quantities are performed and traces are taken. Once the analytical expressions for each diagram are reduced to scalar integrals the program MATAD [27] is called in order to handle vacuum integrals up to three loops.

The three-loop calculation requires the inclusion of counterterms up to two-loop order (see, e.g., Ref. [28]) which we implement in the $\overline{\rm MS}$ scheme both for the parameters m_1, m_2 and α_s and the renormalization of the current itself. The latter is equal to the mass renormalization for the scalar and pseudo-scalar case and unity for the vector and axial-vector current. At that point all remaining poles are local and can be subtracted in the $\overline{\rm MS}$ scheme in order to arrive at $\bar{\Pi}^\delta(q^2)$. (See Ref. [22] for details.)

Following this procedure we were able to evaluate the coefficients $\bar{C}_n(x)$ with $n \leq 4$ where terms up to order x^8 and $(1-x)^9$ are included in the expansions around $x = 0$ and $x = 1$, respectively.

3 Results

In this Section we discuss the quality of our expansions in x. At one and two loops we can compare to the exact result which provides confidence concerning the validity of the three-loop approximations. In the main text we restrict ourselves to the pseudo-scalar case. Results for the remaining three correlators are presented in the Appendix and in Ref. [21].

3.1 One- and two-loop results for the pseudo-scalar moments

The exact dependence on x of the one-loop results for the pseudo-scalar moments $\bar{C}_n^p(x)$ with $n \leq 4$ is given by

$$
\begin{aligned}
\bar{C}_{-1}^{(0),p}(x) &= -4\,(1+l_\mu)\left(1-x+x^2\right) + l_x\,\frac{8\,x^3}{1+x}\,,\\
\bar{C}_{0}^{(0),p}(x) &= \frac{1+4\,x+x^2}{(1+x)^2} - l_x\,\frac{4\,x^3\,(2+x)}{(-1+x)\,(1+x)^3} + 2\,l_\mu\,,\\
\bar{C}_{1}^{(0),p}(x) &= \frac{2\,(1-x+x^2)\,(1+4\,x+x^2)}{3\,(-1+x)^2\,(1+x)^4} - l_x\,\frac{8\,x^3}{(-1+x)^3\,(1+x)^5}\,,\\
\bar{C}_{2}^{(0),p}(x) &= \frac{1+4\,x-7\,x^2+40\,x^3-7\,x^4+4\,x^5+x^6}{6\,(-1+x)^4\,(1+x)^6} - l_x\,\frac{4\,x^3\,(2-x+2\,x^2)}{(-1+x)^5\,(1+x)^7}\,,\\
\bar{C}_{3}^{(0),p}(x) &= \frac{1+5\,x-14\,x^2+145\,x^3-94\,x^4+145\,x^5-14\,x^6+5\,x^7+x^8}{15\,(-1+x)^6\,(1+x)^8}\\
&\quad - l_x\,\frac{8\,x^3\,(1-x+3\,x^2-x^3+x^4)}{(-1+x)^7\,(1+x)^9}\,,
\end{aligned}
$$

$$
\begin{aligned}
\bar{C}_4^{(0),p}(x) = & \frac{1+6\,x-23\,x^2+356\,x^3-398\,x^4+956\,x^5}{30\,(-1+x)^8\,(1+x)^{10}} \\
& + \frac{-398\,x^6+356\,x^7-23\,x^8+6\,x^9+x^{10}}{30\,(-1+x)^8\,(1+x)^{10}} \\
& - l_x\,\frac{4\,x^3\,(2-3\,x+12\,x^2-8\,x^3+12\,x^4-3\,x^5+2\,x^6)}{(-1+x)^9\,(1+x)^{11}}\,,
\end{aligned}
\tag{10}
$$

where $l_\mu = \ln(\mu^2/m_1^2)$ and $l_x = \ln(x^2)/2$. These results have been extracted from Ref. [15] where $\Pi^p(q^2)$ is given in analytic form up to two loops. We have checked the results of Eq. (10) evaluating directly the moments both exactly in x and restricting ourselves to expansions in x^n and $(1-x)^n$. Note that the latter method has to be applied at three loops.

In Fig. 2 we discuss $\bar{C}_n^{(0),p}(x)$ for $n = 1, 2, 3$ and 4 and show the curves including x^0, x^7 and x^8 for the small-x expansion as dashed lines where longer dashes correspond to approximations including higher order power corrections. The dotted lines correspond to expansions around the equal-mass case where curves including $(1-x)^0$, $(1-x)^8$ and $(1-x)^9$ (decreasing distance between dots) are shown. The exact result is shown as solid curve. One observes that the approximation including corrections up to order x^8 coincides with the exact result up to $x \approx 0.2$, for $n = 1$ even up to $x \approx 0.3$. On the other hand, the approximation based on the $\mathcal{O}((1-x)^9)$ terms is indistinguishable from the solid line for $x \gtrsim 0.2-0.3$. These limits are close to the ones obtained by considering the approximations including x^7 and $(1-x)^8$ terms only. It is obvious that the small gap in between can easily be closed with the help of a polynomial interpolation.

The exact analytic results for the two-loop moments of the pseudo-scalar current are already quite lengthy. Thus we exemplify the structure by showing $\bar{C}_1^{(1),p}(x)$ in the main text. The results for $\bar{C}_n^{(1),p}(x)$ for $n \neq 1$ can be found in Appendix A.1. For $\bar{C}_1^{(1),p}(x)$ we have

$$
\begin{aligned}
\bar{C}_1^{(1),p}(x) = & \frac{1+36\,x-22\,x^2+36\,x^3+x^4}{9\,(-1+x)^2\,(1+x)^4} + \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right]\frac{8\,(1+3\,x+x^2)}{9\,(-1+x)\,(1+x)^3} \\
& + \frac{16\,l_x\,x^2\,(1+6\,x+x^2)}{9\,(-1+x)^3\,(1+x)^5} - \frac{16\,l_x^2\,x^3\,(18+9\,x^2-2\,x^3+3\,x^4+x^5)}{9\,(-1+x)^4\,(1+x)^6}\,,
\end{aligned}
\tag{11}
$$

where $l_{1-x^2} = \ln(1-x^2)$ and $\zeta_2 = \pi^2/6$. Note that in Eq. (11) the coefficient of $\ln(\mu^2/m_1^2)$ is zero. Using the same notation as in Fig. 2 we show in Fig. 3 the results for $\bar{C}_n^{(1),p}(x)$. The same conclusions as in the one-loop case can be drawn: The expansion for $x \to 0$ provides reliable results for $x \lesssim 0.2$ whereas the expansion around $x = 1$ is trustworthy for $x \gtrsim 0.2-0.4$. This is true both for the approximations containing the ninth and the ones containing only the eighth order in x and $1-x$, respectively. Thus it is promising to proceed at three-loop level, where the computation of the exact dependence on x is quite complex, in the following way: using the combination of computer programs described in Section 2 it is straightforward to compute expansions in x and $(1-x)$. Afterwards we combine the expansions applying a simple interpolation.

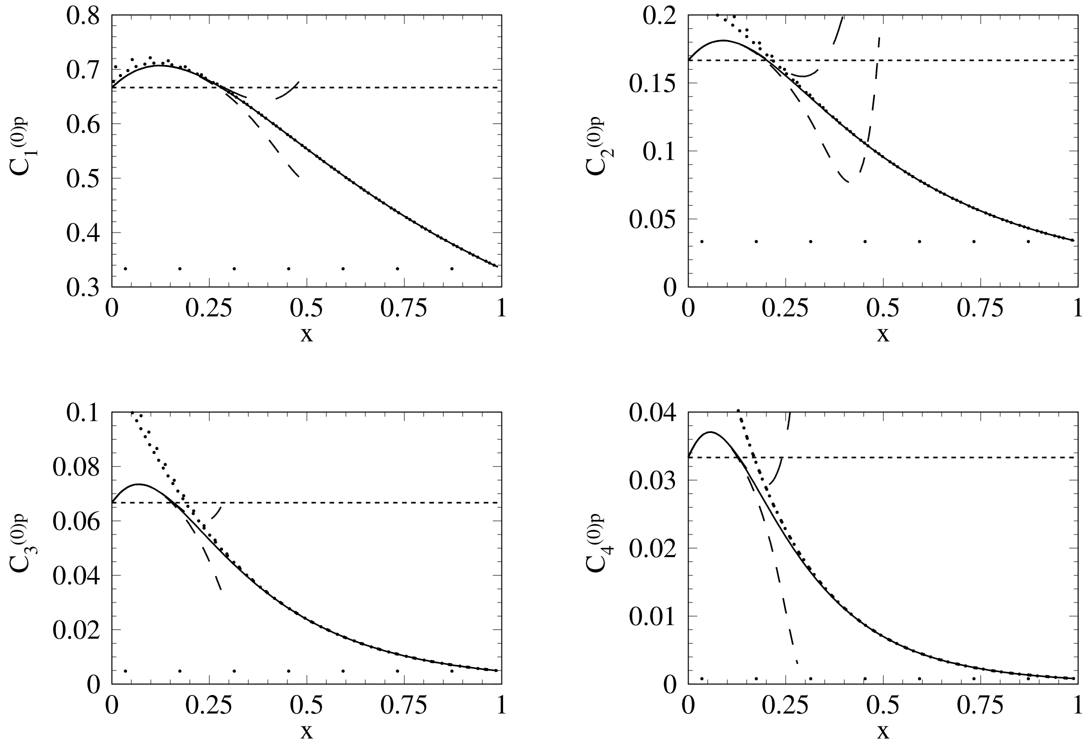

Figure 2: One-loop contribution to $\bar{C}_n^p$. The dashed and dotted lines correspond to the expansions in x and $(1-x)$, respectively, where the lowest and the two highest approximations are shown. The solid line represents the exact result. Note that for $n=4$ the curve including terms up to order $(1-x)^8$ and the one including terms up to order $(1-x)^9$ lie on top of each other in the shown interval.

The approximations for moments of the vector, axial-vector and scalar correlators have the same quality as in the pseudo-scalar case, i.e. they approximate the exact result in the same regions of x. We refrain from an explicit discussion in this paper, however, provide the figures corresponding Fig. 2 and 3 in Appendix B.

3.2 Three-loop results

Let us in a first step present the expansions around $x=0$ and $x=1$. Since they are quite lengthy we show in the main part of the paper only results for the first moment adopting $\mu^2 = m_1^2$ to exemplify the structure of the result. Furthermore, we restrict ourselves to the first three terms in the expansion. The complete expressions for general μ and the results for the moments with $n = -1, 0, 2, 3$ and 4 are provided in the `Mathematica` file [21]. For $x \to 0$ we have

$$\bar{C}_1^{(2),p}(x) = \frac{35}{16} + \frac{1055}{162}\,\zeta_2 - \frac{88}{9}\,\zeta_3 - \frac{134}{27}\,\zeta_4 + \frac{2}{27}\,D_3 + \frac{105}{4}\,S_2$$

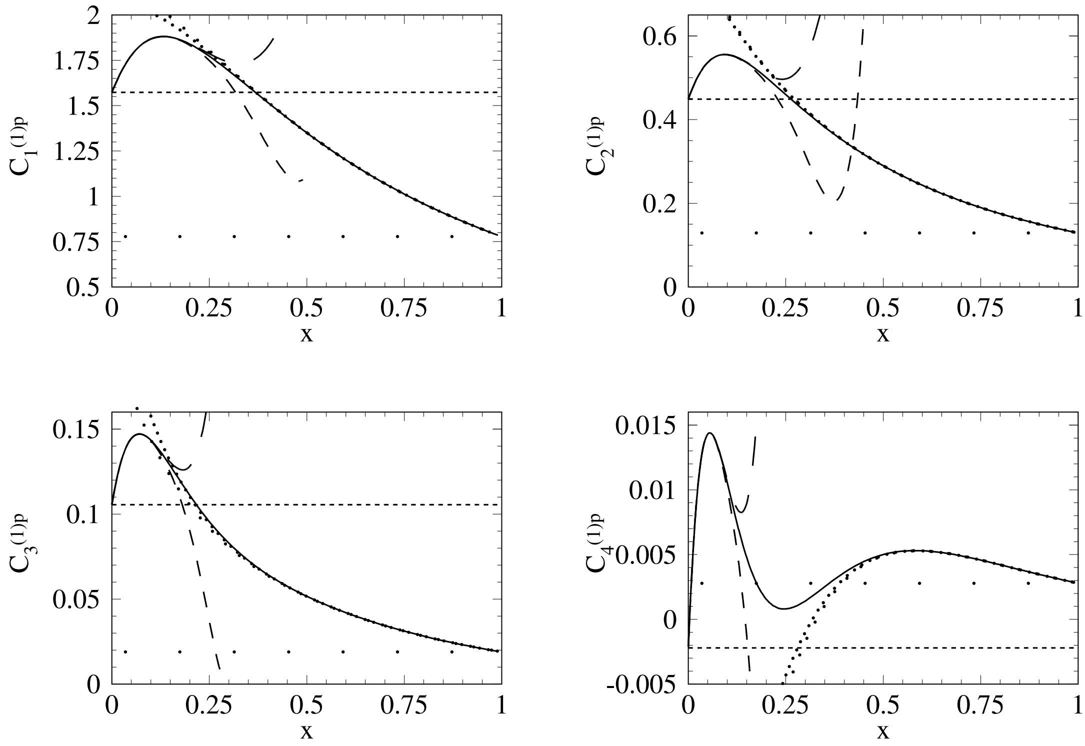

Figure 3: Two-loop contribution to $\bar{C}_n^p$. The same notation as in Fig. 2 is adopted.

$$
\begin{aligned}
&+ x\left(\frac{13465}{972} + \frac{4493}{243}\,\zeta_2 + \frac{409}{27}\,\zeta_3 - \frac{137}{27}\,\zeta_4 + \frac{D_3}{9} - \frac{39}{4}\,S_2\right)\\
&+ x^2\left(-\frac{176603}{3888} - \frac{12557}{486}\,\zeta_2 - \frac{5564}{81}\,\zeta_3 + \frac{94}{27}\,\zeta_4 - \frac{10}{27}\,D_3 + \frac{761}{4}\,S_2\right)\\
&+ x^3\left(\frac{163867}{1944} + \frac{401977}{1620}\,\zeta_2 + \frac{221}{81}\,\zeta_3 + \frac{5327}{324}\,\zeta_4\right.\\
&\qquad\left. + \frac{50}{27}\,D_3 - \frac{2228}{243}\,S_2^\epsilon + 600\,S_2 + \frac{1114}{81}\,T_1^\epsilon\right)\\
&+ l_x\,x^3\left(\frac{2368}{27} - \frac{1114}{3}S_2 - \frac{80}{27}\,\zeta_2 - \frac{16}{3}\,\zeta_3\right) - l_x^2\,\frac{1024\,x^3}{9} + l_x^3\,\frac{1264\,x^3}{9}\\
&+ n_l\left[-\frac{5}{72} - \frac{\zeta_2}{9} + \frac{8}{27}\,\zeta_3 + x\left(-\frac{55}{36} - \frac{8}{27}\,\zeta_2 + \frac{8}{27}\,\zeta_3\right)\right.\\
&\qquad + x^2\left(\frac{823}{216} + \frac{\zeta_2}{9} - \frac{8}{27}\,\zeta_3\right) + x^3\left(-\frac{571}{54} + \frac{28}{9}\,\zeta_2 + \frac{176}{27}\,\zeta_3\right)\\
&\qquad\left. + l_x\,x^3\left(\frac{88}{27} + \frac{16}{3}\,\zeta_2\right) + l_x^2\,\frac{80\,x^3}{9} - l_x^3\,\frac{32\,x^3}{9}\right] + \mathcal{O}(x^4)\,, \qquad (12)
\end{aligned}
$$

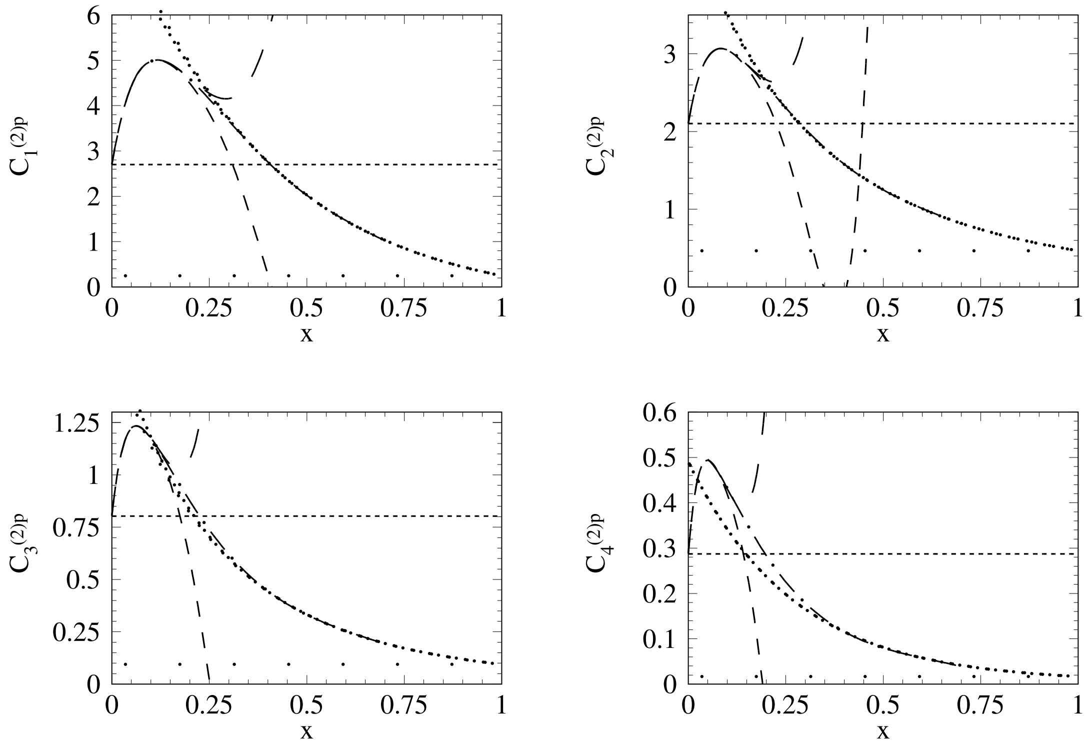

Figure 4: Three-loop contribution to $\bar{C}_n^p$. The dash-doted line represents the result based on the fit as described in the text. For the rest the same notation as in Fig. 2 is adopted.

where $D_3 \approx -3.0270$, $S_2 \approx 0.2604$, $S_2^\epsilon \approx 7.8517$, $T_1^\epsilon \approx -24.2089$ [27] and $\zeta_3 \approx 1.2021$. n_l is the number of massless quarks and the total number of quarks is $n_f = n_l + 2$. For $x \to 1$ one obtains

$$\begin{aligned}
\bar{C}_1^{(2),p}(x) = \; & -\frac{13139}{2592} + \frac{6977}{1728}\zeta_3 + (1-x)^3\left(\frac{14229845}{93312} - \frac{13342139}{103680}\zeta_3\right) \\
& + (1-x)\left(\frac{8267}{2592} - \frac{6977}{1728}\zeta_3\right) + (1-x)^2\left(-\frac{72234997}{933120} + \frac{13760759}{207360}\zeta_3\right) \\
& + n_l\left[\frac{25}{162} - \frac{2}{81}(1-x) - \frac{289}{4860}(1-x)^2 + \frac{409}{4860}(1-x)^3\right] + \mathcal{O}\left((1-x)^4\right) .
\end{aligned} \tag{13}$$

In Fig. 4 we show $\bar{C}_n^{(2),p}$ as a function of x using the same notation as at one and two loops. We have chosen $n_l = 3$ which corresponds to a massive charm and bottom quark. We again observe that the approximations based on the x^7 terms and the one including x^8 terms coincide up to $x \approx 0.1 - 0.2$. Furthermore, also for the expansion around the equal-mass case the two highest approximations overlap for $x \gtrsim 0.2 - 0.3$, depending on n. From the experience gained from the one- and two-loop considerations we can conclude that in these regions our expressions constitute perfect approximations to the exact result. Furthermore, it is again straightforward to obtain approximations which are valid for all x values by a simple interpolation. In order to obtain handy expressions we perform a polynomial fit which is valid for the intermediate region $x \in [0.1, 0.5]$. Allowing also for half-integer exponents of x we obtain

$$\begin{aligned}
\bar{C}^{(2),p}_{1,\mathrm{appr}}(x)\Big|_{\mu^2=m_1^2} &= 0.382 + 22.191\,x^{1/2} + 5.126\,x - 138.613\,x^{3/2} + 149.094\,x^2 - 39.042\,x^3\,,\\
\bar{C}^{(2),p}_{2,\mathrm{appr}}(x)\Big|_{\mu^2=m_1^2} &= 0.284 + 21.369\,x^{1/2} - 45.897\,x + 11.032\,x^{3/2} + 26.207\,x^2 - 13.199\,x^3\,,\\
\bar{C}^{(2),p}_{3,\mathrm{appr}}(x)\Big|_{\mu^2=m_1^2} &= -0.307 + 15.358\,x^{1/2} - 50.366\,x + 61.164\,x^{3/2} - 27.39\,x^2 + 1.475\,x^3\,,\\
\bar{C}^{(2),p}_{4,\mathrm{appr}}(x)\Big|_{\mu^2=m_1^2} &= 0.129 + 4.414\,x^{1/2} - 16.562\,x + 20.537\,x^{3/2} - 8.586\,x^2 - 0.031\,x^3\,. \qquad (14)
\end{aligned}$$

These results are valid for $n_l = 3$ and are shown as dash-dotted lines in Fig. 4. We applied the same procedure to the one- and two-loop moments. The comparison with the exact result provides an estimate for the accuracy of the approximations in Eqs. (14) to better than 5% for $x \in [0.1, 0.5]$. For $x < 0.1$ or $x > 0.5$ the corresponding expansions should be used for the evaluation of the moments since in these regions they provide excellent approximations to the (unknown) exact result. Analogue results for the vector, axial-vector and scalar correlators can be found in [21] where again $n_l = 3$ has been chosen. If necessary it is straightforward to obtain formulae for other values for n_l.

4 Summary

We have computed three-loop QCD corrections to the moments of the current correlators formed by quark fields with different masses. Our final expressions are based on expansions around the known limits where either one mass is zero or both masses are equal. We present results which are valid for arbitrary quark masses by combining the expansions and a simple polynomial fit in the intermediate region.

In the main part of the paper we concentrate on the discussion of the pseudo-scalar correlator. However, numerical results for the vector, axial-vector and scalar correlator can be found in the Appendix and analytical expressions are provided in a `Mathematica` file which comes together with this paper.

One- and two-loop results with exact quark mass dependence and analytic results for the expansions at three-loop order are presented in `Mathematica` format on the internet page [21]. Also the exact renormalization scale dependence of the three-loop moments are provided and numerical approximations valid for arbitrary m_2/m_1.

The results obtained in this paper constitute important input for lattice calculations in the context of semileptonic and leptonic B meson decays. For such simulations non-perturbative Z factors for heavy-light currents are needed which can be extracted from the corresponding current-current correlators. Whereas in the case of the B_s meson expansions for small strange quark mass might be sufficient this is not true in the case of B_c since $m_c/m_b \approx 0.23$ is in a region where the small-mass expansion already breaks down.

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A Results for the moments of the pseudo-scalar correlator

In this Appendix we want to complete the results for the moments of the pseudo-scalar correlator. The one-loop expressions can be found in Eq. (10) and $\bar{C}_1^{(1),p}(x)$ is given in Eq. (11). In Subsection A.1 we provide the results for $\bar{C}_n^{(1),p}(x)$ with $n = -1, 0, 2, 3$ and 4. As far as the three-loop results are concerned we list in Subsection A.2 the exact x-dependence of the μ-dependent terms which complement the results in Eq. (14).

A.1 Exact two-loop results

$$
\begin{aligned}
\bar{C}_{-1}^{(1),p}(x) =& -\frac{40\left(1-x+x^2\right)}{3} + l_x \frac{80\,x^3}{3\,(1+x)} - l_x^2 \frac{32\,x^3}{1+x} \\
&+ l_\mu \left[-\frac{40\left(1-x+x^2\right)}{3} + l_x \frac{32\,x^3}{1+x}\right] - 8\,l_\mu^2\left(1-x+x^2\right)\,, \qquad (15)\\
\bar{C}_{0}^{(1),p}(x) =& \frac{13+66\,x+13\,x^2}{6\,(1+x)^2} - \left[\mathrm{Li}_2\left(x^2\right) + 2\,l_x\,l_{1-x^2} - \zeta_2\right]\frac{4\,(-1+x)}{3\,(1+x)} \\
&- l_x \frac{8\,x^2\left(1+5\,x+x^2\right)}{3\,(-1+x)\,(1+x)^3} + l_x^2 \frac{16\,x^3\left(-6-3\,x+3\,x^2+2\,x^3\right)}{3\,(-1+x)^2\,(1+x)^4} \\
&+ l_\mu\left[\frac{4\left(1+5\,x+x^2\right)}{3\,(1+x)^2} - l_x \frac{8\,x^3\,(2+x)}{(-1+x)\,(1+x)^3}\right] + 2\,l_\mu^2\,, \qquad (16)\\
\bar{C}_{2}^{(1),p}(x) =& \frac{3+72\,x-127\,x^2+112\,x^3-127\,x^4+72\,x^5+3\,x^6}{36\,(-1+x)^4\,(1+x)^6} \\
&+ l_x \frac{2\,x^2\left(2+122\,x-27\,x^2+230\,x^3-22\,x^4+12\,x^5+3\,x^6\right)}{9\,(-1+x)^5\,(1+x)^7}
\end{aligned}
$$

$$
\begin{aligned}
& - l_x^2\, \frac{4\,x^3\left(72-36\,x+216\,x^2-68\,x^3+136\,x^4-3\,x^5+4\,x^6+x^7\right)}{9\left(-1+x\right)^6\left(1+x\right)^8} \\
& + l_\mu \left[-\frac{1+4\,x-7\,x^2+40\,x^3-7\,x^4+4\,x^5+x^6}{3\left(-1+x\right)^4\left(1+x\right)^6} + l_x\, \frac{8\,x^3\left(2-x+2\,x^2\right)}{\left(-1+x\right)^5\left(1+x\right)^7}\right] \\
& + \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right] \frac{2\left(1+4\,x+4\,x^3+x^4\right)}{9\left(-1+x\right)^3\left(1+x\right)^5}\,, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\bar{C}_3^{(1),p}(x) = & - \frac{11-219\,x+658\,x^2+1479\,x^3+1326\,x^4+1479\,x^5+658\,x^6-219\,x^7+11\,x^8}{270\left(-1+x\right)^6\left(1+x\right)^8} \\
& + l_x \left[\frac{8\,x^2\left(1+263\,x-184\,x^2+1109\,x^3-424\,x^4\right)}{45\left(-1+x\right)^7\left(1+x\right)^9}\right. \\
& \quad \left. + \frac{8\,x^2\left(683\,x^5-44\,x^6+15\,x^7+3\,x^8\right)}{45\left(-1+x\right)^7\left(1+x\right)^9}\right] \\
& - l_x^2 \left[\frac{8\,x^3\left(180-180\,x+1260\,x^2-800\,x^3+1940\,x^4\right)}{45\left(-1+x\right)^8\left(1+x\right)^{10}}\right. \\
& \quad \left. + \frac{8\,x^3\left(-535\,x^5+545\,x^6-4\,x^7+5\,x^8+x^9\right)}{45\left(-1+x\right)^8\left(1+x\right)^{10}}\right] \\
& + \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right] \frac{4\,\left(1+5\,x-x^2+20\,x^3-x^4+5\,x^5+x^6\right)}{45\left(-1+x\right)^5\left(1+x\right)^7} \\
& + l_\mu \left[l_x\, \frac{32\,x^3\left(1-x+3\,x^2-x^3+x^4\right)}{\left(-1+x\right)^7\left(1+x\right)^9}\right. \\
& \quad \left. - \frac{4\left(1+5\,x-14\,x^2+145\,x^3-94\,x^4+145\,x^5-14\,x^6+5\,x^7+x^8\right)}{15\left(-1+x\right)^6\left(1+x\right)^8}\right]\,, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\bar{C}_4^{(1),p}(x) = & - \frac{61-201\,x+978\,x^2+19344\,x^3-7591\,x^4+51570\,x^5}{810\left(-1+x\right)^8\left(1+x\right)^{10}} \\
& - \frac{-7591\,x^6+19344\,x^7+978\,x^8-201\,x^9+61\,x^{10}}{810\left(-1+x\right)^8\left(1+x\right)^{10}} \\
& + l_x \left[\frac{2\,x^2\left(2+1512\,x-1780\,x^2+11358\,x^3-7758\,x^4+16758\,x^5\right)}{45\left(-1+x\right)^9\left(1+x\right)^{11}}\right. \\
& \quad \left. + \frac{2\,x^2\left(-5155\,x^6+4662\,x^7-214\,x^8+54\,x^9+9\,x^{10}\right)}{45\left(-1+x\right)^9\left(1+x\right)^{11}}\right] \\
& - l_x^2 \left[\frac{4\,x^3\left(360-540\,x+4500\,x^2-4400\,x^3+13200\,x^4-7055\,x^5\right)}{45\left(-1+x\right)^{10}\left(1+x\right)^{12}}\right. \\
& \quad \left. + \frac{4\,x^3\left(9810\,x^6-2169\,x^7+1476\,x^8-5\,x^9+6\,x^{10}+x^{11}\right)}{45\left(-1+x\right)^{10}\left(1+x\right)^{12}}\right] \\
& + \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right] \\
& \quad \frac{2\,\left(1+6\,x-2\,x^2+54\,x^3-18\,x^4+54\,x^5-2\,x^6+6\,x^7+x^8\right)}{45\left(-1+x\right)^7\left(1+x\right)^9}
\end{aligned}
$$

$$+ l_\mu \left[-\frac{1+6\,x-23\,x^2+356\,x^3-398\,x^4+956\,x^5}{5\,(-1+x)^8\,(1+x)^{10}} \right.$$
$$- \frac{-398\,x^6+356\,x^7-23\,x^8+6\,x^9+x^{10}}{5\,(-1+x)^8\,(1+x)^{10}}$$
$$\left. +l_x\,\frac{24\,x^3\left(2-3\,x+12\,x^2-8\,x^3+12\,x^4-3\,x^5+2\,x^6\right)}{(-1+x)^9\,(1+x)^{11}} \right] . \tag{19}$$

A.2 Analytic three-loop results

Let us decompose the the three-loop coefficients as

$$\bar{C}_n^{(2),p}(x) \;=\; \bar{C}_n^{(2),p}(x)\Big|_{\mu^2=m_1^2} + l_\mu \bar{C}_{n,1}^{(2),p}(x) + l_\mu^2 \bar{C}_{n,2}^{(2),p}(x)\,, \tag{20}$$

with $l_\mu = \ln(\mu^2/m_1^2)$. For $n_l = 3$ one can find approximations for $\bar{C}_n^{(2),p}(x)\big|_{\mu^2=m_1^2}$ in Eq. (14). The exact x-dependence of $\bar{C}_{n,1}^{(2),p}(x)$ and $\bar{C}_{n,2}^{(2),p}(x)$, which can be obtained using renormalization group techniques, reads

$$\bar{C}_{1,1}^{(2),p}(x) = \left(-\frac{29}{2}+n_l\right)\left\{-\frac{1+36\,x-22\,x^2+36\,x^3+x^4}{54\,(-1+x)^2\,(1+x)^4}\right.$$
$$-\left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right]\frac{4\left(1+3\,x+x^2\right)}{27\,(-1+x)\,(1+x)^3}$$
$$\left. -l_x\,\frac{8\,x^2\left(1+6\,x+x^2\right)}{27\,(-1+x)^3\,(1+x)^5} + l_x^2\,\frac{8\,x^3\left(18+9\,x^2-2\,x^3+3\,x^4+x^5\right)}{27\,(-1+x)^4\,(1+x)^6}\right\} , \tag{21}$$

$$\bar{C}_{1,2}^{(2),p}(x) = 0\,,$$

$$\bar{C}_{2,1}^{(2),p}(x) = -\frac{19\left(1+4\,x-7\,x^2+40\,x^3-7\,x^4+4\,x^5+x^6\right)}{36\,(-1+x)^4\,(1+x)^6} + l_x\,\frac{38\,x^3\left(2-x+2\,x^2\right)}{3\,(-1+x)^5\,(1+x)^7}$$
$$+\left(-\frac{5}{2}+n_l\right)\left\{\frac{7-32\,x+57\,x^2+288\,x^3+57\,x^4-32\,x^5+7\,x^6}{216\,(-1+x)^4\,(1+x)^6}\right.$$
$$-\left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right]\frac{1+4\,x+4\,x^3+x^4}{27\,(-1+x)^3\,(1+x)^5}$$
$$-\,l_x\,\frac{x^2\left(2+182\,x-57\,x^2+290\,x^3-22\,x^4+12\,x^5+3\,x^6\right)}{27\,(-1+x)^5\,(1+x)^7}$$
$$\left. +l_x^2\,\frac{2\,x^3\left(72-36\,x+216\,x^2-68\,x^3+136\,x^4-3\,x^5+4\,x^6+x^7\right)}{27\,(-1+x)^6\,(1+x)^8}\right\} , \tag{22}$$

$$\bar{C}_{2,2}^{(2),p}(x) = \left(-\frac{5}{2}+n_l\right)\left[\frac{1+4\,x-7\,x^2+40\,x^3-7\,x^4+4\,x^5+x^6}{36\,(-1+x)^4\,(1+x)^6}\right.$$
$$\left. -l_x\,\frac{2\,x^3\left(2-x+2\,x^2\right)}{3\,(-1+x)^5\,(1+x)^7}\right] , \tag{23}$$

$$
\begin{aligned}
\bar{C}^{(2),p}_{3,1}(x) = & -\frac{13\left(1+5\,x-14\,x^2+145\,x^3-94\,x^4+145\,x^5-14\,x^6+5\,x^7+x^8\right)}{15\,(-1+x)^6\,(1+x)^8} \\
& + l_x\,\frac{104\,x^3\left(1-x+3\,x^2-x^3+x^4\right)}{(-1+x)^7\,(1+x)^9} + \left(\frac{19}{2}+n_l\right) \\
& \left\{\frac{71+81\,x-182\,x^2+10179\,x^3-4314\,x^4+10179\,x^5-182\,x^6+81\,x^7+71\,x^8}{1620\,(-1+x)^6\,(1+x)^8}\right. \\
& \quad - \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right]\frac{2\left(1+5\,x-x^2+20\,x^3-x^4+5\,x^5+x^6\right)}{135\,(-1+x)^5\,(1+x)^7} \\
& \quad - l_x\,\frac{4\,x^2\left(1+413\,x-334\,x^2+1559\,x^3-574\,x^4+833\,x^5-44\,x^6+15\,x^7+3\,x^8\right)}{135\,(-1+x)^7\,(1+x)^9} \\
& \quad \left. +l_x^2\,\frac{4\,x^3\left(180-180\,x+1260\,x^2-800\,x^3+1940\,x^4-535\,x^5+545\,x^6-4\,x^7+5\,x^8+x^9\right)}{135\,(-1+x)^8\,(1+x)^{10}}\right\},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\bar{C}^{(2),p}_{3,2}(x) = & \left(\frac{19}{2}+n_l\right)\left[\frac{1+5\,x-14\,x^2+145\,x^3-94\,x^4+145\,x^5-14\,x^6+5\,x^7+x^8}{45\,(-1+x)^6\,(1+x)^8}\right. \\
& \quad \left. -l_x\,\frac{8\,x^3\left(1-x+3\,x^2-x^3+x^4\right)}{3\,(-1+x)^7\,(1+x)^9}\right],
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\bar{C}^{(2),p}_{4,1}(x) = & -\frac{59\left(1+6\,x-23\,x^2+356\,x^3-398\,x^4+956\,x^5-398\,x^6+356\,x^7-23\,x^8+6\,x^9+x^{10}\right)}{60\,(-1+x)^8\,(1+x)^{10}} \\
& + l_x\,\frac{118\,x^3\left(2-3\,x+12\,x^2-8\,x^3+12\,x^4-3\,x^5+2\,x^6\right)}{(-1+x)^9\,(1+x)^{11}} + \left(\frac{43}{2}+n_l\right) \\
& \left\{\frac{196+609\,x-2127\,x^2+67404\,x^3-61321\,x^4+180630\,x^5}{4860\,(-1+x)^8\,(1+x)^{10}}\right. \\
& \quad + \frac{-61321\,x^6+67404\,x^7-2127\,x^8+609\,x^9+196\,x^{10}}{4860\,(-1+x)^8\,(1+x)^{10}} \\
& \quad - \left[\mathrm{Li}_2\left(x^2\right)+2\,l_x\,l_{1-x^2}-\zeta_2\right]\frac{1+6\,x-2\,x^2+54\,x^3-18\,x^4+54\,x^5-2\,x^6+6\,x^7+x^8}{135\,(-1+x)^7\,(1+x)^9} \\
& \quad - l_x\,x^2\left[\frac{2+2412\,x-3130\,x^2+16758\,x^3-11358\,x^4+22158\,x^5}{135\,(-1+x)^9\,(1+x)^{11}}\right. \\
& \qquad \left. +\frac{-6505\,x^6+5562\,x^7-214\,x^8+54\,x^9+9\,x^{10}}{135\,(-1+x)^9\,(1+x)^{11}}\right] \\
& \quad + 2\,l_x^2\,x^3\left[\frac{360-540\,x+4500\,x^2-4400\,x^3+13200\,x^4-7055\,x^5}{135\,(-1+x)^{10}\,(1+x)^{12}}\right. \\
& \qquad \left.\left. +\frac{9810\,x^6-2169\,x^7+1476\,x^8-5\,x^9+6\,x^{10}+x^{11}}{135\,(-1+x)^{10}\,(1+x)^{12}}\right]\right\},
\end{aligned} \tag{26}
$$

$$
\bar{C}^{(2),p}_{4,2}(x) = \left(\frac{43}{2}+n_l\right)\left[-\frac{2\,l_x\,x^3\left(2-3\,x+12\,x^2-8\,x^3+12\,x^4-3\,x^5+2\,x^6\right)}{(-1+x)^9\,(1+x)^{11}}\right.
$$

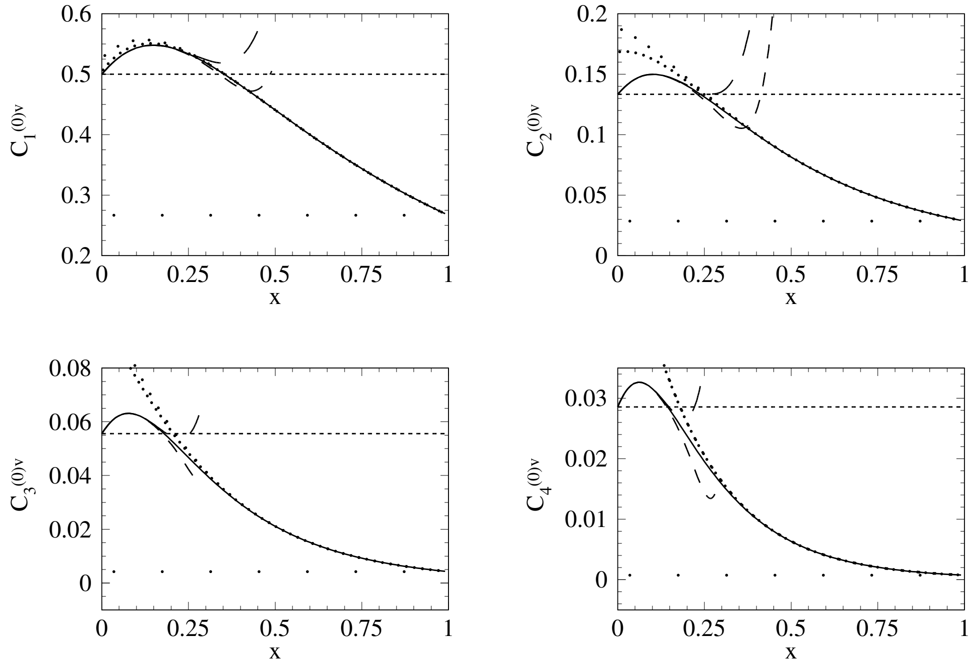

Figure 5: One-loop contribution to $\bar{C}_n^v$. The same notation as in Fig. 2 is adopted.

$$+\frac{1+6\,x-23\,x^2+356\,x^3-398\,x^4+956\,x^5-398\,x^6+356\,x^7-23\,x^8+6\,x^9+x^{10}}{60\,(-1+x)^8\,(1+x)^{10}}\Bigg]\,. \tag{27}$$

B Results for vector, axial-vector and scalar correlator

In this appendix we present the results for the moments $\bar{C}_n^{\delta}(x)$ for $n = 1,\ldots,4$ of the vector, axial-vector and scalar correlator. For the analytic results and numerical approximations for the three-loop moments in the intermediate x-region we refer to [21] and show in the following the corresponding numerical result in analogy to Figs. 2, 3 and 4.

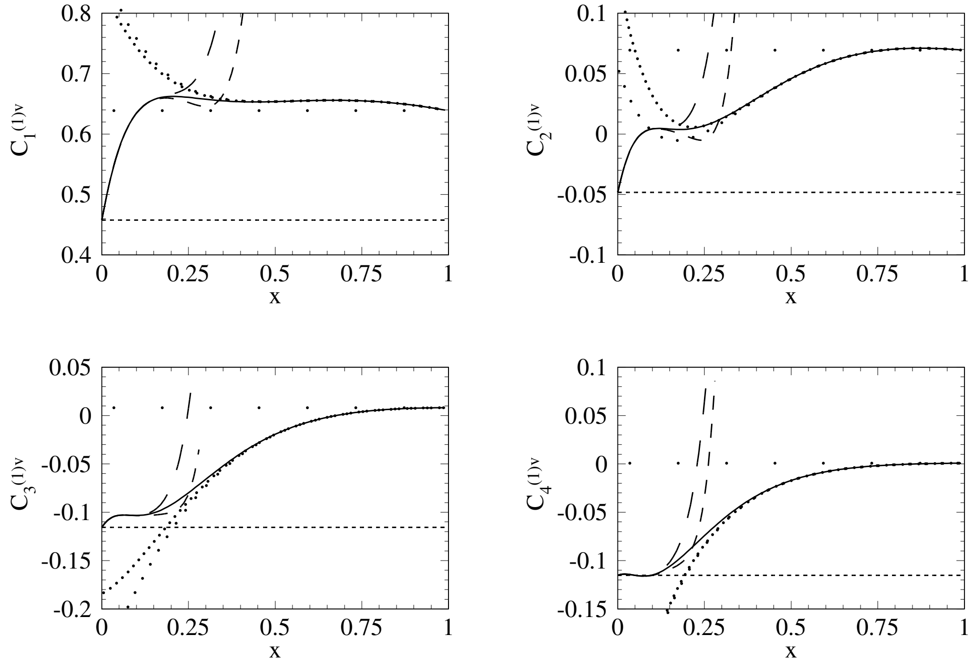

Figure 6: Two-loop contribution to $\bar{C}_n^v$. The same notation as in Fig. 2 is adopted.

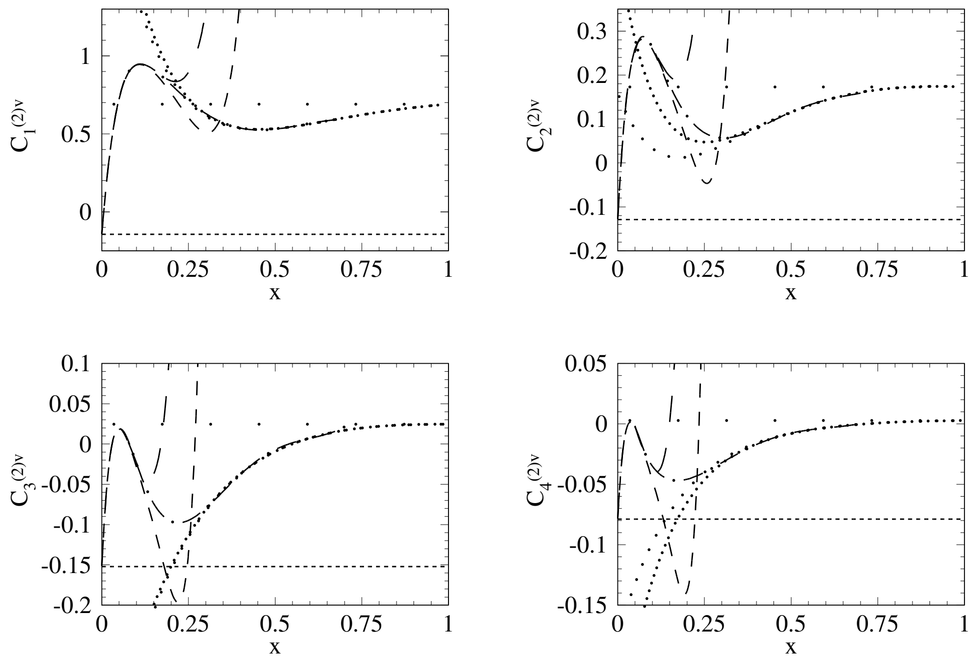

Figure 7: Three-loop contribution to $\bar{C}_n^v$. The same notation as in Fig. 4 is adopted.

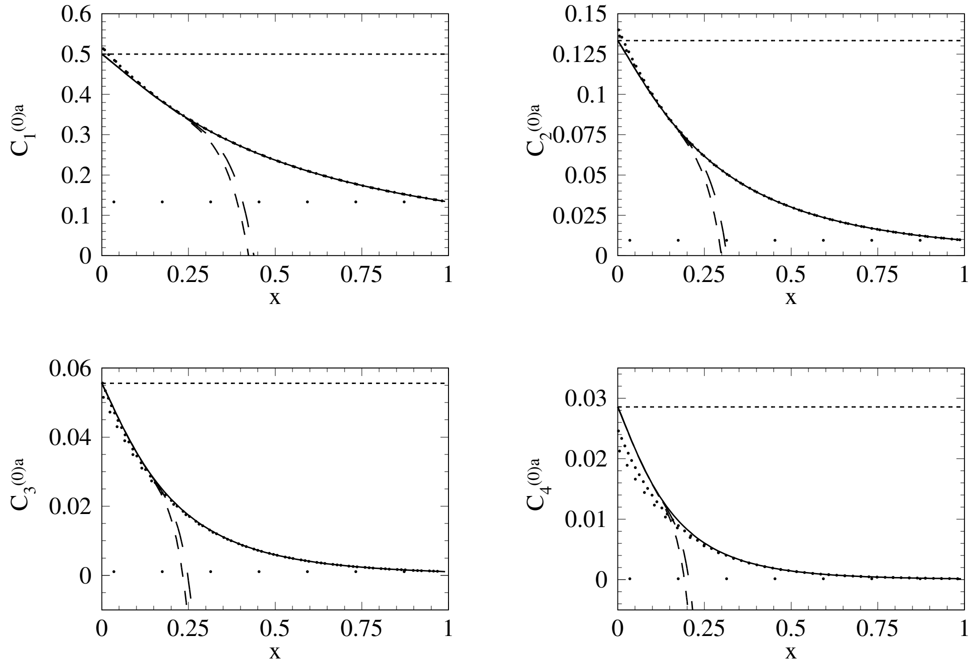

Figure 8: One-loop contribution to $\bar{C}_n^a$. The same notation as in Fig. 2 is adopted.

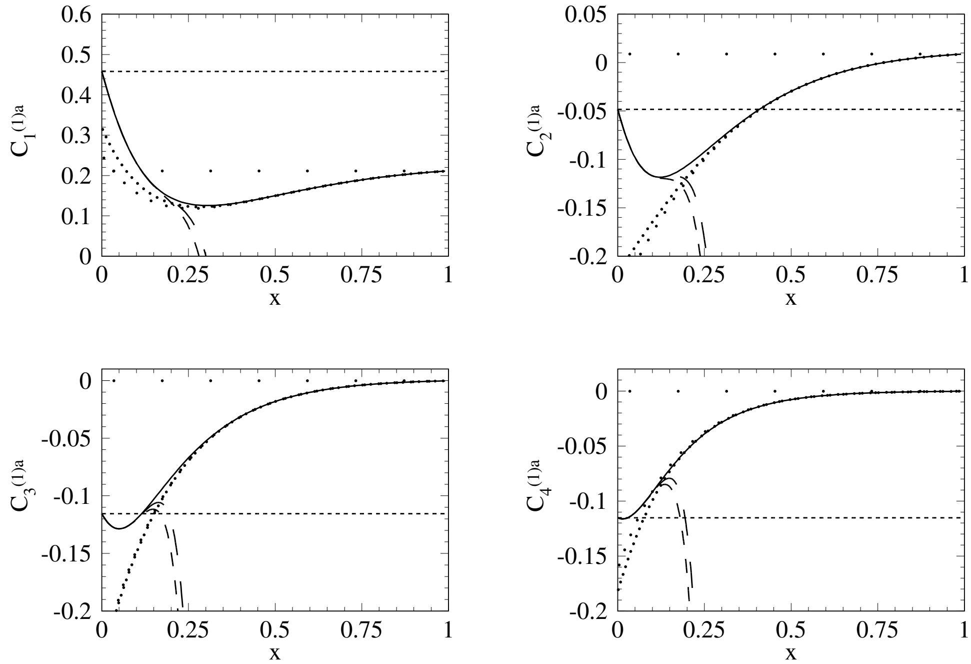

Figure 9: Two-loop contribution to $\bar{C}_n^a$. The same notation as in Fig. 2 is adopted.

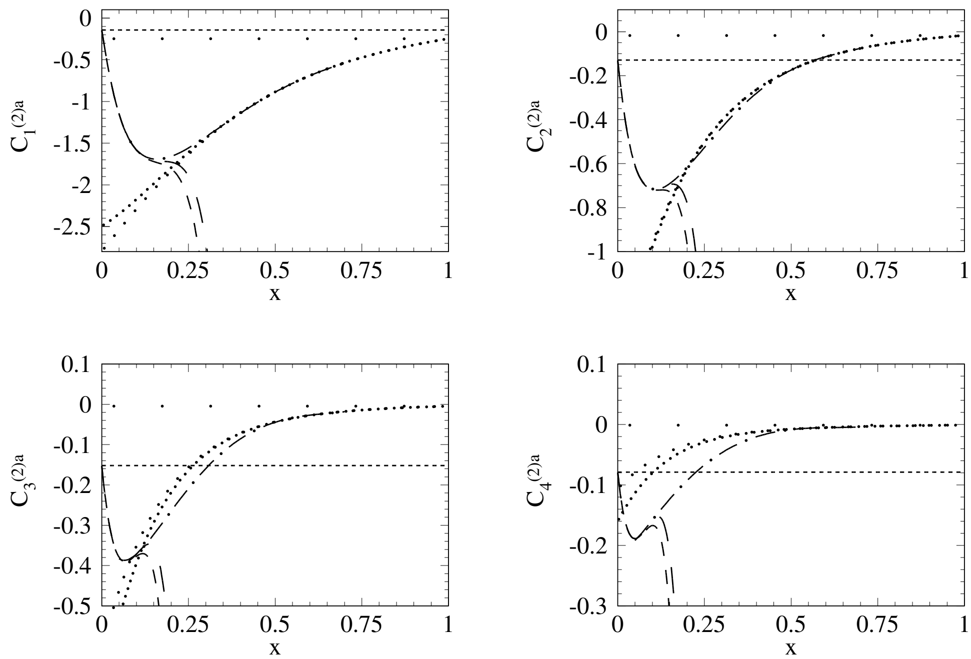

Figure 10: Three-loop contribution to $\bar{C}_n^a$. The same notation as in Fig. 4 is adopted.

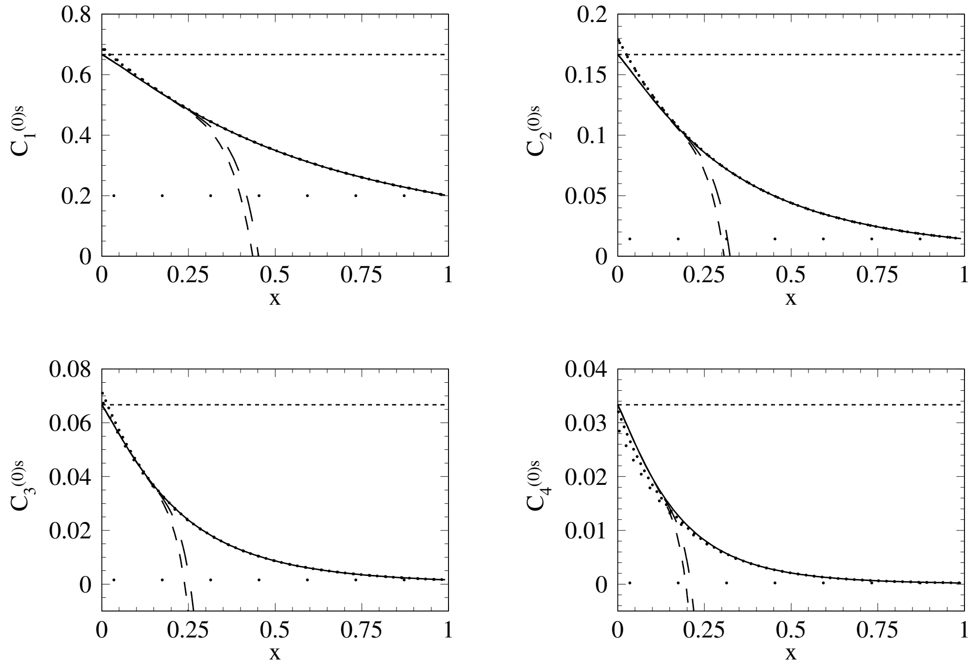

Figure 11: One-loop contribution to $\bar{C}_n^s$. The same notation as in Fig. 2 is adopted.

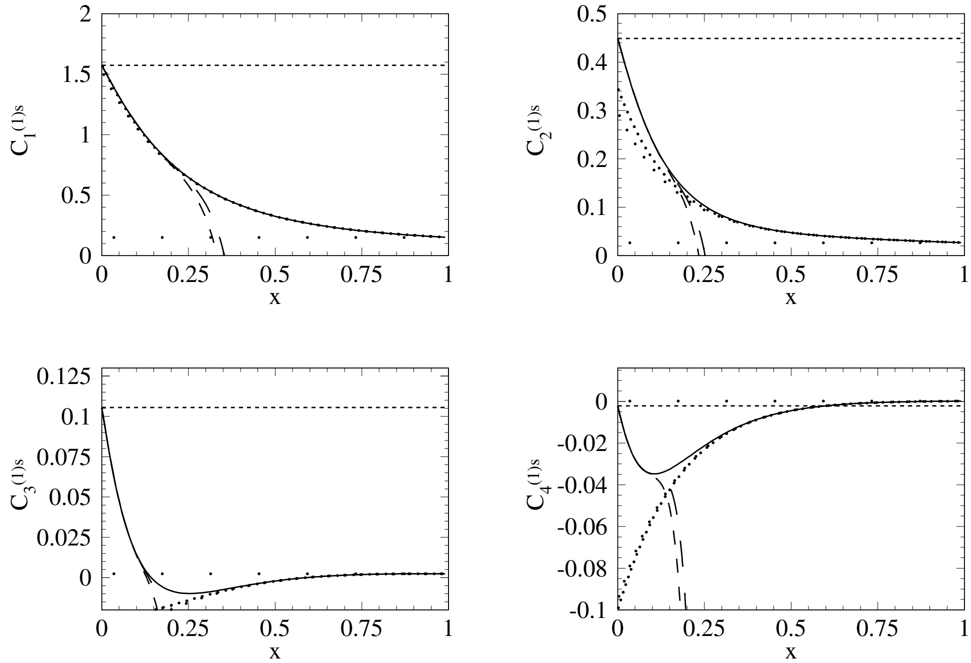

Figure 12: Two-loop contribution to $\bar{C}_n^s$. The same notation as in Fig. 2 is adopted.

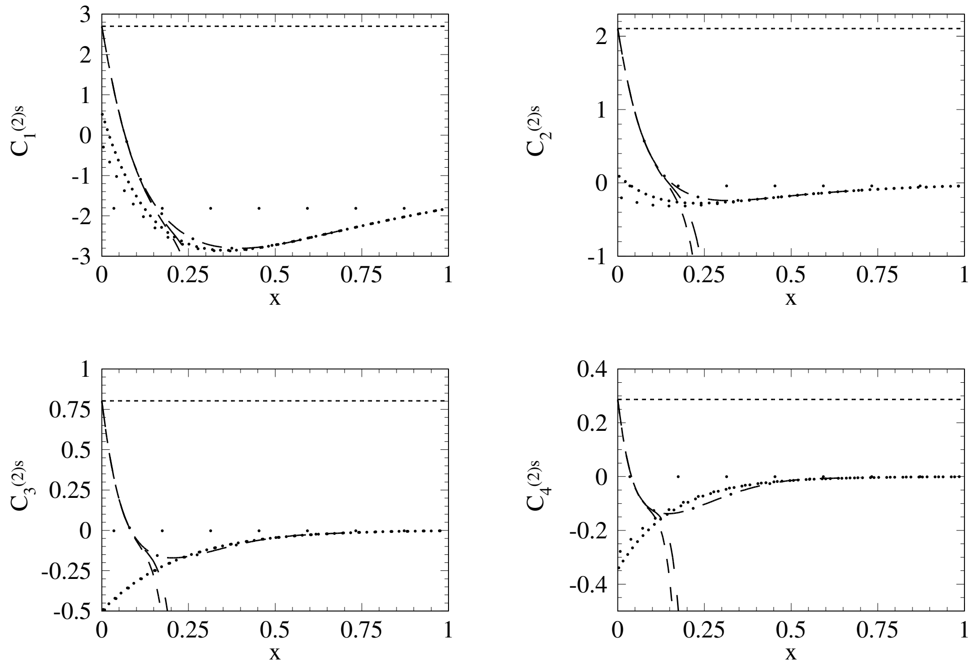

Figure 13: Three-loop contribution to $\bar{C}_n^s$. The same notation as in Fig. 4 is adopted.

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