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Complete (O_7, O_8) contribution to $\bar{B} \to X_s\gamma$ at $O(\alpha_s^2)$

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Abstract

We calculate the set of $O(\alpha_s^2)$ corrections to the branching ratio and to the photon energy spectrum of the decay process $\bar{B} \to X_s\gamma$ originating from the interference of diagrams involving the electromagnetic dipole operator O_7 with diagrams involving the chromomagnetic dipole operator O_8. The corrections evaluated here are one of the elements needed to complete the calculations of the $\bar{B} \to X_s\gamma$ branching ratio at next-to-next-to-leading order in QCD. We conclude that this set of corrections does not change the central value of the Standard Model prediction for $\mathrm{Br}(\bar{B} \to X_s\gamma)$ by more than 1%.

1 Introduction

The first estimate of the $\bar{B} \to X_s\gamma$ branching ratio within the Standard Model at the next-to-next-to-leading order (NNLO) level was published some years ago [1]:

$$\mathrm{Br}(\bar{B} \to X_s\gamma)_{\mathrm{SM},\,E_\gamma>1.6\,\mathrm{GeV}} = (3.15 \pm 0.23) \times 10^{-4}\,. \tag{1.1}$$

This estimate combines a number of different corrections which were calculated by several groups [2–12]. The prediction given in Eq. (1.1) must be compared with the current world averages,

$$\mathrm{Br}(\bar{B} \to X_s\gamma)_{\mathrm{exp},\,E_\gamma>1.6\,\mathrm{GeV}} = \begin{cases} (3.55 \pm 0.24 \pm 0.09) \times 10^{-4}\,, & \text{(HFAG) [13]} \\ (3.50 \pm 0.14 \pm 0.10) \times 10^{-4}\,, & \text{[14]} \end{cases} \tag{1.2}$$

which include measurements from CLEO, BaBar and Belle [15–17]. The central values of the theoretical prediction and of the HFAG average are compatible at the 1.2σ level, while both the theoretical and experimental uncertainties are very similar in size (about 7%). Since the experimental uncertainty is expected to decrease to 5% by the end of the B-factory era (which is already indicated by the average given in the second line of Eq. (1.2)), it is also desirable to reduce the theoretical uncertainty accordingly.

Unfortunately, at this level of accuracy, the theoretical uncertainty is dominated by non-perturbative contributions. As long as one restricts the analysis to processes mediated by the electromagnetic dipole operator $O_7 = \alpha_{\mathrm{em}}/(4\pi) m_b\, (\bar{s}\sigma^{\mu\nu}P_R b)\, F_{\mu\nu}$ alone, non-perturbative effects are well under control [18–22]. However, as soon as operators other than O_7 (such as the chromomagnetic dipole operator $O_8 = g_s/(16\pi^2) m_b\, (\bar{s}\sigma^{\mu\nu}P_R T^a b)\, G^a_{\mu\nu}$) are involved, one encounters non-perturbative effects of $O(\alpha_s\Lambda_{\mathrm{QCD}}/m_b)$. At present, the latter can only be estimated [23]. Hence a 5% uncertainty related to all of the unknown non-perturbative effects has been included in Eq. (1.1). A further reduction of the theoretical uncertainty below the 5% level seems to be rather difficult [24]. Still, given the importance of $\mathrm{Br}(\bar{B} \to X_s\gamma)$ in constraining physics scenarios beyond the Standard Model [25], it is worth to reduce the perturbative uncertainties as much as possible.

In particular, it would be desirable to reduce the uncertainty associated to the interpolation in m_c which was employed to obtain Eq. (1.1) [11]. To get rid of the interpolation in m_c in the calculation of the branching ratio is a highly challenging task and it would represent a clear improvement of the theoretical prediction. Indeed, considering the work that has been done since the publication of [1], and the work that is still in progress, an update of the estimate given in Eq. (1.1) will soon be warranted. Here we would like to mention that the effects of charm and bottom quark masses on gluon lines are now completely known (provided that one neglects on-shell amplitudes that are proportional to the small Wilson coefficients of the four-quark operators O_3-O_6) [26–29]. Therefore this part could be removed from the interpolation. Also the $O(\alpha_s^2\beta_0)$-effects in the (O_2, O_2), (O_2, O_7) and (O_7, O_8)-interference, which are known [30], were not considered in [1,11]. Finally, the complete calculation of the (O_2, O_7)-interference for $m_c = 0$ is well underway [31]. The latter calculation in particular will help to fix the boundary for the m_c interpolation for vanishing m_c; this in turn would allow one to reduce the 3% uncertainty in Eq. (1.1) due to the interpolation. For complete up-to-date lists of needed perturbative and non-perturbative corrections to the branching ratio we refer the reader to the reviews [32–35].

In this paper we calculate the complete (O_7, O_8)-interference corrections at $O(\alpha_s^2)$ to the photon energy spectrum $d\Gamma(b \to X_s^{\rm partonic}\gamma)/dE_\gamma$ and to the total decay width $\Gamma(b \to X_s^{\rm partonic}\gamma)|_{E_\gamma > E_0}$, where E_0 denotes the lower cut in the photon energy. The contributions containing massless and massive quark loops were already presented in [10, 30] and [28], respectively; the contributions which are not yet available in the literature are the ones proportional to the color factors C_F^2 and $C_F C_A$. From the technical point of view, the latter are the most complicated to evaluate and are the main subject of the present work.

The paper is organized as follows: In Sec. 2 we present our results for the (integrated) photon energy spectrum. In Sec. 3 we provide some details about the calculation of the corrections proportional to $\alpha_s^2 C_F^2$ and $\alpha_s^2 C_F C_A$ by analyzing the contribution of a particular Feynman diagram. The numerical impact of the (O_7, O_8) interference on the theoretical prediction for $\mathrm{Br}(\bar{B} \to X_s\gamma)$ at NNLO is estimated in Sec. 4. Finally, we present our conclusions in Sec. 5.

2 Results for the (integrated) photon energy spectrum

Within the low-energy effective theory, the partonic $b \to X_s\gamma$ decay rate can be written as

$$\Gamma(b \to X_s^{\rm parton}\gamma)_{E_\gamma > E_0} = \frac{G_F^2 \alpha_{\rm em} \overline{m}_b^2(\mu) m_b^3}{32\pi^4} |V_{tb}V_{ts}^*|^2 \sum_{i \le j} C_i^{\rm eff}(\mu)\, C_j^{\rm eff}(\mu) \int_{z_0}^1 dz\, \frac{dG_{ij}(z,\mu)}{dz}\,, \tag{2.1}$$

where m_b and $\overline{m}_b(\mu)$ denote the pole and the running $\overline{\rm MS}$ mass of the b quark, respectively, $C_i^{\rm eff}(\mu)$ indicates the effective Wilson coefficients at the low-energy scale, $z = 2E_\gamma/m_b$ is the rescaled photon energy, and $z_0 = 2E_0/m_b$ is the rescaled energy cut in the photon energy spectrum.[1]

As already anticipated in the introduction, we will focus on the function $dG_{78}(z,\mu)/dz$ corresponding to the interference of the electro- and the chromomagnetic dipole operators

$$O_7 = \frac{e}{16\pi^2}\, \overline{m}_b(\mu)\, (\bar{s}\sigma^{\mu\nu} P_R b)\, F_{\mu\nu}\,, \tag{2.2}$$

$$O_8 = \frac{g}{16\pi^2}\, \overline{m}_b(\mu)\, (\bar{s}\sigma^{\mu\nu} P_R T^a b)\, G^a_{\mu\nu}\,. \tag{2.3}$$

In NNLO approximation G_{78} can be rewritten as follows,

$$\frac{dG_{78}(z,\mu)}{dz} = \frac{\alpha_s(\mu)}{4\pi}\, C_F \widetilde{Y}^{(1)}(z,\mu) + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 C_F \widetilde{Y}^{(2)}(z,\mu) + O(\alpha_s^3)\,, \tag{2.4}$$

where $\alpha_s(\mu)$ indicates the running coupling constant in the $\overline{\rm MS}$ scheme and

$$\widetilde{Y}^{(1)}(z,\mu) = \left[\frac{2}{9}\left(33 - 2\pi^2\right) + \frac{16}{3}\ln\left(\frac{\mu}{m_b}\right)\right] \delta(1-z)$$

[1] In this paper we assume that the products $C_i^{\rm eff}(\mu)\, C_j^{\rm eff}(\mu)$ are real quantities. Therefore our formulas are not applicable to physics scenarios beyond the Standard Model which produce complex short distance couplings.

$$+\frac{2}{3}\left(z^2+4\right)-\frac{8}{3}\left(1-\frac{1}{z}\right)\ln(1-z)\,. \tag{2.5}$$

The function $\widetilde{Y}^{(2)}(z,\mu)$ can be split further into a sum of contributions proportional to different color factors:

$$\begin{aligned}\widetilde{Y}^{(2)}(z,\mu) &= C_F\widetilde{Y}^{(2,\mathrm{CF})}(z,\mu)+C_A\widetilde{Y}^{(2,\mathrm{CA})}(z,\mu)\\ &\quad+T_RN_L\widetilde{Y}^{(2,\mathrm{NL})}(z,\mu)+T_RN_H\widetilde{Y}^{(2,\mathrm{NH})}(z,\mu)+T_RN_V\widetilde{Y}^{(2,\mathrm{NV})}(z,\mu)\,.\end{aligned} \tag{2.6}$$

Here, N_L, N_H and N_V denote the number of light ($m_q=0$), heavy ($m_q=m_b$), and purely virtual ($m_q=m_c$) quark flavors, respectively; C_F, C_A and T_R are the SU(3) color factors with numerical values given by 4/3, 3 and 1/2, respectively. The expressions for the functions $\widetilde{Y}^{(2,i)}(z,\mu)$ with $i=$ NL, NH, NV can be found in [28]. The main result of the present work are the so far unknown functions $\widetilde{Y}^{(2,i)}(z,\mu)$ with $i=$ CF, CA, which are given by

$$\begin{aligned}\widetilde{Y}^{(2,\mathrm{CF})}(z,\mu) &= \left(-37.1831-\frac{64}{3}\,L_\mu-\frac{128}{3}\,L_\mu^2\right)\delta(1-z)-11.7874\left[\frac{\ln(1-z)}{1-z}\right]_+\\ &\quad-20.6279\left[\frac{1}{1-z}\right]_+-41.7874\,\ln(1-z)-6.6667\,\ln^2(1-z)\\ &\quad+f_1(z)-12\,\widetilde{Y}^{(1)}(z,m_b)L_\mu+\frac{64}{3}H^{(1)}(z,m_b)L_\mu\,,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\widetilde{Y}^{(2,\mathrm{CA})}(z,\mu) &= \left(4.7666+\frac{808}{27}\,L_\mu+\frac{272}{9}\,L_\mu^2\right)\delta(1-z)\\ &\quad-6.5024\,\ln(1-z)+f_2(z)+\frac{34}{3}\,\widetilde{Y}^{(1)}(z,m_b)L_\mu\,,\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}H^{(1)}(z,m_b) &= -\left(\frac{5}{4}+\frac{\pi^2}{3}\right)\delta(1-z)-\left[\frac{\ln(1-z)}{1-z}\right]_+\\ &\quad-\frac{7}{4}\left[\frac{1}{1-z}\right]_+-\frac{z+1}{2}\,\ln(1-z)+\frac{7+z-2z^2}{4}\,,\end{aligned} \tag{2.9}$$

$$\begin{aligned}f_1(z) &= 20.6279-108.484\,z+13.264\,z^2+16.1268\,z^3-33.2188\,z^4\\ &\quad+69.8819\,z^5-111.088\,z^6+118.405\,z^7-79.6963\,z^8+29.929\,z^9\\ &\quad-4.76579\,z^{10}-56.8265\,(1-z)\ln(1-z)\\ &\quad-8.11265\,(1-z)\ln^2(1-z)-5.77146\,(1-z)\ln^3(1-z)\,,\end{aligned} \tag{2.10}$$

$$\begin{aligned}f_2(z) &= 17.0559\,z+20.9072\,z^2-0.471626\,z^3+10.1494\,z^4\\ &\quad-17.4241\,z^5+24.7733\,z^6-20.4582\,z^7+8.47394\,z^8-0.173599\,z^9\end{aligned}$$

$$
\begin{aligned}
&- 0.657813\, z^{10} + 5.66536\, (1-z) \ln(1-z) \\
&- 11.1319\, (1-z) \ln^2(1-z) + 1.3999\, (1-z) \ln^3(1-z)\,.
\end{aligned} \tag{2.11}
$$

Note that function $H^{(1)}(z, \mu)$ also appeared in Eq. (2.11) of Ref. [9], and that we introduced the short-hand notation $L_\mu = \ln(\mu/m_b)$.

In Eqs. (2.7) and (2.8) the z-dependence of the μ-dependent terms and of those terms which become singular in the limit $z \to 1$ is exact. The functions $f_1(z)$ and $f_2(z)$ were instead obtained by making an ansatz for our numerical results of the non-singular parts, using the functional form

$$
f_i(z) = \sum_{j=0}^{10} c_{i,j}\, z^j + c_{i,11}(1-z)\ln(1-z) + c_{i,12}(1-z)\ln^2(1-z) + c_{i,13}(1-z)\ln^3(1-z)\,. \tag{2.12}
$$

The coefficients $c_0, \ldots, c_{13}$ were then determined by performing a least-square fit, using 100 specific 'data'-points. We checked that the fit-functions remain essentially the same when changing the set of data-points. In particular, the integrals of the fit-functions, taken in an interval $[z_0, 1]$ $(0 \leq z_0 < 1)$, remain basically unchanged. The same holds true when changing the functional ansatz given in Eq. (2.12), e.g., to contain additional terms proportional to $(1-z)^2 \ln^n(1-z)$, with $n = 1, 2, 3$.

The plus distributions appearing in Eq. (2.7) are defined as

$$
\int_0^1 dz \left[\frac{\ln^n(1-z)}{1-z}\right]_+ g(z) = \int_0^1 dz\, \frac{\ln^n(1-z)}{1-z} \left[g(z) - g(1)\right] , \tag{2.13}
$$

where $g(z)$ is an arbitrary test function which is regular at $z = 1$, and $n = 0, 1$. In case the integration does not include the endpoint $z = 1$, we have $(c < 1)$

$$
\int_0^c dz \left[\frac{\ln^n(1-z)}{1-z}\right]_+ g(z) = \int_0^c dz\, \frac{\ln^n(1-z)}{1-z} g(z)\,. \tag{2.14}
$$

We observe that the plus distributions are present only in the part of the spectrum proportional to C_F^2, see Eq. (2.7). This is in agreement with the results reported in [36]; following the procedure presented in that work, it is possible to conclude that the plus distributions appearing in the (O_7, O_8) component of the photon energy spectrum at $\mathcal{O}(\alpha_s^2)$ must be the same ones as in the (O_7, O_7) component of the spectrum at $\mathcal{O}(\alpha_s)$ (up to an overall factor). In particular

$$
\begin{aligned}
\widetilde{Y}^{(2,\mathrm{CF})}(z, m_b)\Big|_{\text{plus distrib.}} &= -\frac{8}{9}\left(33 - 2\pi^2\right) \left\{ \left[\frac{\ln(1-z)}{1-z}\right]_+ + \frac{7}{4}\left[\frac{1}{1-z}\right]_+ \right\} , \\
&= -11.7874 \left[\frac{\ln(1-z)}{1-z}\right]_+ - 20.6279 \left[\frac{1}{1-z}\right]_+ .
\end{aligned} \tag{2.15}
$$

The structure in Eq. (2.15) emerges in our diagrammatic calculation from delicate cancellations among several contributions, and it provides a valuable test for our result.

3 Technical details about the calculation

In order to obtain the results in Eqs. (2.7) and (2.8) one needs to evaluate Feynman diagrams contributing to the process $b \to s\gamma$ up to two loops, one-loop Feynman diagrams contributing to the process $b \to s\gamma g$, and tree-level Feynman diagrams contributing to the processes $b \to s\gamma gg$ (plus corresponding diagrams involving unphysical ghosts in the final state) and $b \to s\gamma s\bar{s}$. As discussed in detail in [9], the interferences among the various partonic diagrams with 2, 3 and 4 particles in the final state are in one-to-one correspondence with the 2-, 3- and 4-particle cuts of the three-loop b-quarks self-energy diagrams shown in Figs. 3, 4, 5, 6, and 7, provided that the cut goes through the photon propagator[2]. The 2-particle cuts of the self-energy graphs correspond to the interference of a tree-level and a two-loop diagrams for the process $b \to s\gamma$, or to the interference of two one-loop diagrams for the same process. The 3-particle cuts split a self-energy graph into a one-loop and a tree-level diagram contributing to the process $b \to s\gamma g$. Finally, the 4-particle cuts correspond to the interference of two tree-level diagrams for the process $b \to sgg$ or for the process $b \to s\gamma s\bar{s}$.

The contribution of each single cut to the photon energy spectrum can be obtained by employing the Cutkosky rules [37–39]. In order to keep the energy of the photon fixed, it is necessary to insert in the integrands a factor [7]

$$\delta\left(E_\gamma - \frac{p_b \cdot p_\gamma}{m_b}\right) = 2m_b\, \delta\left((p_b - p_\gamma)^2 - (1-z)m_b^2\right) \,, \tag{3.1}$$

where p_b and p_γ denote the four momenta of the b quark and the photon, respectively. Finally, the delta function in Eq. (3.1) and all of the delta functions originating from the Cutkosky rules for the propagators crossed by a cut can be rewritten as differences of propagators as follows [40, 41],

$$\delta\left(q^2 - m^2\right) = \frac{1}{2\pi i}\left(\frac{1}{q^2 - m^2 - i0} - \frac{1}{q^2 - m^2 + i0}\right) \,. \tag{3.2}$$

It is then possible to evaluate all of the relevant cuts of the three-loop self-energy diagrams by first identifying a set of Master Integrals for each cut, and then by evaluating those Master Integrals by means of the tools and techniques usually employed in the calculation of multi-loop Feynman integrals. As usual, we work in $d = 4 - 2\epsilon$ space-time dimensions to regularize ultraviolet, infrared and collinear singularities. In the rest of this section we discuss some features of this procedure by considering one of the simplest Feynman diagrams we encountered in the course of the calculation. More technical details concerning the parameterization of the phase-space integrals can be found in [8, 9, 42].

Let us consider the topology displayed in Fig. 1, which corresponds to the last diagram shown in Fig 6. We carried out the reduction to Master Integrals by means of the package AIR [43] and by means of a private code written by one of us (in order to have a cross check). In the diagram corresponding to the topology in Fig. 1, only a 2- and a 3-particle cut are present (they are indicated by the dashed red and blue lines in Fig. 1, respectively). We will first concentrate on the 3-particle cut; the 2-particle cut will be discussed at the end of this section. Because of the delta function in Eq. (3.1), one finds ($k_1 = p_\gamma$)

$$(p_b - k_1)^2 = m_b^2(1-z) \,; \tag{3.3}$$

[2] We only display the diagrams that contribute to the functions $\widetilde{Y}^{(2,i)}(z,\mu)$ with $i =$ CF, CA.

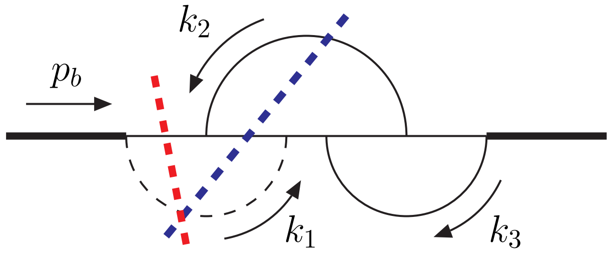

Figure 1: *A three-loop self-energy topology and the corresponding 2- and 3-particle cuts. Thin lines represent massless propagators; the dashed line corresponds to the photon propagator; the 2- and the 3-particle cut are indicated by the dashed red and blue lines, respectively; k_1, k_2 and k_3 are the loop momenta.*

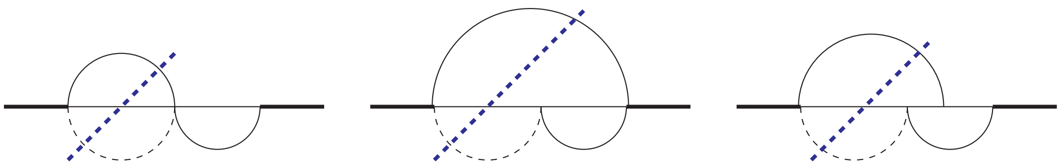

Figure 2: *Master Integrals for 3-particle cut of the topology displayed in Fig. 1.*

therefore the corresponding propagator can be immediately factored out of the integrals. The reduction indicates that the 3-particle cut of the topology in Fig. 1 has three Master Integrals, which are shown in Fig. 2.

Those Master Integrals have only single poles in ϵ (as long as $z \neq 1$), and are sufficiently simple to be calculated analytically by direct integration of a suitable Feynman parameterization of the virtual loop and of the 3-particle phase-space. By combining the output of the reduction to Master Integrals with the analytic expressions of the latter, it is possible to find an expression for the contribution of this particular cut to the photon energy spectrum. It is then straightforward to check that the coefficient of the single pole in ϵ, the finite part, and the $O(\epsilon)$ term vanish in the $z \to 0$ limit, as expected from dimensional reasoning [32]. On the contrary, in the $z \to 1$ limit this particular cut shows divergences of the form $\ln^n(1-z)/(1-z)$ $(n = 0, 1)$. Such divergences give origin to the plus distribution functions which survive in the part of the photon energy spectrum which is proportional to C_F^2. To see this one needs to extract a factor $(1-z)^{b\epsilon}$ (where b is an arbitrary integer) out of each Master Integral, to combine it with possible factors $1/(1-z)$ which emerge from the reduction procedure, and then to replace the resulting expression by using

$$(1-z)^{-1+b\epsilon} = \frac{1}{b\epsilon}\,\delta(1-z) + \sum_{n=0}^{\infty} \frac{(b\epsilon)^n}{n!} \left[\frac{\ln^n(1-z)}{(1-z)}\right]_+ . \tag{3.4}$$

The relation above explains why one needs to calculate the cofactor of $(1-z)^{-1+b\epsilon}$ including terms of $O(\epsilon)$: the latter terms contribute to the $O(\epsilon^0)$ part of the coefficient of $\delta(1-z)$. For the definition of the plus distributions see Eqs. (2.13) and (2.14).

Very often we had to deal with Master Integrals which we were not able to evaluate

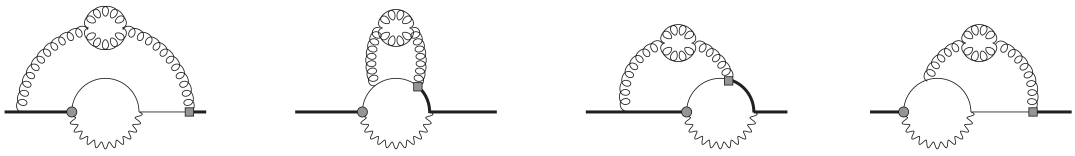

Figure 3: *Diagrams with a closed gluon loop. Similar diagrams with a closed ghost loop are also present. The gray square indicates the operator O_8, while the gray circle indicates the operator O_7. Thick lines represent the massive b quark, thin lines the massless s quark, wavy lines photons and curly lines gluons. The color factor of these diagrams is proportional to $C_F C_A$.*

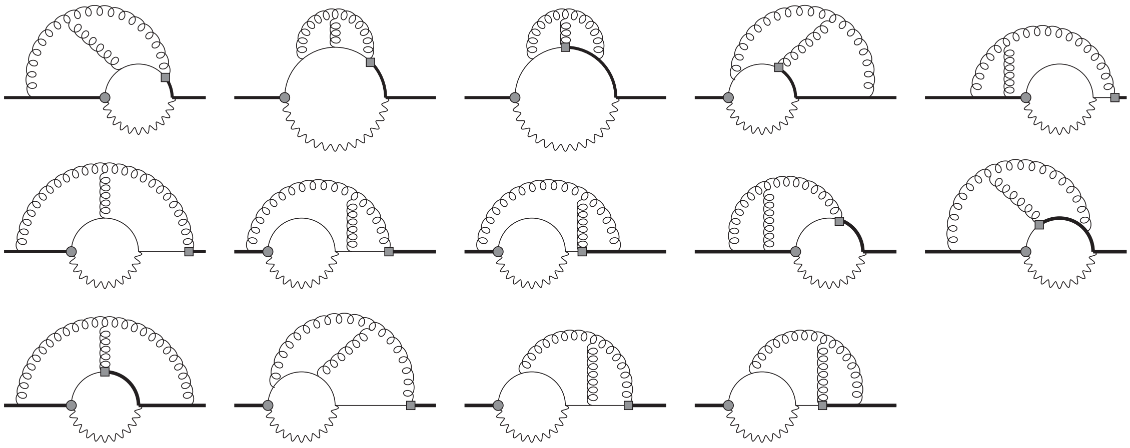

Figure 4: *Same as in Fig. 3 for diagrams containing a triple gluon vertex. The color factor of these diagrams is proportional to $C_F C_A$.*

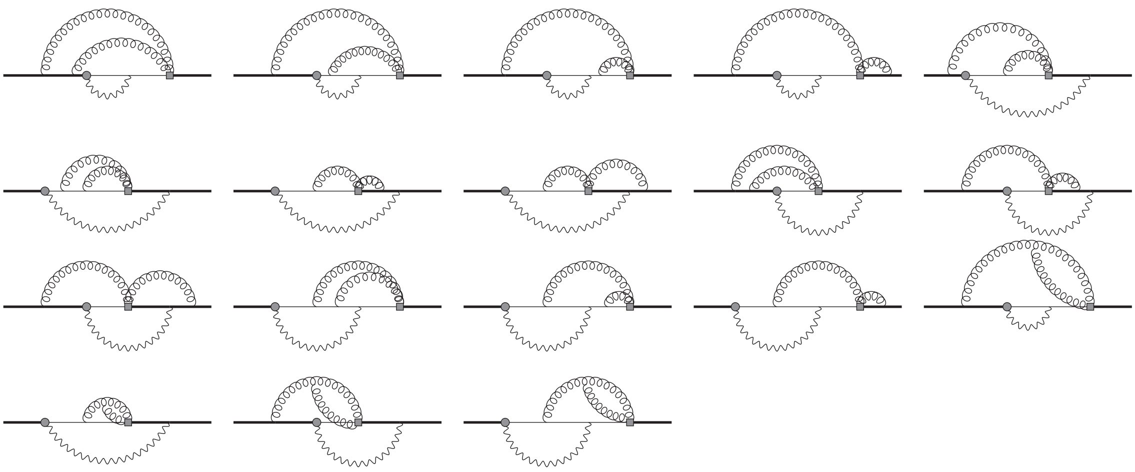

Figure 5: *Same as in Fig. 3 for diagrams involving double gluon emission from the O_8 operator. The color factor of these diagrams is proportional to $C_F C_A$.*

Figure 6: *Same as in Fig. 3 for diagrams with color factor proportional to* $C_F(C_F - C_A/2)$.

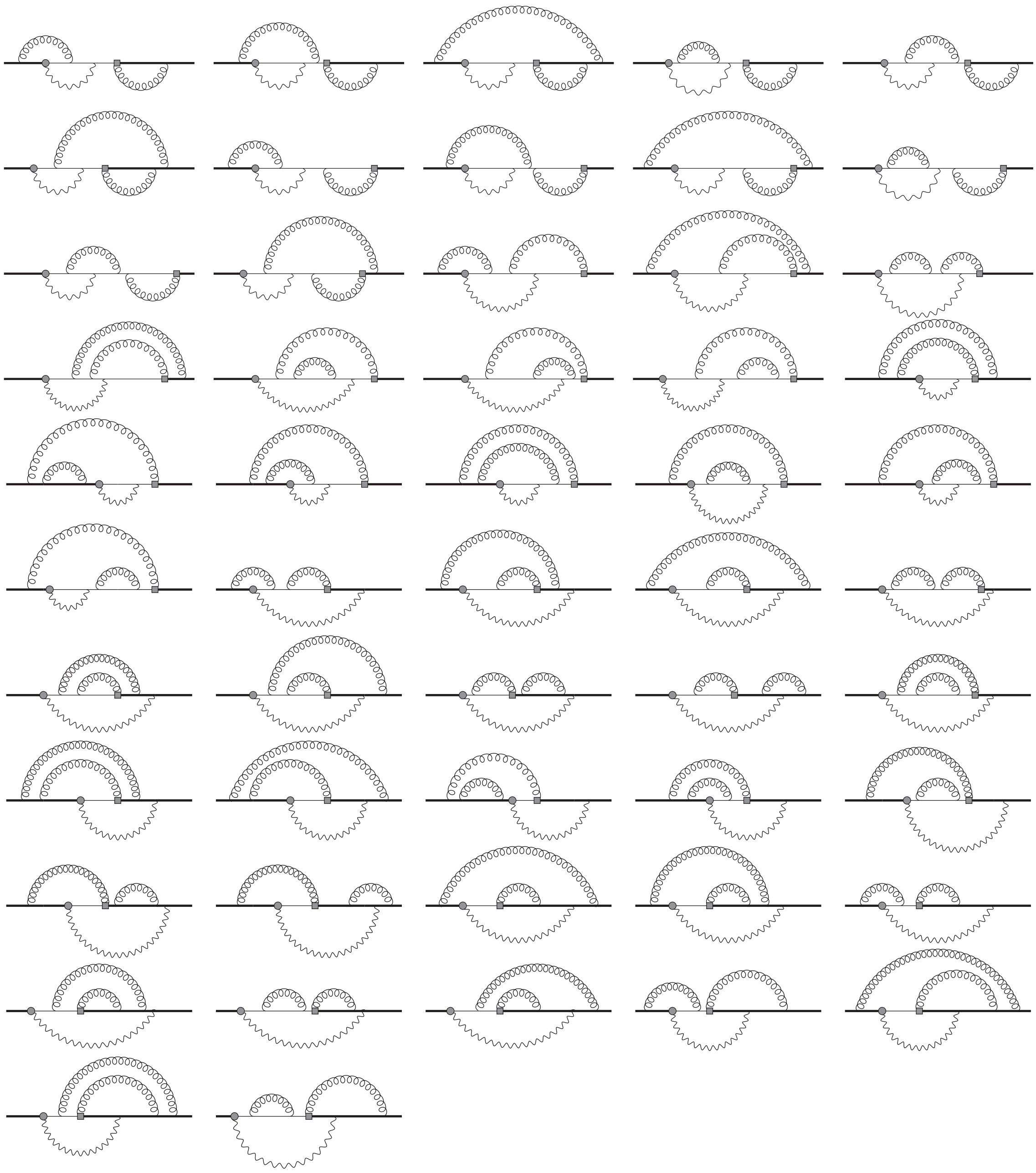

Figure 7: *Same as in Fig. 3 for diagrams proportional to the color factor* C_F^2.

by direct integration of their integral representation in terms of Feynman parameters. A powerful tool to be used in these cases is the differential equation method [40, 41, 44, 45]. The goal of the method is to employ the output of the reduction procedure for a given topology to build differential equations which are satisfied by the Master Integrals of that topology. In our case, we consider differential equations with respect to z, the only variable which appears in the Master Integrals. In order to generate these differential equations, it is necessary to take the derivative of the integrand of a given Master Integral with respect to z. It is interesting to observe that the only factor in the integrand which depends on z is the propagator associated to the delta function given in Eq. (3.1).

A weakness of the differential equation method is the fact that there is no general strategy which allows to fix the integration constants which are not determined by the solution of the differential equations. In a number of cases it was possible to fix the missing constants by directly evaluating the Master Integral after setting $z = 0$ from the start. However, we also found Master Integrals which are singular in $z = 0$; they appear in the reduction of the 3- and 4-particle cuts of the 3rd and the last diagram given in 4th line in Fig. 6. To fix the integration constants in those cases we exploited the fact that the coefficient of each term in the ϵ expansion of each single cut contributing to the photon energy spectrum must vanish in the $z \to 0$ limit. However, we had to deal with cuts for which this procedure did not provide enough conditions to fix all of the unknown integration constants. Therefore we calculated some of the Master Integrals for $z = 1$ to reduce the number of unknown integration constants; subsequently we fixed the remaining ones by considering the $z \to 0$ limit of the cuts involving those integrals. Another procedure which we employed to reduce the number of unknown integration constants consists in integrating some Master Integral over z from 0 to 1. With these methods we were able to obtain analytic expressions for all of the poles in ϵ appearing in the calculation of the various cuts. For a few cuts we calculated the finite parts only numerically. Some of the diagrams with non-trivial endpoint behavior were checked by an independent calculation of the integrated spectrum. We would like to mention that for the (O_7, O_8)-interference the endpoint singularities are only present in 3-particle cuts of the b-quark self energies; 4-particle cuts are free of endpoint singularities since the 4-particle phase space is proportional to $(1 - z)$.

Now we will turn briefly to the evaluation of the 2-particle cut in Fig. 1. The reduction to Master Integrals is carried out along the lines of the reduction of the 3-particle cut discussed above. However, the insertion of the delta function given in Eq. (3.1) in the integrand is not necessary, since the 2-particle process takes place at fixed photon energy, $z = 1$. We calculated the four Master Integrals which appear in this case by using a numerical method based on sector decomposition [46]; this technique allows to disentangle overlapping infrared, collinear and ultraviolet divergences. We applied this numerical method in order to evaluate all of the Master Integrals arising from 2-particle cuts. However, due to the presence of internal thresholds, the integration over some of the Feynman parameters can only be done numerically after a suitable contour deformation [47–49].

The infrared and collinear divergences appearing in the 2-particle cuts will cancel after adding the 3- and 4-particle cuts, which also suffer from infrared and collinear divergences. The remaining divergences are ultraviolet. They are removed by adding counterterm diagrams with appropriate Z-factor insertions (see App. B).

4 Estimating the numerical impact on $\mathrm{Br}(\bar{B} \to X_s\gamma)$

In this section we investigate the numerical size of the (O_7, O_8)-interference at $O(\alpha_s^2)$ at the level of the branching ratio of the decay process $\bar{B} \to X_s\gamma$. In order to do so, we adopt the notation and conventions introduced in [11]. The $O(\alpha_s^2)$ correction to the function $K_{78}(E_0, \mu)$ in Eqs. (2.6) and (3.1) of [11] is given by

$$K_{78}^{(2)}(E_0,\mu) = \frac{C_F}{2}\int_{z_0}^{1} dz \left\{ \widetilde{Y}^{(2)}(z,\mu) - C_F^2 \frac{8\pi}{3}\alpha_\Upsilon \delta(1-z) \right.$$
$$\left. - C_F \left[\frac{41}{2} - 2\pi^2 + 12\, L_\mu\right] \widetilde{Y}^{(1)}(z,\mu) \right\}, \tag{4.1}$$

with $\alpha_\Upsilon = 0.22$. Note that Eq. (4.1) refers to the $1S$-scheme for the b-quark mass. The value used for this parameter in the numerics is $m_b^{1S} = 4.68\,\mathrm{GeV}$.

Combining the results of our present paper with those of [28] we are now able to write down the complete expression for $K_{78}^{(2)}$ containing all abelian and non-abelian contributions as well as the effects of the masses of the u, d, s, c and b quarks running in the bubbles inserted in gluon lines. This complete term affects the branching ratio by an amount

$$\Delta\mathrm{Br}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \mathrm{Br}(\bar{B} \to X_c e\bar{\nu})_{\mathrm{exp}} \left|\frac{V_{tb}V_{ts}^*}{V_{cb}}\right|^2 \frac{6\,\alpha_{\mathrm{em}}}{\pi\, C}\,\Delta P(E_0)\,, \tag{4.2}$$

where

$$\Delta P(E_0) = 2C_7^{(0)\mathrm{eff}}(\mu) C_8^{(0)\mathrm{eff}}(\mu) \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 K_{78}^{(2)}(E_0,\mu)\,, \tag{4.3}$$

and C is the so-called semileptonic phase-space factor. In order to compare with Ref. [11], we employ the numerical value for C which was obtained from a fit of the measured spectrum of the $\bar{B} \to X_c l\bar{\nu}$ decay in the 1S scheme[3] [50,51].

One might think that $\Delta\mathrm{Br}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$ in Eq. (4.2) simply represents the shift due to $K_{78}^{(2)}$ of the theoretical prediction given in Eq. (1.1). This is, however, not the case because an approximated version of $K_{78}^{(2)}$ was already included in [1]: While the β_0-part of K_{78} (i.e. $K_{78}^{(2)\beta_0}$ when following the notation of [11]) was fully taken into account, the remaining piece, $K_{78}^{(2)\mathrm{rem}}$, was calculated for large values of $\rho = m_c^2/m_b^2$ and then interpolated (combined with contributions not related to the (O_7, O_8)-interference) to the physical value of ρ. To remove $K_{78}^{(2)\mathrm{rem}}$ from the interpolation procedure and to replace it by the exact result obtained by us, is beyond the scope of the present paper; this issue will be correctly treated in a systematic update of Eq. (1.1) in the near future. To get nevertheless an idea of the numerical size of the $O(\alpha_s^2)$ contribution of the (O_7, O_8)-interference at the level of branching ratio for $\bar{B} \to X_s\gamma$, we can ignore this issue and simply discuss a few numerical aspects of the quantity $\Delta\mathrm{Br}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$, based on $K_{78}^{(2)}$ which we calculated in this paper.

Fig. 8 shows $\Delta\mathrm{Br}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$ as a function of z_0 for $\mu = 2.5\,\mathrm{GeV}$, $\alpha_s(2.5\,\mathrm{GeV}) = 0.271$, $C_7^{(0)\mathrm{eff}}(2.5\,\mathrm{GeV}) = -0.369$, $C_8^{(0)\mathrm{eff}}(2.5\,\mathrm{GeV}) = -0.171$, $N_L = 3$ and $N_H = N_V = 1$.

[3] Using the numerical value for C as obtained from a fit in the kinetic scheme raises the central value given in Eq. (1.1) by approximately 3% [52].

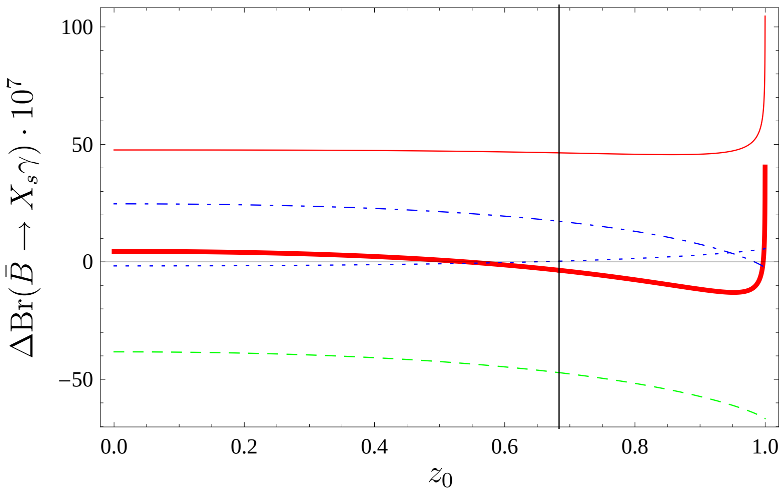

Figure 8: $\Delta Br(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$ *as a function of* z_0. *See text for more details.*

(Note that the scale $\mu = 2.5\,\mathrm{GeV}$ defines the central value of the branching ratio given in Eq. (1.1).) The remaining numerical input parameters are taken from [11]. The thick red solid line shows the complete result, the thin red solid line corresponds to the contribution proportional to C_F^2 (including the numerical value of C_F^2), the green dashed line indicates the contribution proportional to $C_F C_A$ (again including $C_F C_A$), and the blue dotted line is the contribution stemming from massless and massive quark bubbles. The dash-dotted blue line indicates the corrections obtained by applying the large-β_0 approximation [53, 54]. The vertical line indicates the value of z_0 corresponding to the choice $E_0 = 1.6$ GeV. It can be seen from the figure that the contributions proportional to C_F^2 and $C_F C_A$ cancel each other over almost the whole range of z_0, resulting in a contribution of size similar to the one due to the fermionic corrections. Only in the region very close to the endpoint the contribution proportional to C_F^2 dominates, due to its singular behavior for $z_0 \to 1$. We stress, however, that the cancellations mentioned above refer to a value of $\mu = 2.5$ GeV; they do not occur anymore when going to smaller values of μ.

For a photon energy cut-off $E_0 = 1.6\,\mathrm{GeV}$, as the one employed in Eq. (1.1), we find the following numerical value for the quantity in Eq. (4.2) (using $\mu = 2.5$ GeV):

$$\Delta\mathrm{Br}(\bar{B} \to X_s\gamma)_{E_\gamma > 1.6\,\mathrm{GeV}} = \frac{C_F}{2}\Big(52.21\,C_F - 23.57\,C_A - 2.66\,C_F^2 - 2.28\,N_L + 8.73\,N_H - 1.42\,N_V\Big) \times 10^{-7} = -3.57 \cdot 10^{-7}\,, \qquad (4.4)$$

where in the last step we inserted numerical values for the color factors and we set $N_L = 3$ and $N_H = N_V = 1$. By comparing this with the central value of the estimate given in Eq. (1.1), one sees that the $O(\alpha_s^2)$ corrections $K_{78}^{(2)}$ have an impact of -0.11% at the level of the branching ratio. For the analogous effect generated by the large-β_0 approximation of $K_{78}^{(2)}$ only, $K_{78}^{(2)\beta_0}$, we find $+0.56\%$. These results are, however, very strongly dependent on the scale μ. For $\mu = 1.25,\ 2.34,\ 5\,\mathrm{GeV}$ the corresponding numbers read -5.15, -0.23, -0.07% (full) and $+0.10$, $+0.49$, $+1.15\%$ (large-β_0 approximation), respectively. From these results we conclude that the large-β_0 approximation does not provide a good estimate of the full $O(\alpha_s^2)$ correction of the (O_7, O_8)-contribution for $\mu \in [1.25, 5]\,\mathrm{GeV}$. As already mentioned, a more detailed analysis of the effect of the complete calculation of $K_{78}^{(2)}$ on the central value of Eq. (1.1) would require to repeat the interpolation procedure of [11]. While this is beyond the scope of the present work, we can conclude that the correction originating from the (O_7, O_8)-interference at $O(\alpha_s^2)$ will not alter the central value of Eq. (1.1) by more than 1%.

5 Summary and conclusions

In the present work we calculated the set of the $O(\alpha_s^2)$ corrections to the partonic decay process $b \to X_s\gamma$ which originates from the interference of diagrams involving the electromagnetic dipole operator O_7 with diagrams involving the chromomagnetic dipole operator O_8. These corrections are one of the elements needed in order to complete the calculation of the branching ratio for the radiative decay $\bar{B} \to X_s\gamma$ up to NNLO in QCD.

To carry out the calculation, we mapped the interference of diagrams contributing to the processes $b \to s\gamma$, $b \to s\gamma g$, $b \to s\gamma gg$ and $b \to s\gamma s\bar{s}$ onto 2-, 3- , and 4-particle cuts for the three-loop b-quark self-energy diagrams which include insertions of the operators O_7 and O_8. Subsequently, we evaluated each single cut by employing the Cutkosky rules. From the technical point of view, the calculation was made possible by the use of the Laporta Algorithm [55] to identify the needed Master Integrals, and of the differential equation method and sector decomposition method to solve the Master Integrals.

From the phenomenological point of view, it is interesting to estimate the effect of these corrections on the theoretical prediction for the $\bar{B} \to X_s\gamma$ branching ratio. Our conclusion is that they will not change its central value given in Eq. (1.1) by more than 1%.

At present, the largest theoretical uncertainty affecting the prediction in Eq. (1.1) is of non-perturbative origin. It is expected to set a lower limit of about 5% on the total theoretical uncertainty for the prediction of the $\bar{B} \to X_s\gamma$ branching ratio in the near future. The perturbative $O(\alpha_s^2)$ corrections of the (O_7, O_8)-interference presented in this paper are a further contribution to make the perturbative uncertainty negligible with respect to the non-perturbative one.

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A Two-particle cuts

To obtain the contributions to the functions $\widetilde{Y}^{(1)}$, $\widetilde{Y}^{(2,CF)}$ and $\widetilde{Y}^{(2,CA)}$ (see Eqs. (2.4) and (2.6)) originating from the 2-particle cuts of the b-quark self energies discussed in the paper, we first calculated the ultraviolet renormalized on-shell matrix elements

$$\langle O_i \rangle \equiv \langle s\gamma | O_i | b \rangle \,, \qquad (i = 7, 8) \,.$$

For the (O_7, O_8)-interference, $\langle O_7 \rangle$ and $\langle O_8 \rangle$ are needed to one-loop and two-loop accuracy, respectively. By considering only the terms proportional to C_F^2 and $C_F C_A$ to $O(\alpha_s^2)$, which are the ones of interest in order to obtain Eq. (2.7) and (2.8), we write

$$\langle O_i \rangle = \langle O_7 \rangle_{\text{tree}} \left[\delta_{i7} + \frac{\alpha_s}{4\pi} C_F \, D_i^{(1)} + \left(\frac{\alpha_s}{4\pi} \right)^2 C_F \, \left(C_F D_{iF}^{(2)} + C_A D_{iA}^{(2)} + \ldots \right) + O(\alpha_s^3) \right] . \tag{A.1}$$

Note that the operator O_7 in $\langle O_7 \rangle_{\text{tree}}$ contains the b-quark running mass $\overline{m}_b(\mu)$. For O_7 the relevant results are [6, 8]

$$\begin{aligned} D_7^{(1)} &= -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} (2\, L_\mu + 2.5) - 2\, L_\mu^2 - 7\, L_\mu - 6.8225 \\ &\quad - \epsilon \left(1.3333\, L_\mu^3 + 7\, L_\mu^2 + 13.6449\, L_\mu + 13.4779 \right) \\ &\quad - \epsilon^2 \left(0.6667\, L_\mu^4 + 4.6667\, L_\mu^3 + 13.6449\, L_\mu^2 + 26.9559\, L_\mu + 26.1412 \right) , \end{aligned} \tag{A.2}$$

and for O_8 one finds

$$\begin{aligned} D_8^{(1)} &= 2.6667\, L_\mu + 1.4734 + 2.0944\, i \\ &\qquad + \epsilon \left[2.6667\, L_\mu^2 + 2.9468\, L_\mu - 1.1947 + \, i\, (4.1888\, L_\mu + 4.1888) \right] \\ &\qquad + \epsilon^2 \big[1.7778\, L_\mu^3 + 2.9468\, L_\mu^2 - 2.3894\, L_\mu - 5.5373 \\ &\qquad\qquad + i \, \left(4.1888\, L_\mu^2 + 8.3776\, L_\mu + 2.1627 \right) \big] , \\ D_{8F}^{(2)} &= D_8^{(1)} \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} (2\, L_\mu + 2.5) \right) - 5.3333\, L_\mu^3 - 32.2802\, L_\mu^2 - 50.9612\, L_\mu \\ &\qquad -1.8875 - i \left(4.1888\, L_\mu^2 + 31.4159\, L_\mu + 29.8299 \right) , \\ D_{8A}^{(2)} &= 15.111\, L_\mu^2 + 31.6617\, L_\mu + 2.38332 + i\, (23.7365\, L_\mu + 28.0745) \, . \end{aligned} \tag{A.3}$$

Taking into account the phase-space factors in $d = 4 - 2\epsilon$ dimensions, we easily obtain the contributions to the functions $\widetilde{Y}^{(1)}$, $\widetilde{Y}^{(2,\text{CF})}$ and $\widetilde{Y}^{(2,\text{CA})}$ which originate from the 2-particle cuts only:

$$\widetilde{Y}^{(1)}_{2-\text{cuts}}(z,\mu) = \left[\frac{2}{9} \left(33 - 2\pi^2 \right) + \frac{16}{3} L_\mu \right] \delta(1-z) \,,$$

$$\widetilde{Y}^{(2,\text{CF})}_{2-\text{cuts}}(z,\mu) = \left[\frac{1}{\epsilon^2} (-5.89368 - 10.6667\, L_\mu) + \frac{1}{\epsilon} \left(-15.849 - 72.6954\, L_\mu - 53.3333\, L_\mu^2 \right) \right.$$

$$+ 3.01591 - 246.244\, L_\mu - 335.42\, L_\mu^2 - 135.111\, L_\mu^3 \Big] \, \delta(1-z)\,,$$

$$\widetilde{Y}^{(2,\mathrm{CA})}_{2-\mathrm{cuts}}(z,\mu) = \left[4.76664 + 63.3235\, L_\mu + 30.2222\, L_\mu^2\right] \, \delta(1-z)\,. \tag{A.4}$$

The results given in this appendix were already used by one of us in [57].

B Renormalization constants

In this appendix, we collect the explicit expressions of the renormalization constants needed for the ultraviolet renormalization in our calculation. The strong coupling constant is renormalized in the $\overline{\mathrm{MS}}$ scheme:

$$\alpha_s^{\mathrm{bare}} = \mu^{2\epsilon} \left(\frac{e^\gamma}{4\pi}\right)^{\epsilon} Z_\alpha^{\overline{\mathrm{MS}}}\, \alpha_s(\mu)\,, \tag{B.1}$$

with

$$Z_\alpha^{\overline{\mathrm{MS}}} = 1 - \frac{1}{\epsilon}\left(\frac{11}{3}\, C_A - \frac{4}{3}\, T_R N_F\right) \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,, \tag{B.2}$$

and $N_F = N_L + N_H + N_V$. The b-quark mass which appears in the operators $O_{7,8}$, as well as the Wilson coefficients $C_{7,8}^{\mathrm{eff}}$ themselves, are also renormalized in the $\overline{\mathrm{MS}}$ scheme [58]:

$$\begin{aligned}
Z_{m_b}^{\overline{\mathrm{MS}}} &= 1 - \frac{3\, C_F}{\epsilon} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,, \\
Z_{77}^{\overline{\mathrm{MS}}} &= 1 + \frac{4\, C_F}{\epsilon} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,, \\
Z_{87}^{\overline{\mathrm{MS}}} &= -\frac{4\, C_F}{3\,\epsilon} \frac{\alpha_s(\mu)}{4\pi} + \left\{ \frac{1}{\epsilon^2}\left(\frac{34}{9}\, C_F C_A - 8\, C_F^2 - \frac{8}{9}\, C_F T_R N_F\right) \right. \\
&\quad \left. + \frac{1}{\epsilon}\left(-\frac{101}{27}\, C_A C_F + \frac{8}{3}\, C_F^2 + \frac{28}{27}\, C_F T_R N_F\right) \right\} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 + O(\alpha_s^3)\,, \\
Z_{88}^{\overline{\mathrm{MS}}} &= 1 + \frac{2}{\epsilon}\,(4\, C_F - C_A)\, \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,.
\end{aligned} \tag{B.3}$$

All the remaining fields and parameters are renormalized in the on-shell scheme. The on-shell renormalization constant for the b-quark mass is given by

$$Z_{m_b}^{\mathrm{OS}} = 1 - C_F\, \Gamma(\epsilon)\, e^{\gamma\epsilon}\, \frac{3-2\epsilon}{1-2\epsilon} \left(\frac{\mu}{m_b}\right)^{2\epsilon} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,. \tag{B.4}$$

The renormalization constants for the gluon field and the s- and b-quark fields are

$$\begin{aligned}
Z_3^{\mathrm{OS}} &= 1 - \frac{4}{3}\, T_R \left(N_H + N_V \rho^{-\epsilon}\right) \Gamma(\epsilon)\, e^{\gamma\epsilon} \left(\frac{\mu}{m_b}\right)^{2\epsilon} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,, \\
Z_{2s}^{\mathrm{OS}} &= 1 + O(\alpha_s^2)\,,
\end{aligned}$$

$$Z_{2b}^{\rm OS} = 1 - C_F\,\Gamma(\epsilon)\,e^{\gamma\epsilon}\,\frac{3-2\epsilon}{1-2\epsilon}\left(\frac{\mu}{m_b}\right)^{2\epsilon}\frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2)\,, \tag{B.5}$$

where $\rho = m_c^2/m_b^2$.

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