Matching QCD and HQET at three loops

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QCD/HQET matching for the heavy-quark field [1] and heavy–light quark currents [2] with three-loop accuracy is discussed.

1. Heavy-quark field

QCD problems with a single heavy quark Q can be treated in a simpler effective theory — HQET, if there exists a 4-velocity v such that the heavy-quark momentum is $p = mv+k$ (m is the on-shell mass) and the characteristic residual momentum is small: $k \ll m$. QCD operators can be written as series in $1/m$ via HQET operators; the coefficients in these series are determined by matching on-shell matrix elements in both theories.

At the tree level, the heavy-quark field Q is related to the corresponding HQET field Q_v (satisfying $\rlap{/}vQ_v = Q_v$) by [3,4]

$$\begin{aligned} &Q(x) = e^{-imv\cdot x}\left(1+\frac{i\rlap{/}D_\perp}{2m}+\cdots\right)Q_v(x)\,,\\ &D_\perp^\mu = D^\mu - v^\mu v\cdot D\,. \end{aligned} \tag{1}$$

The matrix elements of the bare fields between the on-shell quark with momentum $p = mv + k$ and the vacuum in both theories are given by the on-shell wave-function renormalization constants:

$$\begin{aligned} &<0|Q_0|Q(p)> = \left(Z_Q^{\mathrm{os}}\right)^{1/2} u(p)\,,\\ &<0|Q_{v0}|Q(p)> = \left(\tilde{Z}_Q^{\mathrm{os}}\right)^{1/2} u_v(k) \end{aligned} \tag{2}$$

(HQET renormalization constants are denoted by $\tilde{Z}$). The Dirac spinors are related by the Foldy–Wouthuysen transformation

$$u(mv+k) = \left[1+\frac{\rlap{/}k}{2m}+\mathcal{O}\left(\frac{k^2}{m^2}\right)\right]u_v(k)\,.$$

Therefore, the bare fields are related by

$$\begin{aligned} Q_0(x) = e^{-imv\cdot x}\Bigg[&z_0^{1/2}\left(1+\frac{i\rlap{/}D_\perp}{2m}\right)Q_{v0}(x)\\ &+\mathcal{O}\left(\frac{1}{m^2}\right)\Bigg]\,, \end{aligned} \tag{3}$$

where the bare matching coefficient is

$$z_0 = \frac{Z_Q^{\mathrm{os}}(g_0^{(n_l+1)}, a_0^{(n_l+1)})}{\tilde{Z}_Q^{\mathrm{os}}(g_0^{(n_l)}, a_0^{(n_l)})} \tag{4}$$

(we use the covariant gauge: the gauge-fixing term in the Lagrangian is $-(\partial_\mu A_0^{a\mu})/(2a_0)$, and the free gluon propagator is $(-i/p^2)(g_{\mu\nu} - (1-a_0)p_\mu p_\nu/p^2)$; the number of flavours in QCD is $n_f = n_l + 1$). The $\mathcal{O}(1/m)$ matching coefficient in (3) is equal to the leading one, z_0; this reflexes the reparametrization invariance [5]. The $\overline{\mathrm{MS}}$ renormalized fields are related by the formula similar to (3), with the renormalized decoupling coefficient

$$z(\mu) = \frac{\tilde{Z}_Q(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))}{Z_Q(\alpha_s^{(n_l+1)}(\mu), a^{(n_l+1)}(\mu))} z_0\,. \tag{5}$$

If there are no massive flavours except Q, then $\tilde{Z}_Q^{\mathrm{os}} = 1$ because all loop corrections are scale-free. The QCD on-shell renormalization constant Z_Q^{os} contains the single scale m in this case; it has been calculated [6] up to three loops. The three-loop $\overline{\mathrm{MS}}$ anomalous dimensions of Q_v [6,7] and Q [8] are also known. We have to express all three quantities $Z_Q^{\mathrm{os}}(g_0^{(n_l+1)}, a_0^{(n_l+1)})$, $Z_Q(\alpha_s^{(n_l+1)}(\mu), a^{(n_l+1)}(\mu))$, $\tilde{Z}_Q(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))$ via the same variables, say,

$\alpha_s^{(n_l)}(\mu)$, $a^{(n_l)}(\mu)$, see [9]. The explicit result for the renormalized matching coefficient $z(\mu)$ can be found in [1]. Gauge dependence first appears at three loops, as in $Z_Q^{\rm os}$ [6]. The requirement of finiteness of the renormalized matching coefficient (5) at $\varepsilon \to 0$ has allowed the authors of [6] to extract $\tilde{Z}_Q$ from their result for $Z_Q^{\rm os}$.

In the large-β_0 limit (see Chapter 8 in [10] for a pedagogical introduction):

$$z(\mu) = 1 + \int_0^\beta \frac{d\beta}{\beta}\left(\frac{\gamma(\beta)}{2\beta} - \frac{\gamma_0}{2\beta_0}\right) + \frac{1}{\beta_0}\int_0^\infty du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (6)$$

where $\beta = \beta_0\alpha_s/(4\pi)$, $\gamma = \gamma_0\alpha_s/(4\pi) + \cdots$ (differences of n_l-flavour and (n_l+1)-flavour quantities can be neglected at the $1/\beta_0$ order). The difference of the QCD and HQET anomalous dimensions $\gamma = \gamma_Q - \tilde{\gamma}_Q$ (it is gauge invariant at this order) and the Borel image $S(u)$ are [11,12,10]

$$\begin{aligned} \gamma(\beta) &= -2\frac{\beta}{\beta_0}F(-\beta,0) = \\ &2C_F\frac{\beta}{\beta_0}\frac{(1+\beta)(1+\frac{2}{3}\beta)}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)}, \\ S(u) &= \frac{F(0,u)-F(0,0)}{u} = \\ &-6C_F\left[e^{(L+5/3)u}\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u^2) - \frac{1}{2u}\right]. \end{aligned} \quad (7)$$

This Borel image has infrared renormalon poles at each positive half-integer u and at $u=2$. Therefore, the integral in (6) is not well defined. Comparing its residue at the leading pole $u=1/2$ with the residue of the static-quark self-energy at its ultraviolet pole $u=1/2$ [13], we can express the renormalon ambiguity of $z(\mu)$ as

$$\Delta z(\mu) = \frac{3}{2}\frac{\Delta\bar{\Lambda}}{m} \quad (8)$$

($\bar{\Lambda}$ is the ground-state meson residual energy). The matching coefficient is gauge invariant at the order $1/\beta_0$. Expanding $\gamma(\beta)$ and $S(u)$ and integrating, we obtain confirm the contributions with the highest power of n_l in each term in our three-loop result, and predict such a contribution at α_s^4.

Numerically, in the Landau gauge at $n_l = 4$

$$\begin{aligned} z(m) &= 1 - \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} \\ &- (16.6629 - 4.5421)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 \\ &- (153.4076 + 42.6271 - 61.5397)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\ &- (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots \\ &= 1 - \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} - 12.1208\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 \\ &- 134.4950\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\ &- (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots \end{aligned} \quad (9)$$

(β_0 is for $n_l = 4$ flavours). Naive nonabelianization [11] works rather well at two and three loops. Numerical convergence of the series is very poor; this is related to the infrared renormalon at $u = 1/2$.

Now let us consider the relation between the $\overline{\rm MS}$ renormalized electron field in QED and the Bloch–Nordsieck electron field. The bare matching coefficient $z_0 = Z_\psi^{\rm os}$ is gauge invariant to all orders, see [6]. In the Bloch-Nordsieck model, due to exponentiation, $\log\tilde{Z}_\psi = (3 - a^{(0)})\alpha^{(0)}/(4\pi\varepsilon)$ (where the 0-flavour $\alpha^{(0)}$ is equal to the on-shell $\alpha \approx 1/137$). In the full QED, it is supposed that $\log Z_\psi = a^{(1)}\alpha^{(1)}/(4\pi\varepsilon)$ + (gauge-invariant higher terms) (this is equivalent to the similar statement for the anomalous dimension γ_ψ, because $d\log(a^{(1)}\alpha^{(1)})/d\log\mu = -2\varepsilon$ exactly). This has been demonstrated up to four loops by the direct calculation [14], but there is no general proof. The gauge dependence cancels in $\log(\tilde{Z}_\psi/Z_\psi)$ because of the QED decoupling relation $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$. Therefore, the renormalized matching coefficient $z(\mu)$ in QED is gauge invariant at least up to four loops. The three-loop result is presented in [1].

2. Heavy–light currents

Now we shall consider [2] $\overline{\text{MS}}$ renormalized heavy–light QCD quark currents

$$j(\mu) = Z_j^{-1}(\mu) j_0\,, \quad j_0 = \bar{q}_0 \Gamma Q_0\,, \tag{10}$$

where Γ is a Dirac matrix. They can be expressed via operators in HQET

$$j(\mu) = C_\Gamma(\mu)\tilde{\jmath}(\mu) + \frac{1}{2m}\sum_i B_i(\mu) O_i(\mu) + \mathcal{O}\left(\frac{1}{m^2}\right)\,, \tag{11}$$

where

$$\tilde{\jmath}(\mu) = \tilde{Z}_j^{-1}(\mu)\tilde{\jmath}_0\,, \quad \tilde{\jmath}_0 = \bar{q}_0 \Gamma Q_{v0}\,, \tag{12}$$

and O_i are dimension-4 HQET operators with appropriate quantum numbers.

There are 8 Dirac structures giving non-vanishing quark currents in 4 dimensions:

$$\begin{gathered}\Gamma = 1\,, \quad \rlap/v\,, \quad \gamma_\perp^\alpha\,, \quad \gamma_\perp^\alpha \rlap/v\,, \\ \gamma_\perp^{[\alpha}\gamma_\perp^{\beta]}\,, \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta]}\rlap/v\,, \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta}\gamma_\perp^{\gamma]}\,, \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta}\gamma_\perp^{\gamma]}\rlap/v\,,\end{gathered} \tag{13}$$

where $\gamma_\perp^\alpha = \gamma^\alpha - \rlap/v v^\alpha$. The last four of them can be obtained from the first four by multiplying by the 't Hooft–Veltman γ_5^{HV}. We are concerned with flavour non-singlet currents only, therefore, we may also use the anticommuting γ_5^{AC} (there is no anomaly). The currents renormalized at a scale μ with different prescriptions for γ_5 are related by [15]

$$\begin{aligned}&\left(\bar{q}\gamma_5^{\text{AC}} Q\right)_\mu = Z_P(\mu)\left(\bar{q}\gamma_5^{\text{HV}} Q\right)_\mu\,, \\ &\left(\bar{q}\gamma_5^{\text{AC}}\gamma^\alpha Q\right)_\mu = Z_A(\mu)\left(\bar{q}\gamma_5^{\text{HV}}\gamma^\alpha Q\right)_\mu\,, \\ &\left(\bar{q}\gamma_5^{\text{AC}}\gamma^{[\alpha}\gamma^{\beta]} Q\right)_\mu = Z_T(\mu)\left(\bar{q}\gamma_5^{\text{HV}}\gamma^{[\alpha}\gamma^{\beta]} Q\right)_\mu\,,\end{aligned} \tag{14}$$

where the finite renormalization constants $Z_{P,A,T}$ can be reconstructed from the differences of the anomalous dimensions of the currents. Multiplying Γ by γ_5^{AC} does not change the anomalous dimension. In the case of $\Gamma = \gamma^{[\alpha}\gamma^{\beta]}$, multiplying it by γ_5^{HV} just permutes its components, and also does not change the anomalous dimension, therefore,

$$Z_T(\mu) = 1\,; \tag{15}$$

$Z_{P,A}(\mu)$ are known up to three loops [15].

The anomalous dimension of the HQET current (12) does not depend on the Dirac structure Γ. Therefore, there are no factors similar to $Z_{P,A}$ in HQET. Multiplying Γ by γ_5^{AC} does not change the matching coefficient. Therefore, the matching coefficients for the currents in the second row of (13) can be obtained from those for the first row. In the v rest frame

$$\begin{aligned}Z_P(\mu) &= \frac{C_{\gamma_5^{\text{AC}}}(\mu)}{C_{\gamma_5^{\text{HV}}}(\mu)} = \frac{C_1(\mu)}{C_{\gamma^0\gamma^1\gamma^2\gamma^3}(\mu)}\,, \\ Z_A(\mu) &= \frac{C_{\gamma_5^{\text{AC}}\gamma^0}(\mu)}{C_{\gamma_5^{\text{HV}}\gamma^0}(\mu)} = \frac{C_{\gamma^0}(\mu)}{C_{\gamma^1\gamma^2\gamma^3}(\mu)} \\ &= \frac{C_{\gamma_5^{\text{AC}}\gamma^3}(\mu)}{C_{\gamma_5^{\text{HV}}\gamma^3}(\mu)} = \frac{C_{\gamma^3}(\mu)}{C_{\gamma^0\gamma^1\gamma^2}(\mu)}\,, \\ Z_T(\mu) &= \frac{C_{\gamma_5^{\text{AC}}\gamma^0\gamma^1}(\mu)}{C_{\gamma_5^{\text{HV}}\gamma^0\gamma^1}(\mu)} = \frac{C_{\gamma^0\gamma^1}(\mu)}{C_{\gamma^2\gamma^3}(\mu)} \\ &= \frac{C_{\gamma_5^{\text{AC}}\gamma^2\gamma^3}(\mu)}{C_{\gamma_5^{\text{HV}}\gamma^2\gamma^3}(\mu)} = \frac{C_{\gamma^2\gamma^3}(\mu)}{C_{\gamma^0\gamma^1}(\mu)} = 1\,.\end{aligned} \tag{16}$$

In particular, $C_{\gamma_\perp \rlap/v}(\mu) = C_{\gamma_\perp^{[\alpha}\gamma_\perp^{\beta]}}(\mu)$. In the following we shall consider only the matching coefficients for the first 4 Dirac structures in (13).

In order to find the coefficients $C_\Gamma(\mu)$, we equate matrix elements of the left- and right-hand side of (11) from the heavy quark with momentum $p = mv + k$ to the light quark with momentum k_q:

$$\begin{aligned}&<q(k_q)|j(\mu)|Q(mv+k)> = \\ &C_\Gamma(\mu)<q(k_q)|\tilde{\jmath}(\mu)|Q_v(k)> + \mathcal{O}\left(\frac{k, k_q}{m}\right)\,.\end{aligned} \tag{17}$$

The on-shell matrix elements are

$$\begin{aligned}<q(k_q)|j(\mu)|Q(p)> &= \bar{u}_q(k_q)\Gamma(p,k_q)u(p) \\ &\quad\times Z_j^{-1}(\mu) Z_Q^{1/2} Z_q^{1/2}\,, \\ <q(k_q)|\tilde{\jmath}(\mu)|Q_v(k)> &= \bar{u}_q(k_q)\tilde{\Gamma}(k,k_q)u_v(k) \\ &\quad\times \tilde{Z}_j^{-1}(\mu)\tilde{Z}_Q^{1/2}\tilde{Z}_q^{1/2}\,,\end{aligned} \tag{18}$$

where $\Gamma(p,k_q)$ and $\tilde{\Gamma}(k,k_q)$ are the bare vertex functions, and $\tilde{Z}_q$ differs from Z_q because there are no Q loops in HQET. The difference between $u(mv+k)$ and $u_v(k)$ is of order k/m, and can be neglected. It is most convenient to use $k = k_q = 0$, then the $\mathcal{O}(1/m)$ term is absent. The QCD vertex has two Dirac structures:

$$\Gamma(mv,0) = \Gamma \cdot (A + B\not{v}) .$$

This leads to

$$\bar{u}(0)\Gamma(mv,0)u(mv) = \bar{\Gamma}(mv,0)\,\bar{u}(0)\Gamma u(mv) ,$$
$$\bar{\Gamma}(mv,0) = A + B .$$

The HQET vertex has just one Dirac structure. Therefore,

$$C_\Gamma(\mu) = \frac{\bar{\Gamma}(mv,0) Z_j^{-1}(\mu) Z_Q^{1/2} Z_q^{1/2}}{\tilde{\Gamma}(0,0)\tilde{Z}_j^{-1}(\mu)\tilde{Z}_Q^{1/2}\tilde{Z}_q^{1/2}} . \quad (19)$$

If all flavours except Q are massless, all loop corrections to $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ contain no scale and hence vanish: $\tilde{\Gamma}(0,0) = 1$, $\tilde{Z}_Q = 1$, $\tilde{Z}_q = 1$. The quantities $\Gamma(mv,0)$, Z_Q, and Z_q contain a single scale m; Z_Q has been calculated up to 3 loops in [6], Z_q in [9], and $\Gamma(mv,0)$ in the present work [2]. The $\overline{\text{MS}}$ renormalization constants $\tilde{Z}_j$ [7] and Z_j [16] (for all Γ) are also known to 3 loops.

If there is another massive flavour (c in b-quark HQET), then $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ contain a single scale m_c. The first two quantities have been calculated up to 3 loops in [17]; the last one is known from [9]. The quantities $\Gamma(mv,0)$, Z_Q, and Z_q now contain 2 scales, and are non-trivial functions of $x = m_c/m$. The renormalization constant Z_Q has been calculated in this case, up to 3 loops, in [18] (the master integrals appearing in this case are discussed in Ref. [19]). The other two quantities are found in this work [2].

The bare on-shell QCD quantities $\bar{\Gamma}(mv,0)$, Z_Q, and Z_q are expressed via $g_0^{(n_f)}$ (and $m_{c0}^{(n_f)}$ if it is non-zero; we re-express it via the on-shell mass m_c). They don't contain μ. The $\overline{\text{MS}}$ QCD renormalization constant Z_j is expressed via $\alpha_s^{(n_f)}(\mu)$. The bare on-shell HQET quantities $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ are expressed via $g_0^{(n_f-1)}$ and $m_{c0}^{(n_f-1)}$ (they are trivial at $m_c = 0$); we re-express $m_{c0}^{(n_f-1)}$ via the on-shell mass m_c (which is the same in both theories). These bare quantities also don't contain μ. Finally, the $\overline{\text{MS}}$ HQET renormalization constant $\tilde{Z}_j$ is expressed via $\alpha_s^{(n_f-1)}(\mu)$. We re-express all the quantities in (19) via $\alpha_s^{(n_f-1)}(\mu)$, see [9].

From equation of motion we have

$$i\partial_\alpha j^\alpha = i\partial_\alpha j_0^\alpha = m_0 j_0 = m(\mu) j(\mu) , \quad (20)$$

where $m(\mu)$ is the $\overline{\text{MS}}$ mass of the heavy quark Q. Taking the on-shell matrix element between the heavy quark with $p = mv$ and the light quark with $k_q = 0$ and re-expressing both QCD matrix elements via the matrix element of the HQET current with $\Gamma = 1$, we obtain [11]

$$m C_{\not{v}}(\mu) = m(\mu) C_1(\mu) . \quad (21)$$

The ratio $m(\mu)/m$ has been calculated at three loops in [20] (for $m_c \neq 0$ in [18]).

The matching coefficients have been calculated up to 2 loops in [11], and to 3 loops in the present work [2]. Analytical expressions are long; numerically, at $m_c = 0$ and $\mu = m$ we have

$$\begin{aligned}
C_1^{(2)} &= 7.55 + 1.09 = 8.64 , \\
C_{\not{v}}^{(2)} &= -5.47 + 3.06 = -2.41 , \\
C_{\gamma_\perp}^{(2)} &= -9.87 + 1.53 = -8.34 , \\
C_{\gamma_\perp \not{v}}^{(2)} &= -14.13 + 2.42 = -11.70 , \\
C_1^{(3)} &= 64.74 + 75.34 - 38.16 = 101.92 , \\
C_{\not{v}}^{(3)} &= -37.25 - 10.72 + 29.74 = -18.23 , \\
C_{\gamma_\perp}^{(3)} &= -88.92 - 46.34 + 45.34 = -89.92 , \\
C_{\gamma_\perp \not{v}}^{(3)} &= -123.61 - 63.57 + 63.22 = -123.96
\end{aligned}$$

(in the middle part of each formula, terms with descending powers of $\beta_0^{(n_f-1)}$ are shown separately). Naive nonabelianization [11] works reasonably well.

At $m_c \neq 0$, results are expressed via the master integrals depending on $x = m_c/m$ [19]. Their status is summarized in the Tables 1–4 in this paper. In the present work [2], we were able to obtain exact analytical expressions (via harmonic polylogarithms of x) for $\mathcal{O}(1)$ terms in the master integrals 5.2, 5.2a, from the requirement of finiteness of the matching coefficients. Therefore, the Table 3 in [19] should be now replaced with the following Table 1 (DE means the method of differential equations, and MB the Mellin–Barnes representation). Unfortunately, $\mathcal{O}(\varepsilon)$ terms in 4 master integrals are still known only as truncated series in x (the entries x in the table). Therefore, the m_c corrections to the 3-loop matching coefficients are also known only as truncated series in x (or numerical approximations).

We let's apply our results to the matrix elements between a B or B^* meson with momentum p and the vacuum:

$$\begin{aligned}
&<0|\left(\bar{q}\gamma_5^{\mathrm{AC}}Q\right)_\mu|B> = -im_B f_B^P(\mu)\,, \qquad (22)\\
&<0|\bar{q}\gamma^\alpha\gamma_5^{\mathrm{AC}}Q|B> = if_B p^\alpha\,,\\
&<0|\bar{q}\gamma^\alpha Q|B^*> = im_{B^*}f_{B^*}e^\alpha\,,\\
&<0|\left(\bar{q}\sigma^{\alpha\beta}Q\right)_\mu|B^*> = f_{B^*}^T(\mu)(p^\alpha e^\beta - p^\beta e^\alpha)\,.
\end{aligned}$$

The corresponding HQET matrix elements in the v rest frame are

$$\begin{aligned}
&<0|\left(\bar{q}\gamma_5^{\mathrm{AC}}Q_v\right)_\mu|B(\vec{k}\,)>_{\mathrm{nr}} = -iF(\mu)\,,\\
&<0|\left(\bar{q}\vec{\gamma}Q_v\right)_\mu|B^*(\vec{k}\,)>_{\mathrm{nr}} = iF(\mu)\vec{e}, \qquad (23)
\end{aligned}$$

where the single-meson states are normalized by the non-relativistic condition

$$_{\mathrm{nr}}<B(\vec{k}\,')|B(\vec{k}\,)>_{\mathrm{nr}} = (2\pi)^3\delta(\vec{k}\,'-\vec{k}\,)\,.$$

These two matrix elements are characterized by a single hadronic parameter $F(\mu)$ due to the heavy-quark spin symmetry. From (20) we have [11]

$$\frac{f_B^P(\mu)}{f_B} = \frac{m_B}{m(\mu)}\,, \qquad (24)$$

where we may replace m_B by the on-shell b-quark mass m, neglecting power corrections.

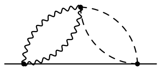
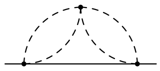
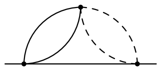
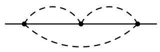
Our main result is the ratio f_{B^*}/f_B. At $m_c = 0$

$$\begin{aligned}
\frac{f_{B^*}}{f_B} &= 1 - \frac{1}{2}C_F\frac{\alpha_s^{(4)}(m)}{\pi} +\\
&(C_F r_F + C_A r_A + T_F n_l r_l + T_F r_h)\,C_F\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2\\
&+ \bigl(C_F^2 r_{FF} + C_F C_A r_{FA} + C_A^2 r_{AA} + C_F T_F n_l r_{Fl}\\
&+ C_F T_F r_{Fh} + C_A T_F n_l r_{Al} + C_A T_F r_{Ah}\\
&+ T_F^2 n_l^2 r_{ll} + T_F^2 n_l r_{lh} + T_F^2 r_{hh}\bigr)C_F\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3\\
&+ \mathcal{O}\left(\alpha_s^4, \frac{\Lambda}{m}\right)\,, \qquad (25)
\end{aligned}$$

where

$$\begin{aligned}
r_F &= \frac{1}{3}\pi^2\log 2 - \frac{1}{2}\zeta_3 - \frac{4}{9}\pi^2 + \frac{31}{48}\,,\\
r_A &= -\frac{1}{6}\pi^2\log 2 + \frac{1}{4}\zeta_3 + \frac{1}{6}\pi^2 - \frac{263}{144}\,,\\
r_l &= \frac{19}{36}\,, \quad r_h = \frac{1}{9}\pi^2 - \frac{41}{36}\,,\\
r_{FF} &= -\frac{8}{3}a_4 - \frac{1}{9}\log^4 2 - \frac{2}{9}\pi^2\log^2 2\\
&+ \frac{19}{6}\pi^2\log 2 + \frac{25}{12}\zeta_5 - \frac{1}{9}\pi^2\zeta_3 + \frac{11}{8}\zeta_3\\
&- \frac{43}{1080}\pi^4 - \frac{43}{24}\pi^2 - \frac{289}{192}\,,\\
r_{FA} &= -\frac{20}{9}a_4 - \frac{5}{24}\log^4 2 - \frac{5}{27}\pi^2\log^2 2\\
&+ \frac{305}{108}\pi^2\log 2 - \frac{115}{48}\zeta_5 + \frac{1}{12}\pi^2\zeta_3 - \frac{899}{144}\zeta_3\\
&+ \frac{817}{12960}\pi^4 - \frac{2233}{648}\pi^2 + \frac{4681}{864}\,,\\
r_{AA} &= \frac{16}{9}a_4 + \frac{2}{27}\log^4 2 + \frac{4}{27}\pi^2\log^2 2\\
&- \frac{119}{54}\pi^2\log 2 + \frac{5}{6}\zeta_5 - \frac{11}{144}\pi^2\zeta_3 + \frac{343}{144}\zeta_3\\
&- \frac{17}{3240}\pi^4 + \frac{2839}{1728}\pi^2 - \frac{48125}{5184}\,,\\
r_{Fl} &= \frac{16}{9}a_4 + \frac{2}{27}\log^4 2 + \frac{4}{27}\pi^2\log^2 2\\
&- \frac{28}{27}\pi^2\log 2 + \frac{25}{9}\zeta_3 - \frac{11}{324}\pi^4 + \frac{179}{162}\pi^2 - \frac{815}{864}\,,\\
r_{Fh} &= -\frac{32}{9}a_4 - \frac{4}{27}\log^4 2 + \frac{4}{27}\pi^2\log^2 2\\
&+ \frac{46}{27}\pi^2\log 2 + 5\zeta_3 - \frac{1}{162}\pi^4 - \frac{1439}{1080}\pi^2 - \frac{119}{36}\,,\\
r_{Al} &= \frac{8}{9}a_4 - \frac{1}{27}\log^4 2 - \frac{2}{27}\pi^2\log^2 2\\
&+ \frac{14}{27}\pi^2\log 2 - \frac{13}{18}\zeta_3 + \frac{13}{3240}\pi^4 - \frac{17}{72} + \frac{422}{81}\,,\\
r_{Ah} &= \frac{16}{9}a_4 + \frac{2}{27}\log^4 2 - \frac{2}{27}\pi^2\log^2 2\\
&- \frac{86}{27}\pi^2\log 2 + \frac{55}{48}\zeta_5 - \frac{31}{144}\pi^2\zeta_3 + \frac{43}{36}\zeta_3\\
&+ \frac{8}{405}\pi^4 + \frac{577}{270}\pi^2 - \frac{1121}{648}\,,\\
r_{ll} &= -\frac{1}{27}\pi^2 - \frac{203}{324}\,, \quad r_{lh} = \frac{5}{81}\pi^2 - \frac{101}{162}\,,
\end{aligned}$$

Table 1
Master integrals with 5 lines

	 5.1, 5.1a	 5.2, 5.2a	 5.3, 5.3a	 5.4, 5.4a
ε^{-3}	DE	DE	DE	DE
ε^{-2}	DE	DE	DE	DE
ε^{-1}	DE	DE	DE	DE
1	DE	NEW	MB	DE
ε	DE	x	x	DE
ε^2				DE

$$r_{hh} = -\frac{8}{9}\zeta_3 + \frac{8}{405}\pi^2 + \frac{277}{324}$$

($a_4 = \mathrm{Li}_4(1/2)$). The result for $f^T_{B^*}(m)/f_{B^*}$ is similar.

Numerically,

$$\left(\frac{f_{B^*}}{f_B}\right)^{(2)} = -4.40 - 1.97 = -6.37\,,$$
$$\left(\frac{f^T_{B^*}(m_b)}{f_{B^*}}\right)^{(2)} = -4.26 + 0.89 = -3.37\,,$$
$$\left(\frac{f_{B^*}}{f_B}\right)^{(3)} = -51.67 - 42.21 + 16.33 = -77.55\,,$$
$$\left(\frac{f^T_{B^*}(m_b)}{f_{B^*}}\right)^{(3)} = -34.69 - 22.91 + 19.07$$
$$= -38.53\,.$$

Naive nonabelianization [11] works reasonably well.

Asymptotics of the perturbative coefficients for the matching coefficients at a large number of loops $l \gg 1$ have been investigated in Ref. [21] in a model-independent way. The results contain three unknown normalization constants $N_{0,1,2} \sim 1$. The asymptotics of the perturbative coefficients for f_{B^*}/f_B contain N_0 and N_2; in the case of $m/\hat{m}$ it contains only N_0:

$$\left(\frac{f_{B^*}}{f_B}\right)^{(n+1)}_{L=-5/3} = -\frac{14}{27}\left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right.$$
$$\left. + \frac{2}{7}\left(\frac{50}{3}n\right)^{-9/25}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]\frac{N_2}{N_0}\right\}$$
$$\times \left(\frac{m}{\hat{m}}\right)^{(n+1)}_{L=-5/3}\,. \qquad (26)$$

The coefficient of N_2/N_0 is about 0.08 at $n = 2$, and it seems reasonable to neglect this contribution. Neglecting also $1/n$ corrections, we obtain [21]

$$\left(\frac{f_{B^*}}{f_B}\right)^{(3)}_{L=-5/3} = -\frac{14}{27} \cdot 56.37 = -29.23\,.$$

Our exact result -37.787 agrees with this prediction reasonably well. However, $1/n$ corrections are large and tend to break this agreement. It is natural to expect that $1/n^2$ (and higher) corrections are also substantial at $n = 2$.

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