

Four Loop Massless Propagators: a Numerical Evaluation of All Master Integrals

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Abstract

We present numerical results which are needed to evaluate all non-trivial master integrals for four-loop massless propagators, confirming the recent analytic results of [1] and evaluating an extra order in ε expansion for each master integral.

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1 Introduction

Sector decomposition in its practical aspect is a constructive method used to evaluate Feynman integrals numerically. The goal of sector decomposition is to decompose the initial integration domain into appropriate subdomains (sectors) and introduce, in each sector, new variables in such a way that the integrand factorizes, i.e. becomes equal to a monomial in new variables times a non-singular function.

Originally it was used as a tool for analyzing the convergence and proving theorems on renormalization and asymptotic expansions of Feynman integrals [2, 3, 4, 5, 6]. After a pioneering work [7] sector decomposition has become an efficient tool for numerical evaluating Feynman integrals (see Ref. [8] for a recent review). At present, there are two public codes performing the sector decomposition [9] and [10].

The latter one was named **FIESTA** which stands for “Feynman Integral Evaluation by a Sector decomposiTion Approach”. Last year **FIESTA** has been greatly improved in various aspects [11]. The code is capable of evaluating many classes of integrals that one would not be able to evaluate with the original **FIESTA** 1. Moreover the code can now be applied to solve the problem of obtaining asymptotic expansions of Feynman integrals in various limits of momenta and masses and to find a list of all poles of an integral in space-time dimension d . During the last year **FIESTA** was widely used, some of application are listed in [12].

In the current paper we present numerical results for *master integrals* (MI's) for four-loop massless propagators which are relevant for many important physical applications, like the calculation of the total cross-section of e^+e^- annihilation into hadrons, the Higgs decay rate into hadrons, the semihadronic decay rate of the τ lepton and the running of the fine structure coupling constant (see [13, 14] for details). We confirm numerically the recent analytic results of work [1] and evaluate an extra order in epsilon expansion for each MI.

2 Theoretical background and software structure

FIESTA calculates Feynman integrals with the sector decomposition approach. It is based on the α -representation of Feynman integrals. After performing Dirac and Lorentz algebra one is left with a scalar dimensionally regularized Feynman integral [15]

$$F(a_1, \dots, a_n) = \int \cdots \int \frac{d^d k_1 \dots d^d k_l}{E_1^{a_1} \dots E_n^{a_n}}, \quad (1)$$

where $d = 4 - 2\varepsilon$ is the space-time dimension, a_n are indices, l is the number of loops and $1/E_n$ are propagators. We work in Minkowski space where the standard propagators are the form $1/(m^2 - p^2 - i0)$. Other propagators are permitted, for example, $1/(v \cdot k \pm i0)$ may appear in diagrams contributing to static quark potentials

or in HQET³ where v is the quark velocity (see, e.g [16]). Substituting

$$\frac{1}{E_i^{a_i}} = \frac{e^{ai\pi/2}}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a_i-1} e^{-iE_i\alpha}, \quad (2)$$

changing the integration order, performing the integration over loop momenta, replacing α_i with $x_i\eta$ and integrating over η one arrives at the following formula (see e.g. [17]):

$$F(a_1, \dots, a_n) = \frac{\Gamma(A - ld/2)}{\prod_{j=1}^n \Gamma(a_j)} \int_{x_j \geq 0} dx_1 \dots dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \left(\prod_{j=1}^n x_j^{a_j-1}\right) \frac{U^{A-(l+1)d/2}}{F^{A-l d/2}}, \quad (3)$$

where $A = \sum_{i=1}^n a_i$ and U and F are constructively defined polynomials of x_i . The formula (3) has no sense if some of the indices are non-positive integers, so in case of those the integration is performed according to the rule

$$\int_0^\infty dx \frac{x^{(a-1)}}{\Gamma(a)} f(x) = f^{(n)}(0)$$

where a is a non-positive integer.

After performing the decomposition of the integration region into the so-called *primary sectors* [7] and making a variable replacement, one results in a linear combination of integrals of the following form:

$$\int_{x_j=0}^1 dx_1 \dots dx_{n'} \left(\prod_{j=1}^{n'} x_j^{a_j-1}\right) \frac{U^{A-(l+1)d/2}}{F^{A-l d/2}} \quad (4)$$

If the functions $\frac{U^{A-(l+1)d/2}}{F^{A-l d/2}}$ had no singularities in ε , one would be able to perform the expansion in ε and perform the numerical integration afterwards. However, in general one has to resolve the singularities first, which is not possible for general U and F . Thus, one starts a process the sector decomposition aiming to end with a sum of similar expressions, but with new functions U and F which have no singularities (all the singularities are now due to the part $\prod_{j=1}^n x_j^{a_j-1}$). Obviously it is a good idea to make the sector decomposition process constructive and to end with a minimally possible number of sectors. The way sector decomposition is performed is called a *sector decomposition strategy* and is an essential part of the algorithm.

After performing the sector decomposition one can resolve the singularities by evaluating the first terms of the Taylor series: in those terms one integration is taken analytically, and the remainder has no singularities. Afterwards the ε -expansion can be performed and finally one can do the numerical integration and return the result.

³Heavy-Quark Effective Theory

Please keep in mind that this approach works only using numerical integration: numeric values for all invariants should be specified at the very early stage, after generating the functions U and F .

FIESTA is written in `Mathematica` [19] and C. The user is not supposed to use the C part directly as it is launched from `Mathematica` via the Mathlink protocol. When the integrand expressions are ready, `Mathematica` submits long strings representing integrands for integration; the C part translates them into an internal representation optimizing evaluation speed. Afterwards it uses some numerical integrator to perform the numerical integration of the integrand. The original FIESTA employed a `Fortran` implementation of `Vegas` as an integrator. Later we plugged in the `Cuba` library [18]. By default FIESTA uses the `Vegas` integrator, but this behavior can be easily controlled by the user. Both `Mathematica` and C parts can be efficiently parallelized on modern multi-core computers; the C part also parallelizable on clusters.

The FIESTA user interface is based on `Mathematica`. To run FIESTA, the user has to load the `FIESTA.2.0.0.m` into `Mathematica` 6 or 7. In order to evaluate a Feynman integral one has to use the command

```
SDEvaluate[U,F, $\ell$ ,indices,order],
```

where U and F are the functions from formula (3), ℓ is the number of loops, `indices` is the set of indices and `order` is the required order of the ε -expansion.

To avoid manual construction of U and F one can use a build-in function `UF` and launch the evaluation as follows:

```
SDEvaluate[UF[loop_momenta,propagators,subst],indices,order],
```

where `subst` is a set of substitutions for external momenta, masses and other values (please note that the code performs numerical integrations, therefore the functions U and F should not depend on any external kinematic invariants).

Example:

```
SDEvaluate[UF[{k},{-k2,-(k+p1)2,-(k+p1+p2)2,-(k+p1+p2+p4)2},
{p12→0,p22→0,p42→0,p1 p2→-s/2,p2 p4→-t/2,p1 p4→-(s+t)/2,
s→-3,t→-1}],{1,1,1,1},0]
```

performs an evaluation of the massless on-shell box diagram where the Mandelstam variables are equal to $s = -3$ and $t = -1$.

3 Numerical results for four-loop massless propagators

In [20] a full set of the four-loop massless propagator-like MI's was identified. There exist 28 independent MI's. Analytical results for these integrals were obtained in [1].

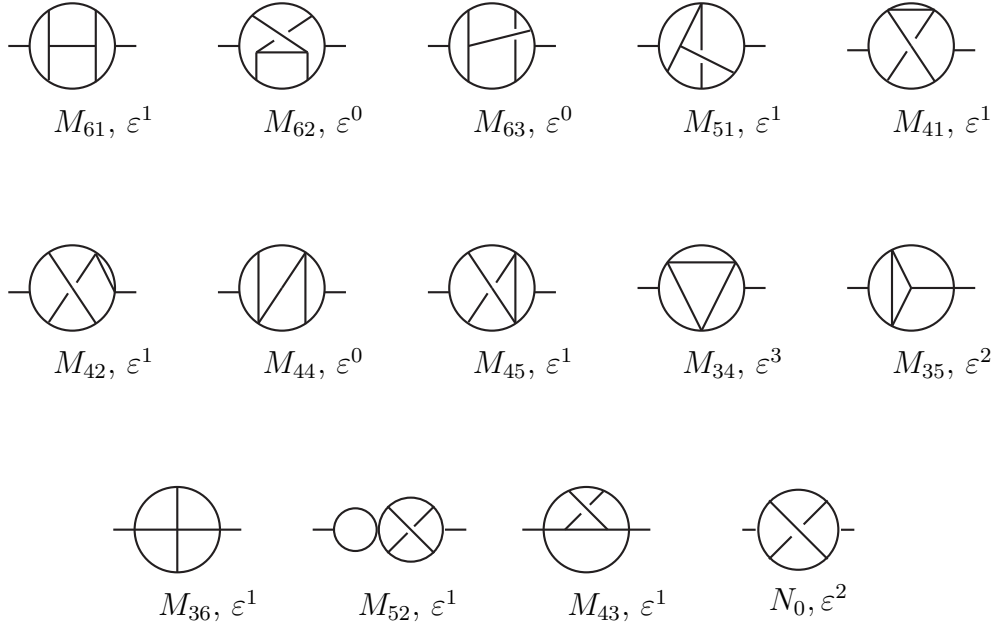


Figure 1: M_{61} — M_{43} : the thirteen complicated four-loop master integrals according to [1]. The integrals are ordered (if read from left to right and then from top to bottom) according to their complexity. The two MI's M_{52} and M_{43} can be identically expressed through the three-loop nonplanar MI N_0 .

By an analytical result is meant *not* an analytical expression for a master integral taken at a generic value of the space-time dimension d (which is usually not possible except for the simplest cases), but rather the analytic expressions for *proper* number of terms in its Laurent expansion in d around the physical value $d = 4$.

As it was shown in [1], the full set of all 28 integrals can be divided in three parts:

1. 9 “primitive” integrals which are expressible in terms of Γ -functions;
2. 6 “simple” integrals which could be expressed in terms of Γ -functions and the so-called generalized two-loop diagram with insertions;
3. remaining 13 “complicated” integrals are quite difficult for both analytical and numerical evaluation.

The complicated MI's are pictured in Fig. 1. The MI's are labeled as M_{ij} where the first index, i , stands for the number of internal lines minus five while the second index, j , numerates (starting from 1) different integrals with the same i . ε^m after M_{ij} stands for the maximal term in ε -expansion of M_{ij} which one needs to know for

evaluation of the contribution of the integral to the final result for a four-loop integral after reduction is done, see [1]. That is, m stands for the maximal power of a spurious pole $1/\varepsilon^m$ which could appear in front of M_{ij} in the process of reduction to masters.

Primitive and simple integrals are known analytically. Two of the complicated integrals (M_{43} and M_{52}) are related by a simple factor with the three-loop MI N_0 [1] so it is enough to evaluate remaining eleven complicated MI's $M_{61} - M_{36}$ as well as first three terms of the ε -expansion of N_0 .

We have calculated all them by means of FIESTA. We used the **Cuba Vegas** integrator with different parameters used for the numerical integration. Evaluations were performed on 8-core (2x4) Intel Xeon E5472 3.0 GHz, 4GB/core RAM, 4.6TB disk/node computers in fully parallel mode, i.e., both Mathematica and C parts were completely parallelized. The square of the external momentum q was chosen as -1 : $q^2 = -1$. The FIESTA input for, say, the integral M_{44} reads:

```
Fm44= {
  -k1^2, -k3^2, -k4^2, -(k1+q)^2, -(k2+q)^2, -(k4+q)^2,
  -(k1-k2)^2, -(k2-k3)^2, -(k3-k4)^2
};

SDEvaluate[UF[{k1,k2,k3,k4}, Fm44,{q^2-> -1}],
  {1,1,1,1,1,1,1,1,1},1];
```

Our results alongside with the corresponding analytical expressions (transformed to the numerical form) from [1] are presented in Tables 1 and 2.

Within FIESTA it is implied that Feynman integrals are with the $-k^2 - i0$ dependence of propagators and results are presented, in a Laurent expansion in ε , by pulling out the factor $i\pi^{d/2}e^{-\gamma_E\varepsilon}$ per loop, where γ_E is the Euler constant. Please, note that the overall normalization used by FIESTA is *different* from the one employed by the authors of [1]. We denote by \overline{M}_i a FIESTA result for an ℓ -loop MI M_i . The connection between both values reads:

$$\overline{M}_i = \left[e^{\gamma_E\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} \right]^\ell M_i.$$

Numerically, for $\ell = 4$ the conversion factor is:

$$\begin{aligned} \overline{M}_i = & \left[1 + 8\varepsilon + 36.710132\varepsilon^2 + 122.46185717\varepsilon^3 + 329.99310668\varepsilon^4 + \right. \\ & \left. + 758.778374\varepsilon^5 + 1543.7276075\varepsilon^6 + 2848.0962405\varepsilon^7 + \mathcal{O}(\varepsilon^8) \right] M_i \end{aligned} \quad (5)$$

and for $\ell = 3$

$$\overline{N}_0 = \left[1 + 6\varepsilon + 21.53259889972766\varepsilon^2 + \mathcal{O}(\varepsilon^3) \right] N_0. \quad (6)$$

Int. id:	Degree of ε :	Exact Value:	Cuba Vegas 500 000 result:	Cuba Vegas 1 500 000 result:
\overline{M}_{34}	ε^{-4}	0.08333	0.08333 ± 0	0.08333 ± 0
	ε^{-3}	0.91666	0.916668 ± 0.00003	0.916667 ± 0.000018
	ε^{-2}	5.6425109	5.64252 ± 0.00038	5.64251 ± 0.00022
	ε^{-1}	27.6412581	27.6413 ± 0.0013	27.6413 ± 0.00077
	ε^0	98.637928	98.638 ± 0.0058	98.638 ± 0.0034
	ε^1	342.7349920	342.738 ± 0.021	342.736 ± 0.012
	ε^2	857.8735165	857.88 ± 0.081	857.88 ± 0.048
	ε^3	2659.825402	2659.86 ± 0.32	2659.84 ± 0.19
	ε^4	—	4344.27 ± 1.3	4344.28 ± 0.75
\overline{M}_{35}	ε^{-2}	0.601028	0.601030 ± 0.000024	0.601028 ± 0.000012
	ε^{-1}	7.4230554	7.4232 ± 0.0004	7.4231 ± 0.00024
	ε^0	44.91255	44.9128 ± 0.0012	44.9127 ± 0.00073
	ε^1	217.0209011	217.026 ± 0.0062	217.023 ± 0.0037
	ε^2	780.4321125	780.439 ± 0.022	780.436 ± 0.013
\overline{M}_{36}	ε^{-1}	5.1846388	5.18467 ± 0.000072	5.184645 ± 0.000042
	ε^0	38.8946741	38.8950 ± 0.00068	38.8948 ± 0.00039
	ε^1	240.0684359	240.071 ± 0.0032	240.069 ± 0.0019
	ε^2	—	948.630 ± 0.016	948.623 ± 0.0091
\overline{M}_{41}	ε^{-1}	20.7385551	20.7386 ± 0.0004	20.73860 ± 0.00023
	ε^0	102.0326759	102.034 ± 0.0051	102.033 ± 0.003
	ε^1	761.5969858	761.61 ± 0.019	761.60 ± 0.011
	ε^2	—	2326.21 ± 0.11	2326.18 ± 0.062

Table 1: Numerical results for the MI's. In the third column the numerical values of the known analytical results are shown. The last two columns contain the results of evaluation on these integrals by FIESTA using the Cuba Vegas integrator with 500000 and 1500000 sampling points correspondingly. For all MI's we calculate one extra ε -term (not known analytically).

Int. id:	Degree of ε :	Exact Value:	Cuba Vegas 500 000 result:	Cuba Vegas 1 500 000 result:
\overline{M}_{42}	ε^{-1}	20.7385551	20.7386 ± 0.00041	20.73860 ± 0.00024
	ε^0	145.3808999	145.382 ± 0.0049	145.381 ± 0.0029
	ε^1	985.9082306	985.92 ± 0.023	985.91 ± 0.014
	ε^2	—	3930.68 ± 0.13	3930.65 ± 0.076
\overline{M}_{44}	ε^0	55.5852539	55.5858 ± 0.00054	55.58537 ± 0.00031
	ε^1	—	175.325 ± 0.006	175.325 ± 0.004
\overline{M}_{45}	ε^0	52.0178687	52.0184 ± 0.00052	52.0181 ± 0.0003
	ε^1	175.496447	175.50 ± 0.0062	175.50 ± 0.0036
	ε^2	—	1475.29 ± 0.017	1475.272 ± 0.0098
\overline{M}_{51}	ε^{-1}	-5.1846388	-5.18466 ± 0.000081	-5.184651 ± 0.000048
	ε^0	-32.096143	-32.0966 ± 0.00097	-32.0962 ± 0.00057
	ε^1	-91.1614758	-91.157 ± 0.009	-91.158 ± 0.0052
	ε^2	—	119.08 ± 0.074	119.06 ± 0.043
\overline{N}_0	ε^0	20.7385551	20.7387 ± 0.00045	20.73857 ± 0.00026
	ε^1	190.600238	190.60 ± 0.004	190.60 ± 0.0023
	ε^2	1049.194196	1049.20 ± 0.025	1049.20 ± 0.014
	ε^3	—	4423.86 ± 0.12	4423.84 ± 0.072
\overline{M}_{61}	ε^{-1}	-10.3692776	-10.36941 ± 0.00011	-10.36931 ± 0.00006
	ε^0	-70.99081719	-70.989 ± 0.002	-70.990 ± 0.0011
	ε^1	-21.663005	-21.633 ± 0.023	-21.650 ± 0.013
	ε^2	—	2832.86 ± 0.17	$2832.69 \pm 0.096^{(a)}$
\overline{M}_{62}	ε^{-1}	-10.3692776	-10.36940 ± 0.00011	-10.36933 ± 0.00006
	ε^0	-58.6210462	-58.6174 ± 0.0022	-58.6187 ± 0.0013
	ε^1	—	244.69 ± 0.025	244.681 ± 0.015
\overline{M}_{63}	ε^{-1}	-5.1846388	-5.18470 ± 0.000078	-5.18467 ± 0.000042
	ε^0	14.397395	14.40 ± 0.0014	14.3989 ± 0.00081
	ε^1	—	740.00 ± 0.017	739.979 ± 0.0099

^aCalculated with the Fortran Vegas using 1 550 000 samples.

Table 2: Continuation of the table 1.

Int. id:	Degree of ε :	Cuba Vegas 500 000 time:	Cuba Vegas 1 500 000 time:
\overline{M}_{34}	ε^{-4}	60.77s	59.37s
	ε^{-3}	63.56s	65.95s
	ε^{-2}	82.89s	127.82s
	ε^{-1}	211.84s	521.50s
	ε^0	401.53s	1064.87s
	ε^1	586.88s	1608.08s
	ε^2	988.27s	2739.44s
	ε^3	1870.97s	5225.72s
	ε^4	3422.80s	9572.06s
	Total time:	9847.39s	23163.80s
\overline{M}_{35}	ε^{-2}	64.29s	76.14s
	ε^{-1}	178.16s	426.75s
	ε^0	325.72s	855.19s
	ε^1	436.57s	1169.30s
	ε^2	678.91s	1828.51s
	ε^3	1275.64s	3532.42s
	Total time:	3764.31s	8694.65s
\overline{M}_{36}	ε^{-1}	152.96s	375.47s
	ε^0	269.92s	704.23s
	ε^1	354.42s	959.94s
	ε^2	526.79s	1442.97s
	Total time:	1590.61s	3769.25s
\overline{M}_{41}	ε^{-1}	185.32s	274.82s
	ε^0	691.48s	1764.69s
	ε^1	928.23s	2431.03s
	ε^2	1260.22s	3379.72s
	Total time:	3776.42s	8562.06s

Table 3: Timing for calculations of the MI's. The last two columns contain time (in seconds) of numerical integration by the **Cuba Vegas** integrator with 500000 and 1500000 sampling points. Also a total time for evaluation of each integral is given, including the Mathematica part.

Int. id:	Degree of ε :	Cuba Vegas 500 000 time:	Cuba Vegas 1 500 000 time:
\overline{M}_{42}	ε^{-1}	176.02	246.99s
	ε^0	686.57	1762.30s
	ε^1	917.95	2435.75s
	ε^2	1233.20	3289.26s
	Total time:	3753.92s	8485.04s
M_{44}	ε^0	798.20s	2097.39s
	ε^1	1016.19s	2713.11s
	Total time:	2174.60s	5185.72s
\overline{M}_{45}	ε^0	750.16s	1906.93s
	ε^1	975.67s	2533.32s
	ε^2	1246.26s	3256.41s
	Total time:	3713.05s	8416.63s
\overline{M}_{51}	ε^{-1}	516.89s	698.85
	ε^0	1676.80s	4206.21
	ε^1	2881.73s	7672.50
	ε^2	3597.15s	9615.16s
	Total time:	10736.08s	24277.30s
\overline{N}_0	ε^0	42.13s	104.29s
	ε^1	75.01s	201.78s
	ε^2	90.57s	246.34s
	ε^3	129.84s	341.99s
	Total time:	411.12s	967.20s
\overline{M}_{61}	ε^{-1}	2262.86s	3495.28s
	ε^0	15242.10s	39673.48s
	ε^1	61481.36s	162453.52s
	ε^2	202018.31s	1794640.00s ^(a)
	Total time:	768727.00s	—
\overline{M}_{62}	ε^{-1}	3003.05s	4131.07s
	ε^0	16073.09s	39690.66s
	ε^1	63720.52s	163026.12s
	Total time:	156510.00s	280778.00s
\overline{M}_{63}	ε^{-1}	273900.44s	3316.17s
	ε^0	14870.92s	36434.93s
	ε^1	59206.88s	151788.20s
	Total time:	147870.00s	262670.00s

^aIntegration by the Fortran Vegas using 1 550 000 samples.

Table 4: Continuation of the table 3.

A comparison with the analytical results shows that the integration with 500 000 sampling points leads to the numerical result with 3-4 reliable digits in a quite reasonable time (see the tables 3 and 4) while the integration with 1 500 000 sampling points reproduces the analytical results with 4-5 digits. We have also evaluated one extra term in the ε -expansion of each MI which is currently unavailable analytically but is necessary for future five-loop calculations. For some technical reasons⁴, for the highest ε term of the integral \overline{M}_{61} , we restricted ourselves with the value produced by the **Fortran Vegas** integrator which is not supported anymore.

Surprisingly this planar integral (\overline{M}_{61}) appears to be the most complicated one for numerical integration, see the table 4. Non-planar integrals \overline{M}_{62} and \overline{M}_{63} are also complicated for **FIESTA** but much less than M_{61} . Other integrals (including non-planars) are incomparably easier for numerical evaluation by **FIESTA**.

The first thirteen MI's from the Fig. 1 are very difficult for analytical evaluation, and only three of them had been checked in an independent way, see [1]. If even one of the remaining ten MI's was evaluated incorrectly, it would change all physical results obtained with the use of these MI's.

Analytical results were obtained in [1] using so-called “glue-and-cut” symmetry together with the procedure of reduction to MI's. The reduction procedure is extremely complicated, it requires careful computer algebra programming and very large-scale computer evaluations. In the present paper we have performed the independent check of these MI's using completely different approach, namely, sector decomposition, providing a quite strong evidence for the correctness of the algorithms and their implementation in [1].

4 Conclusion

Usually, analytical evaluation of multiloop MI is a kind of art. It requires a lot of efforts (and sometimes CPU time). In many situations, independent checkup is hardly any possible in reasonable time. That is why the simple in use tools for numerical evaluation like **FIESTA** are important.

Some of the integrals presented in this paper are really complicated, and the original **FIESTA 1** was not able to evaluate them at all. This was one of our motivations, in particular, to improve **FIESTA** so that it would cope with these (and, hopefully, many others) integrals. There had been both technical and theoretical complications which had to be solved [11] for this aim.

The successful check of the results of [1] demonstrates that the current version of **FIESTA** is a powerful tool for evaluating integrals numerically and for cross-checking analytical results.

⁴The evaluation was performed before we've implemented the **Cuba** library. The integrator spends 1794640 seconds which is more than 20 days so we wouldn't like to load 8-core machine for such a period by the job which was already done

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