TTP10-18
SFB/CPP-10-24

Four Loop Massless Propagators: an Algebraic Evaluation of All Master Integrals☆

P. A. Baikov[a], K. G. Chetyrkin[*,b,c]

[a] *Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia*
[b] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany*
[c] *Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.*

Abstract

The old "glue–and–cut" symmetry of massless propagators, first established in Ref. [1], leads — *after reduction to master integrals is performed* — to a host of non-trivial relations between the latter. The relations constrain the master integrals so tightly that they all can be analytically expressed in terms of only few, essentially trivial, watermelon-like integrals. As a consequence we arrive at explicit analytical results for all master integrals appearing in the process of reduction of massless propagators at three and four loops. The transcendental structure of the results suggests a clean explanation of the well-known mystery of the absence of even zetas (ζ_{2n}) in the Adler function and other similar functions essentially reducible to massless propagators. Once a reduction of massless propagators at five loops is available, our approach should be also applicable for explicitly performing the corresponding *five-loop master integrals*.

☆In memoriam Sergei Grigorievich Corishny, 1958-1988

*corresponding author

Email addresses: baikov@theory.sinp.msu.ru (P. A. Baikov), konstantin.chetyrkin@kit.edu (K. G. Chetyrkin)

Contents

1 Introduction **2**

2 Massless Propagators **5**

3 Recursively one-loop integrals **8**

4 Three-loop integrals **10**
4.1 Three-loop finite p-integrals and glueing 10
4.2 Three-loop master integrals from glueing 13

5 Four-loop Integrals **16**
5.1 Four-loop master integrals from glueing 16
5.2 Tests of the results 24
5.2.1 Primitive integrals 24
5.2.2 Simple integrals 25
5.2.3 Complicated integrals 26

6 Perspectives **26**
6.1 Five-loop master integrals 26
6.2 General case 27

7 Even zetas **28**
7.1 Four and five loops 28
7.2 Three loops 29

8 Discussion **29**

9 Summary and Conclusions **30**

1. Introduction

Within perturbation theory quantum-theoretical amplitudes are described by Feynman Integrals (FI's). The evaluation of the latter has seen quite a lot of progress during last three decades. In fact, it has been elevated from a collection of loosely related prescriptions to a solid part of mathematical physics as was recently certified by the appearance of Smirnov's bestseller books "Evaluating Feynman integrals" and (even!) "Feynman integral calculus" [2, 3].

A significant number of higher order calculations are performed according to the following "standard" scenario. First, the Feynman amplitudes are reduced to a limited set of so-called *master integrals* (MI's). The particular way of implementing the reduction is not unique and not essential for our discussion[1]. At the second and final step the resulting master integrals should be computed.

An important feature of the standard scenario is that the resulting set of master integrals should be computed only once and forever due to the well-established[2] property of universality:

[1]The so-called Laporta approach [4–6] seems to be most often utilized but a few other promising methods are being now actively developed [7–12].

[2]At least well-established in practice. See below for an instructive particular example of a class of massless propagators and also [13, 14] for an attempt to formalize the concept of the masters integrals and to prove the universality property in general. A related discussion could be found in [15–18].

for every given class of Feynman amplitudes characterized by the number of loops and the pattern of external momenta and masses the corresponding set of master integrals is universal in the following sense:

(a) Every (even extremely complicated) amplitude from the class can be expressed in terms of one and the same (finite!) set of masters integrals.

(b) The knowledge of MI's up to some properly fixed order in the ε-expansion is enough to calculate the finite part of the amplitude. Let us consider an L-loop integral P. The reduction to masters leads to an identity of the form:

$$P = \sum_i C_i(\varepsilon = 2 - D/2)\, M_i, \tag{1}$$

where sum goes over all relevant master integrals and $C_i(\varepsilon)$ is a *rational* function of the space time dimension $D = 4 - 2\varepsilon$ and kinematical parameters likes masses, external momenta, etc. The functions could be singular at the point $D = 4$. The corresponding poles in ε are customarily referred to as *spurious* ones. While the coefficients $C_i(\varepsilon)$ depend, obviously, on the initial integral P, the *maximal powers*, p_i, of the spurious poles inside a given $C_i(\varepsilon)$ depend *only* on the choice of the basis of master integrals[3].

Thus, within the standard scenario, to evaluate an L-loop amplitude F one proceeds in three main steps:
(*i*) Choose a set of master integrals.
(*ii*) Reduce every Feynman integral contributing to the amplitude F to form (1).
(*iii*) Compute the $\varepsilon \to 0$ expansion of each master integral M_i up to (and including) the term of order ε^{p_i}.

The steps (i) and (ii) are, in fact, strongly interrelated. In (almost) all approaches to reduction one first tries to use the traditional method of Integration By Part (IBP) identities[4] in order to reduce (read simplify) initial integrals as much as possible. The remaining basis set of further irreducible (at least in practice) integrals is considered as the set of MI's. As this final set is usually rather small it is not of any practical importance whether the corresponding integrals are really independent or not[5].

Once the set of MI's M_i is fixed, then the corresponding powers p_i can be easily read off from the results of reduction of some test set of initial FI's. Of course if a set of input FI's is too limited, it might happen that a in few cases an "experimentally" determined power p_i will be smaller than its true value. Luckily, the basis set of MI's (together with corresponding maximal values of spurious poles in their coefficient functions) is usually determined in early stage after calculation of relatively small subset of all FI's to be computed.

The choice of MI's is not unique. One of the basic criteria is simplicity of the calculation of MI's. For example, in view of an analytical evaluation it is natural to seek for MI's with minimal number of propagators. On the other hand, for a numerical evaluation it is often advisable to consider more complicated but less singular MI's (see, e.g. [19]).

The standard scenario was first developed for massless propagators [21, 22, 1]. It is no wonder that our understanding of reduction and MI's is most advanced for this case. Indeed, at three-loop level there is an explicit algorithm of reduction [1] to MI's (see, Fig. 1). The existence of such

[3] It was proven in [19] that there always exists such a set master integrals that *all* coefficient functions will be regular at ε around zero.

[4] In addition to the IBP identities the so-called Lorentz-invariance ones [6] are also often employed in practical calculations. In fact, the second set of identities has been proved [12] to be a consequence of the first one.

[5] In addition, sometimes there are *implicit* confirmations of the independence. For instance, if one computes a gauge invariant combination of Feynman integrals, then the gauge independence of a coefficient function of a MI could be only guaranteed if the latter is independent from all the others, see, e.g. [20].

an algorithm proves (a)-universality while the (rather tedious) analysis of the structure of the algorithm demonstrates that (b)-universality is also valid [1].

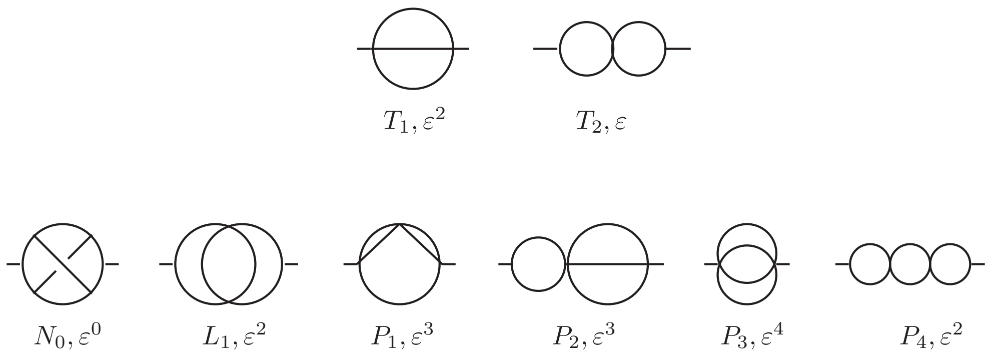


Figure 1: two- and three-loop master p-integrals. ε^m after a master label stands for the maximal term in ε-expansion of the master integral which one needs to know for evaluation of the contribution of the integral to the final result.

Let us consider the next loop level in the same class, that is four-loop massless propagators. Here the full set of *independent* MI's was theoretically constructed in [14]. Then a special procedure of reduction, based on $1/D$ expansion of the coefficient functions of MI's was developed by one of the present authors [14, 23] with the help of a special parametric representation of FI's, elaborated in [7–9]. The $1/D$ method of reduction has been heavily exploited in a series of publications [24–30] in order to compute a number of important physical observables in pQCD. We can not go here into the technical details of the four-loop reduction except for the one: it requires huge computer storage resources and their effective management. As a consequence its practical implementation would hardly be feasible without excellent possibilities for dealing with gigantic data streams offered by the computer algebra language FORM [31] and, especially, its versions ParFORM [32–34] and TFORM [35].

Thus, we conclude that the reduction problem for the four-loop massless propagators is solved in the practical sense. Analytical evaluation of the corresponding MI's is the central theme of the present work.

The plan of the paper is as follows. Next two sections provide the reader with general information about the problem. Section 4 explains the essence of our approach in detail on the (now easy) example of three-loop master integrals. The really new results are described in section 5. It is there we send an expert in multiloop calculations directly. Section 6 discusses perspectives of our method as for its extensions to more loops and other kinematical situations. In section 7 we demonstrate some peculiar properties of our results which help to solve an old puzzle of absence of even zetas from some quantities, like the Adler function, expressible in terms of p-integrals. A discussion of our results is put in section 8. In the last section 9 we summarize the content of the paper and express our gratitude to people and organizations who (which) have been continuously supporting us during the painfully long period of preparation of the present publication.

Our results for all four-loop MI's (together with some auxiliary information) are available (in computer-readable form) in
`http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp10/ttp10-18.`

2. Massless Propagators

Propagators — that is Feynman integrals depending on only one external momentum — appeared in Quantum Field Theory from its very origin and since then form an important class of FI's. Within perturbation theory, every two-point Green function

$$G(q) = \int \mathrm{d}x \, e^{iqx} \, \hat{G}(x), \quad \hat{G}(x) \equiv \langle 0 | T \left[j_2(x) \, j_1(0) \right] | 0 \rangle, \tag{2}$$

with j_1 and j_2 being in general either elementary fields or (local) composite operators, is expressed in terms of propagators. If the momentum transfer q is considered as large with respect to all relevant masses, the propagators contributing to $G(q)$ can be effectively considered as massless. In what follows we will customarily refer to massless propagator-type FI's as *p-integrals*.

p-Integrals appear in many important physical applications. Below we briefly mention some most known/important ones (for more details and examples see, e.g. reviews [36, 37]).

- The total cross-section of e^+e^- annihilation into hadrons, the Higgs decay rate into hadrons, the semihadronic decay rate of the τ lepton and the running of the fine structure coupling are all computable in the high energy limit in terms of p-integrals. This is because these quantities are either defined in terms of a two-point function (2) with properly chosen currents j_1 and j_2 or can be reduced to this form via the optical theorem.

  Note, that by high-energy limit we understand not only the case when all masses can be neglected but also the possibility to take into account mass effects by exploiting a small mass expansion. As a suitable example one could mention the calculation of the power suppressed (of order m_q^2/s, m_q^4/s^2 and so on) corrections for the correlators of (axial)vector quark currents in higher orders of pQCD [38–42].

- Coefficient functions of short distance Operator Product Expansion (OPE) of two composite operators can be always expressed in terms of p-integrals with the help of so-called method of projectors [43, 44]. A good example of an early multiloop OPE calculation is the one of the α_s^3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule [45].

- p-Integrals are extremely useful in Renormalization Group (RG) calculations within the framework of Dimensional Regularization [46–48] and Minimal Subtractions (MS) schemes [49].

The naturalness and convenience of the MS-scheme for RG calculations comes from the following statement [50]:

Theorem 1. *Any UV counterterm for any FI integral and, consequently, any RG function in an arbitrary minimally renormalized model is a polynomial in momenta and masses.*

This observation was effectively employed by A. Vladimirov [51] to simplify considerably the calculation of the RG functions. The method was further developed and named Infrared Rearrangement (IRR) in [21]. It essentially amounts to an appropriate transformation of the IR structure of FI's by setting zero some external momenta and masses (in some cases after the differentiation is performed with respect to the latter). As a result the calculation of UV counterterms is much simplified by reducing the problem to evaluating p-integrals. The method of IRR was ultimately refined and freed from unessential complications by inventing a so-called R^*-operation [52, 53]. The main use of the R^* -operation is in the proof of the following statement [53]:

Theorem 2. *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Theorem 2 is a key tool for multiloop RG calculations as it reduces the general task of evaluation of (L+1)-loop UV counterterms to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals. In the following we shall refer to the latter as the L-loop Problem.

The one-loop Problem is trivial (see eq. (4) in the next section). The two-loop Problem was solved after inventing and developing the Gegenbauer polynomial technique in x-space (GPTX) [21]. *In principle* GTPX is applicable to compute analytically some quite non-trivial three and even higher loop p-integrals[6] (for a review see [57]). However, in practice calculations quickly get clumsy, especially for diagrams with numerators. The main breakthrough at the three-loop level happened with elaborating the method of integration by parts [22, 1] of dimensionally regularized integrals. All (about a dozen) topologically different families of three-loop p-integrals were neatly analyzed in [1] and a explicit calculational algorithm was suggested for every case. As a result the algorithm of integration by parts for three-loop p-integrals was established. Later the algorithm was implemented (and named MINCER) within the computer algebra languages SCHOONSCHIP [58] and FORM [31] (see Refs [59] and [60] respectively). The most recent FORM version of MINCER is freely available from `http://www.nikhef.nl/~form`.

During last two decades MINCER has been used intensively to perform a number of important calculations of higher order radiative corrections in various field theories. As a couple of outstanding examples, characterizing the issue, we mention the analytical evaluation of the $\mathcal{O}(\alpha_s^3)$ correction to the ratio R in massless QCD [61, 62] and recent analytical calculations of three-loop deep-inelastic structure functions[7] [63–65].

Note, that every L-loop Problem is naturally decomposed in two: (A) reduction of a generic L-loop p-integrals to masters and (B) evaluation of the latter. As A-problem has already been discussed, we proceed now to B-problem. For L equal to 1 or 2 problem B degenerates to a trivial one due to the fact that all masters, being primitive ones, are easily evaluated in terms of Γ-functions. At three-loop level there exist only two non-trivial[8] master integrals whose evaluation was rather simple with the help of GPTX and, in fact, was performed well before the algorithm of reduction of three-loop p-integrals was discovered. Thus, in three-loops A and B problems could be considered as two separate ones.

The situation is different in four loops. In this case there exist [14] twenty eight master integrals pictured on Fig. 2 and only 15 of them (all after M_{43}) are simple. We call a four-loop p-integral simple if it is either primitive or reducible to the so-called generalized two-loop F-diagram with insertions, $F(n_1 + a_1\varepsilon, \dots)$ pictured on Fig. 3. The corresponding F-integral has been intensively studied since work [66] and is now in some sense analytically known [67]. The remaining 13 masters (from M_{61} till M_{43}, reading the table from left to right and the from top to bottom) happen to be quite difficult to deal with even numerically, not speaking about analytical evaluation.

The aim of the present work is to demonstrate that there exists a remarkable bootstrap-like connection between parts A and B for the L-loop Problem irrespectively the specific value of L. The connection is powerful enough to result in an explicit solution of problem B for L equal to three and four (which we will demonstrate explicitly) and, in all probability, for L equal to five (we will provide the reader with a strong argument for it).

The only prerequisite for our considerations is the solution of problem A for the corresponding number of loops.

[6] The GPTX is also ideally suited for high-precision numerical calculations of *finite* p-integrals (*with simple or, better, without numerators*) with really many loops. See [54–56] for a number of spectacular examples in four, five, six and even seven loops.

[7] These calculations, in fact, have required development and application of a number of additional technical tools (including highly advanced version of integration by parts algorithm) than just the use of MINCER and the method of projectors; please consult the original works for further details.

[8] More precisely, we mean non-primitive p-integrals, see definitions below in section 3.

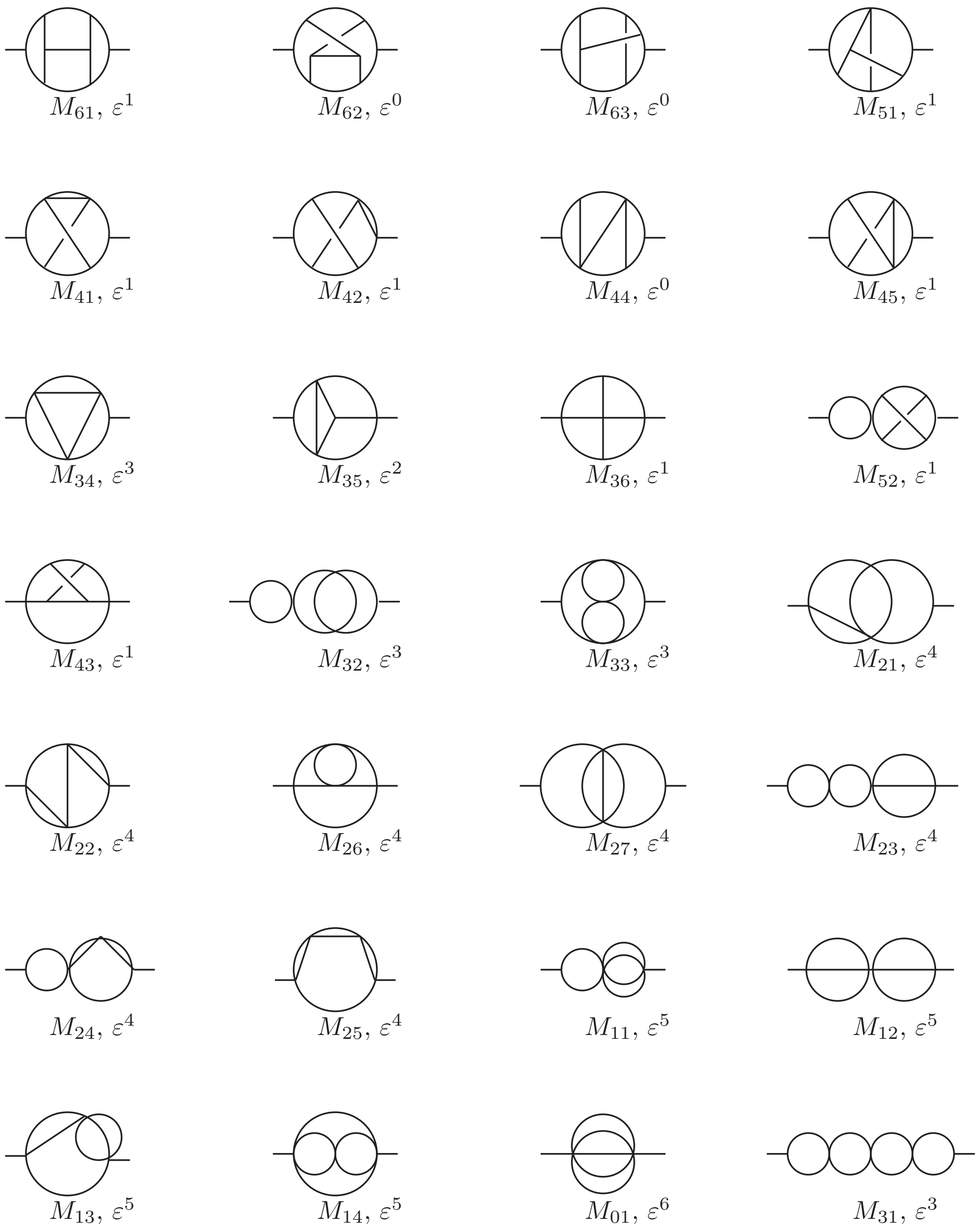


Figure 2: all master p-integrals for the four-loop Problem. In M_{ij} the digit i stands for the number of (internal) lines in the integral minus five and j numerates different integrals with the same value of i. The integrals are ordered (if read from left to right and then from top to bottom) according to their complexity. ε^m after M_{ij} stands for the maximal term in ε-expansion of M_{ij} which one needs to know for evaluation of the contribution of the integral to the final result for a four-loop p-integral after reduction is done. In other words, m stands for the maximal power of a spurious pole $1/\varepsilon^m$ which could appear in front of M_{ij} in the process of reduction to masters.

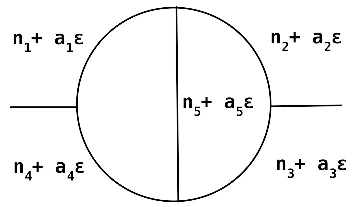


Figure 3: the generalized two-loop p-integral; indexes besides lines show the powers of corresponding massless propagators. n_i and a_i are assumed to be integers.

We will describe how the good old "glue–and–cut" symmetry of massless propagators [21] leads — *after the reduction to master integrals is performed* — to non-trivial relations between the latter. The relations constraint the masters integrals so tightly that they can all be analytically expressed in terms of only few, essentially trivial, watermelon-like integrals (see diagrams $M_{31}, M_{01}, M_{12}, M_{11}$ and M_{23} on Fig 2). This provide us with explicit analytical results for all master integrals appearing in the process of reduction of massless propagators at three and four loops. By an analytical result we mean, of course, not an analytical expression for a master integral taken at a generic value of the space-time dimension D (which is usually not possible except for the simplest cases), but rather the one for *proper* number of terms in its Laurent expansion in D around the physical value $D = 4$ as it was discussed in detail above in section 1.

Note, that for our aims it is completely irrelevant how exactly the part A (reduction to masters) is performed/implemented. In fact, we only need the reduction for relatively simple cases of p-integrals: namely, no squared propagators and relatively low powers of scalar products in numerators. In particular, no knowledge of (admittedly rather complicated) reduction techniques based on the asymptotic $1/D$ expansion is necessary. For understanding of all considerations of the paper it is enough to assume that the reduction (problem A) is done with some implementation of the Laporta algorithm.

3. Recursively one-loop integrals

Without loss of generality we will consider the scalar p-integrals defined in Euclidean space-time. Let $F(q, \varepsilon)$ be a dimensionally regulated scalar L-loop p-integral depending on external momentum q and the space-time dimension $D = 4 - 2\,\varepsilon$. Its dependence on q can be written as

$$F(q, \varepsilon) = f(\varepsilon)\,(q^2)^{\omega/2 - L\varepsilon} \tag{3}$$

where ω is the canonical mass dimension of $F(q, 0)$ and $f(\varepsilon)$ is a meromorphic function of ε.

The complexity of computing of the function of $f(\varepsilon)$ depends on the loop number L. At one loop level the result for the generic integral

$$\int \frac{\underline{d}\,\ell}{(\ell^{2\alpha})(q-\ell)^{2\beta}} = (q^2)^{2-\varepsilon-\alpha-\beta} G(\alpha, \beta) \tag{4}$$

is known since long (see, e.g. [21]) and reads[9]

$$G(\alpha, \beta) = \frac{\Gamma(\alpha+\beta-2+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(2-\alpha-\varepsilon)\,\Gamma(2-\beta-\varepsilon)}{\Gamma(4-\alpha-\beta-2\varepsilon)}. \tag{5}$$

[9]We provide every loop integration $d^D l$ with an extra normalization factor $1/\pi^{D/2}$ and write $\underline{d}\ell = \frac{d^D\,\ell}{\pi^{D/2}}$.

Here "generic" means that the powers α and β could be not only integers but functions of ε. The most useful in applications case is

$$\alpha = m + a\varepsilon, \qquad \beta = n + b\varepsilon \tag{6}$$

with n, m being arbitrary integers, a, b nonnegative ones. Note that *negative values* of a and/or b might lead to ε independent singular factor(s) like $\Gamma(0)$ within the corresponding G-function. On formal grounds $G(\alpha,\beta)$ is not defined in this situation[10]. The reduction formula for G-functions

$$G(\alpha,\beta) = \frac{(\alpha+\beta-3+\varepsilon)(4-\alpha-\beta-2\varepsilon)}{(\beta-1)(2-\beta-\varepsilon)} G(\alpha,\beta-1) \tag{7}$$

as well as the expansion

$$\begin{aligned} G(1+a\varepsilon, 1+b\varepsilon) &= \frac{G_0(\varepsilon)}{\varepsilon(1+a+b)}\Big(1+(a+b)\varepsilon+(a+b)(a+b+2)\varepsilon^2+\dots\Big), & (8)\\ G_0(\varepsilon) \equiv \varepsilon\, G(1,1) &= 1+\varepsilon\,(2-\gamma_E)+\dots & (9) \end{aligned}$$

allows for a convenient evaluation of $G(n+a\varepsilon, m+b\varepsilon)$ without any reference to the awkward formula (5). In fact, the well-known freedom in the definition of the dimensional regularization[11] allows to tune the function $G_0(\varepsilon)$ at will (provided $G_0(0)=1$). The most natural choice

$$G(1,1) \equiv \frac{1}{\varepsilon} \tag{10}$$

or, equivalently,

$$G_0(\varepsilon) \equiv 1 \tag{11}$$

fixes the so-called G-scheme [21] and will be adopted here. Note that the G-scheme is not only extremely convenient from purely calculational point of view; it is also "natural" in the realm of massless propagators. There is evidence that results expressed in the G-scheme usually tend to display a better pattern of "apparent" convergence in comparison to the $\overline{\text{MS}}$ scheme.

In view of eqs. (3) and (4) any recursively one-loop p-integral can be easily performed analytically [21]. We will denote such integrals *primitive* ones. For example, the two-loop MI's T_1 and T_2 (see Fig. 1) are both primitive ones, their ε-expansions (with accuracy necessary for the-two-loop calculation) can be easily computed via G-functions:

$$\begin{aligned} T_1 &= -\frac{1}{4\,\varepsilon} - \frac{5}{8} - \frac{27\varepsilon}{16} + \varepsilon^2\left(-\frac{153}{32}+\frac{3\,\zeta_3}{2}\right) + \mathcal{O}(\varepsilon^3), & (12)\\ T_2 &= \frac{1}{\varepsilon^2} + \mathcal{O}(\varepsilon^2), & (13) \end{aligned}$$

with $\zeta_n \equiv \sum_{i\geq 1}\frac{1}{i^n}$. Here and almost everywhere below we set $q^2=1$.

For future reference we provide below expressions in terms of G-functions for the four watermelon-like primitive three-loop master integrals which serve as building blocks for all other (three-loop) masters (see section 4). To make the formulas shorter we always use the G-scheme defining relation (10) and write everywhere $1/\varepsilon$ instead of the $G(1,1)$:

$$\begin{aligned} P_1 &= \frac{1}{\varepsilon^2}\,G(2\varepsilon,1), & P_2 &= \frac{1}{\varepsilon^2}\,G(\varepsilon,1),\\ P_3 &= \frac{1}{\varepsilon^2}\,G(\varepsilon,\varepsilon), & P_4 &= \frac{1}{\varepsilon^3}. \end{aligned} \tag{14}$$

[10] See, however, [68] for a significantly deeper discussion of such cases.

[11] The freedom amounts to the multiplication of every L-loop integral by a factor $n(\varepsilon)^L$, with $n(\varepsilon) = 1+\mathcal{O}(\varepsilon)$ being a regular (at least in a vicinity of the point $\varepsilon=0$) function of ε [69]. Thus, the formulas (10) and (11) below should be understood in the sense that $n(\varepsilon)$ is chosen as follows $n(\varepsilon) = 1/(\varepsilon\, G(1,1)) \equiv \Gamma(2-2\varepsilon)/(\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)^2)$.

4. Three-loop integrals

In this section we discuss the main idea of our method on a first non-trivial example of three-loop massless propagator-like integrals.

4.1. Three-loop finite p-integrals and glueing

It is easier to explain the glue-and-cut symmetry on a real-life example. Almost exactly thirty years ago one of the present authors was strongly puzzled by the following facts (resulting from first calculations made with the help of just discovered technique of Gegenbauer polynomials in the position space [21]):

$$\begin{aligned} L_0 &= (q^2)^{-2-3\varepsilon}\, 20\,\zeta_5 + \mathcal{O}(\varepsilon), \qquad N_0 = (q^2)^{-2-3\varepsilon}\, 20\,\zeta_5 + \mathcal{O}(\varepsilon), \\ N_1 &= (q^2)^{-1-3\varepsilon}\, 20\,\zeta_5 + \mathcal{O}(\varepsilon), \qquad N_2 = (q^2)^{-1-3\varepsilon}\, 20\,\zeta_5 + \mathcal{O}(\varepsilon), \end{aligned} \tag{15}$$

where L_0, N_0, N_1 and N_2 are scalar three-loop p-integrals (see Fig. 4).

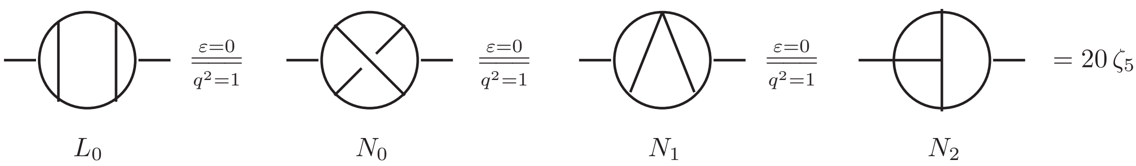
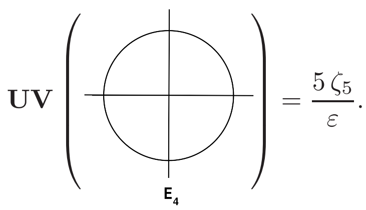


Figure 4: Four finite three-loop p-integrals displaying a remarkable feature of being equal at $\varepsilon = 0$ and $q^2 = 1$.

Indeed, a short look on eqs. (15) immediately leads to an obvious question: why on the Earth four quite different (but all finite) p-integrals have identical values at $D = 4$ (if one set $q^2 = 1$)? Incidentally, by that time a pioneering[12] calculation of the four-loop β-function in the ϕ^4-model [70] had been just finished. One of its results was the UV divergence of the following four-loop vertex-type integral



(16)

The suspicious appearance of one and the same irrational constant ζ_5 with very simple coefficients in eqs. (15) and (16) was suggesting some mysterious connection between three-loop *finite* p-integrals L_0, N_0, N_1, N_2 and the *divergent* part of the four-loop vertex-type integral E_4. In addition, a closer inspection of all five diagrams revealed that the four propagators-type diagram could be formally produced from the vertex graph in two steps (see Fig 5):
(i) Delete all four external lines from the vertex diagram (transforming it, thus, to a vacuum one).
(ii) Cut in the resulting vacuum diagram either a line (there exist only two non-equivalent choices, leading to N_1 and N_2) or delete the central vertex[13] (again, one could do it in two ways, leading to N_0 and L_0).

[12] To our knowledge it was the first full calculation of the β-function in a four-dimensional model in **four** loops.

[13] The operation of deleting of a vertex means that one first transforms the vertex into two new ones by introducing a fictitious line (with the unit propagator) and then cutting the new line. Note that deleting a three-linear vertex does not produce any new diagrams in addition to those coming from cutting the corresponding three incident lines. In general, one can cut a four-linear vertex by a three non-equivalent ways a shown by Fig 5(d,e,f). Due to the high symmetry of the envelope E_4 diagram the possibility (d) and (e) lead to one and the same result.

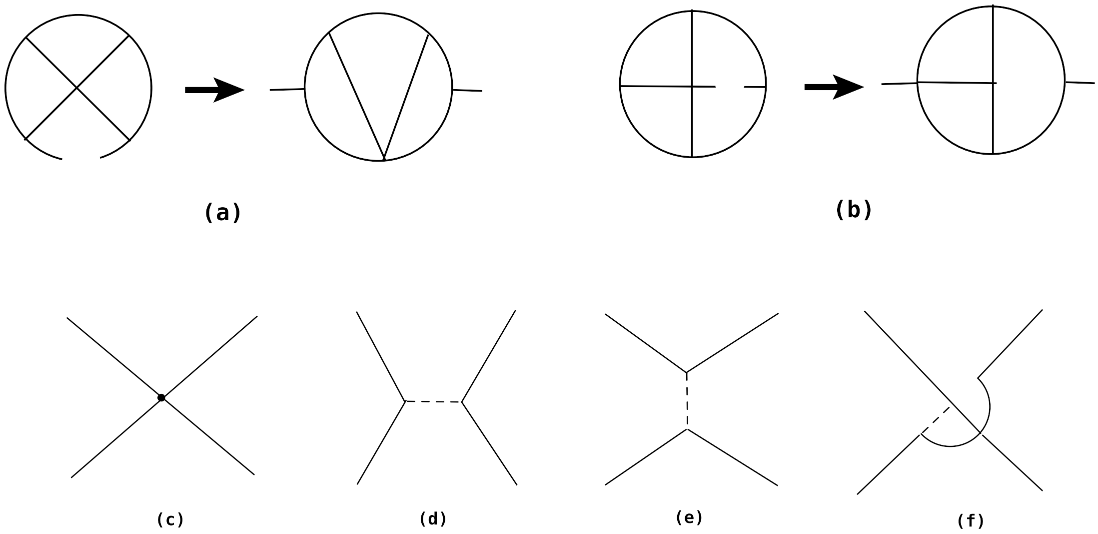


Figure 5: Two ways of cutting a line in the graph E_4 (a,b). The generic four-linear vertex (c) and three ways transforming it into a pair of the three-linear ones connected by an auxiliary propagator (d,e,f).

The puzzle was finally understood after the geometrical construction was provided with analytical content. As a result the Glue-and-Cut (GaC) symmetry of massless propagators was established. Below we prove a theorem [1] which solves the puzzle.

Let $\langle\Gamma\rangle(\varepsilon)$ be a dimensionally regulated massless scalar $L+1$ loop vacuum Feynman amplitude *without* any subdivergences and with the superficial divergence index $\omega_\Gamma = 0$ at four-dimensions. Surely, every expert would cry at this point that such an object is identical zero due to absence of any intrinsic scale which is true, beyond any doubts[14]. Please, be patient! In fact, by a Feynman amplitude we understand a formal triplet consisting of the corresponding Feynman graph Γ, properly constructed Feynman *integrand* and, at last, a function of kinematical parameters (external momenta and masses) resulting after evaluation the integrand. It means that even if the function vanishes for some particular choice of the kinematical parameters it may become nonzero after some modification of the latter.

Without essential loss of generality we assume that the graph Γ contains only triple vertexes. Consider an arbitrary line of ℓ of Γ with

$$P_\ell(q) = \frac{\mathcal{P}(q,\dots)}{q^2}$$

being the corresponding propagator of Γ and $\mathcal{P}(q)$ being some polynomial in the line momentum q. We allow the integral $\langle\Gamma\rangle$ to contain non-trivial numerator; in the case of the line ℓ being a fictitious one with the unit propagator, one can always redefine the propagator as follows:

$$P_\ell(q) = \frac{\mathcal{P}'(q)}{q^2} \quad \text{with} \quad \mathcal{P}'(q) = q^2.$$

Let $\langle\Gamma\rangle(m_0,\varepsilon)$ be the Feynman amplitude obtained from $\langle\Gamma\rangle(\varepsilon)$ by introducing an auxiliary non-

[14] By the way, a real mathematical justification why one could *self-consistently* set zero such massless vacuum integrals *within* dimensional regularization was to best of our knowledge performed only in [71]. It requires first of all a self-consistent and mathematically solid definitions of the very dimensional regularization which, in turn, demands a heavy use of various parametric representations.

zero mass m_0 into the selected propagator, as an infrared regulator, that is,

$$P_\ell(q) \to P_\ell(q)\,\frac{q^2}{q^2+m_0^2} = \frac{\mathcal{P}'(q,\dots)}{q^2+m_0^2}.$$

Assuming that the loop momenta in $\langle\Gamma\rangle$ are chosen in such a way that the momentum q is the loop one, we can formally present the integral $\langle\Gamma\rangle(m_0,\varepsilon)$ as a convolution

$$\langle\Gamma\rangle(m_0,\varepsilon) = \int \langle\Gamma\backslash\ell\rangle(q,\varepsilon)\;\frac{\underline{d}q^D}{q^2+m_0^2}, \tag{17}$$

where $\langle\Gamma\backslash\ell\rangle(q,\varepsilon)$ is the L-loop p-integral obtained from $\langle\Gamma\rangle(m_0,\varepsilon)$, first, by "freezing" integration over q and, second, by multiplying the result by $(q^2+m_0^2)$.

Theorem 3. Under the above listed conditions the following statements are true

(a) The vacuum integral $\langle\Gamma\rangle(m_0,\varepsilon)$ is IR finite and its UV divergence is a simple pole, that is

$$\langle\Gamma\rangle(m_0,\varepsilon) = \frac{C}{(L+1)\varepsilon} + \mathcal{O}(\varepsilon^0), \tag{18}$$

with C being a constant;

(b) for every choice of the line ℓ the p-integral $\langle\Gamma\backslash\ell\rangle(q,\varepsilon)$ is finite and its value at $D=4$ meets the condition

$$\lim_{\varepsilon\to 0}\langle\Gamma\backslash\ell\rangle(q,\varepsilon) \;=\; \frac{C}{q^2}. \tag{19}$$

Proof.

(a) Due to the assumed absence of any subdivergences, the insertion of a mass does not influence the UV divergence, but removes the only possibility for the IR one (related with integration over the region of *all* loop momenta being small). On dimensional grounds we have

$$\langle\Gamma\rangle(m_0,\varepsilon) = (m_0^2)^{-(L+1)\varepsilon}\, f(\varepsilon), \tag{20}$$

with $f(\varepsilon)$ depending *only* on ε. Without any UV subdivergences, the (minimal) UV counterterm corresponding to the integral $\langle G\rangle(m_0,\varepsilon)$ as a whole reduces to its pole part. If $f(\varepsilon)$ would contain a non-simple pole it would lead to appearance a non-polynomial dependence on the mass m_0 of the counterterm in the direct violation of Theorem 1. Thus,

$$C = (L+1)\lim_{\varepsilon\to 0}\varepsilon\, f(\varepsilon). \tag{21}$$

(b) The (scalar) p-integral $\langle\Gamma\backslash\ell\rangle$ has a homogeneous dependence on q^2, namely

$$\langle\Gamma\backslash\ell\rangle(q,\varepsilon) = (q^2)^{-1-L\varepsilon}\, g(\varepsilon) \tag{22}$$

with $g(\varepsilon)$ depending *only* on ε. As a consequence, the integral over q in (17) can be easily performed with the help of a textbook formula (see, e.g. [2]):

$$\int \frac{\underline{d}^D\ell}{(m^2+\ell^2)\,(\ell^2)^{1+n\,\varepsilon}} =$$
$$(m^2)^{D/2-2-n\varepsilon}\;\frac{\Gamma((n+1)\,\varepsilon)\,\Gamma(1-(n+1)\,\varepsilon)}{\Gamma(2-\varepsilon)} = \frac{1}{(n+1)\varepsilon}\cdot(1+\mathcal{O}(\varepsilon))$$

with the result

$$\langle\Gamma\rangle(m_0,\varepsilon) = \frac{g(\varepsilon)}{(L+1)\,\varepsilon} + \mathcal{O}(\varepsilon^0). \tag{23}$$

A comparison of eqs. (20,21) and (23) directly leads to eq.

$$\lim_{\varepsilon\to 0} g(\varepsilon) = C$$

which is equivalent to eq. (19). □.

The GaC symmetry, proven in **Theorem 3**, clearly explains the origin of relations displayed in Fig. 4. Still, considered by itself, it is not especially useful as it does not provide us with the value of the constant C.

4.2. Three-loop master integrals from glueing

The situation is radically changed if one utilizes the GaC symmetry *together* with the reduction to masters[15]. Indeed, let us forget for the moment about eqs. (15) and use *only* the GaC symmetry for the p-integrals shown on Fig. 4 . This leads to four equations, namely:

$$N_0 = L_0 + \mathcal{O}(\varepsilon), \quad N_0 = N_1 + \mathcal{O}(\varepsilon), \quad N_0 = N_2 + \mathcal{O}(\varepsilon), \quad N_0 = \mathcal{O}(\varepsilon^0). \tag{24}$$

On the other hand the reduction of three reducible p-integrals in eq. (24) to masters gives:

$$\begin{aligned} L_0 &= \frac{3\,(3D-10)\,(D-3)}{(D-4)^2}\,L_1 + \frac{4\,(D-3)^2}{(D-4)^2}\,P_4 + \frac{32\,(2D-7)\,(D-3)^2}{(D-4)^3}\,P_1 \\ &- \frac{12(3D-8)\,(3D-10)\,(D-3)}{(D-4)^3}\,P_2 + \frac{4(9D^2-65D+118)\,(3D-8)\,(2D-5)}{(D-4)^4}\,P_3, \end{aligned} \tag{25}$$

$$\begin{aligned} N_1 &= \frac{(3D-10)\,(D-3)}{(D-4)^2}\,L_1 + \frac{8(2D-7)\,(D-3)^2}{(D-4)^3}\,P_1 \\ &- \frac{2(3D-8)\,(3D-10)\,(D-3)}{(D-4)^3}\,P_2 + \frac{4(3D-8)\,(2D-5)}{(D-4)^2}\,P_3, \end{aligned} \tag{26}$$

$$\begin{aligned} N_2 &= -\frac{16(2D-7)\,(D-3)^2}{(D-4)^3}\,P_1 + \frac{(3D-8)\,(3D-10)\,(D-3)}{(D-4)^3}\,P_2 \\ &+ \frac{10(3D-8)\,(3D-10)\,(2D-5)\,(2D-7)}{(D-4)^4}\,P_3. \end{aligned} \tag{27}$$

Now, it is well-known fact that a maximal order of the pole in ε of a (dimensionally regulated) L-loop p-integral can not exceed[16] L. Thus, we can parametrize the coefficients of the three-loop master p-integrals as follows[17]:

$$\begin{aligned} N_0 &= \sum_{i=-3}^{0} N_{0,i}\,\varepsilon^i + \mathcal{O}(\varepsilon), \quad L_1 = \sum_{i=-3}^{2} L_{1,i}\,\varepsilon^i + \mathcal{O}(\varepsilon^3), \\ P_1 &= \sum_{i=-3}^{3} P_{1i}\,\varepsilon^i + \mathcal{O}(\varepsilon^4), \quad P_2 = \sum_{i=-3}^{3} P_{2,i}\,\varepsilon^i + \mathcal{O}(\varepsilon^4), \\ P_3 &= \sum_{i=-3}^{4} P_{3,i}\,\varepsilon^i + \mathcal{O}(\varepsilon^5), \quad P_4 = \frac{P_{4,-3}}{\varepsilon^3}, \quad P_{4,-3} = 1. \end{aligned} \tag{28}$$

Note that the higher term in ε, not shown explicitly on (28), at any case can not, obviously, be constrained by eqs. (28). In addition, they could contribute only to terms of order ε or higher

[15] Unfortunately, this step was overlooked thirty years ago, presumably, because the problem of evaluation of three-loop p-masters had been already solved *before* the idea of glueing appeared.

[16] In [53] the statement was proved for an arbitrary euclidean Feynman integral.

[17] Below the explicit result for the simplest master integral, P_4 is taken as granted; this fixes the global normalization of all remaining integrals.

to the value of an arbitrary three-loop p-integral (this statement follows from the knowledge of maximal power of spurious pole in ε which might appear in front of a master integral in the process of reduction, see Fig. 1).

After substitution of eqs. (28), eqs. (24) produce some non-trivial constraints on coefficients of the ε-expansion of our master p-integrals. To be specific, let consider first eq. from (24). Its expanded in ε form reads

$$\lim_{\varepsilon \to 0} \varepsilon^n (L_0 - N_0) = 0 \tag{29}$$

for $n \geq 0$. Note that (29) is met identically if $n > 7$ (because of the fact that the maximal allowed poles in ε which could appear in eqs. (25,26,27) and eqs (28), are 4 and 3 respectively). For $n = 7, 6, 5$ and 4 the resulting equations are[18]

$$0 = P_{3,-3}, \tag{30}$$

$$0 = 6\,P_{3,-2} + 12\,P_{2,-3} - 4\,P_{1,-3}, \tag{31}$$

$$0 = -59\,P_{3,-2} + 6\,P_{3,-1} - 78\,P_{2,-3} + 12\,P_{2,-2} + 32\,P_{1,-3} - 4\,P_{1,-2} + \frac{3L_{1,-3}}{2} + 1, \tag{32}$$

$$0 = 239\,P_{3,-2} - 59\,P_{3,-1} + 6\,P_{3,0} + 162\,P_{2,-3} - 78\,P_{2,-2} + 12\,P_{2,-1} \tag{33}$$

$$-80\,P_{1,-3} + 32\,P_{1,-2} - 4\,P_{1,-1} - \frac{15L_{1,-3}}{2} + \frac{3L_{1,-2}}{2} - 4. \tag{34}$$

Already now we can see that eqs. (32) and (34) express two coefficients, $L_{1,-3}$ and $L_{1,-2}$, of a non-primitive integral through the coefficients of primitive ones, namely, P_1, P_2 and P_3. Indeed, a solution of eqs. (30-34) is

$$P_{3,\text{-}3} = 0, \tag{35}$$

$$P_{1,\text{-}3} = \frac{3P_{3,\text{-}2}}{2} + 3P_{2,\text{-}3}, \tag{36}$$

$$L_{1,\text{-}3} = \frac{22P_{3,\text{-}2}}{3} - 4P_{3,\text{-}1} - 12P_{2,\text{-}3} - 8P_{2,\text{-}2} + \frac{8P_{1,\text{-}2}}{3} - \frac{2}{3}\ , \tag{37}$$

$$L_{1,\text{-}2} = -\frac{128P_{3,\text{-}2}}{3} + \frac{58P_{3,\text{-}1}}{3} - 4P_{3,0} - 8P_{2,\text{-}3} + 12P_{2,\text{-}2} - 8\ P_{2,\text{-}1} - 8P_{1,\text{-}2} + \frac{8P_{1,\text{-}1}}{3} - \frac{2}{3} \tag{38}$$

Proceeding in the same vein we arrive eventually to a linear system of 31 equations (not necessarily independent) for 32 coefficients $N_{0,i_0}, L_{1,i_1}, P_{1,j_1}, P_{2,j_2}, P_{3,j_3}$. One can solve the system by expressing the coefficients from more complicated masters through those from less complicated ones. A convenient ordering is given by two rules[19]

- $N_{0,i_0} \succ L_{1,i_1} \succ P_{1,j_1} \succ P_{2,j_2} \succ P_{3,j_3} \succ P_{4,j_4}$.
- For two coefficients of a master integral the more complicated one is that with larger value of the second index.

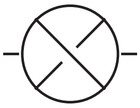
The system is easily solved with the result: coefficients are expressed in terms of only eight coefficients of three *primitive* integrals, namely,

$$P_{3,-1,}, P_{3,0}, P_{3,1}, P_{3,2}, P_{3,3}, P_{3,4}, P_{2,3}, P_{4,-3}. \tag{39}$$

These eight coefficients are trivially determined from eqs. (14) and, finally, we arrive at the following results for the three-loop master integrals:

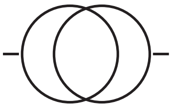
[18] Note, that for brevity in writing eqs. (31-34) we have used eq. 30 to discard the terms proportional to (zero) coefficient $P_{3,-3}$.

[19] The order between primitive integrals is rather arbitrary, except for the natural choice to use P_4 as the easiest one.



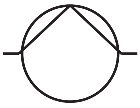
N_0, ε^0

$$= 20\,\zeta_5 + \mathcal{O}(\varepsilon), \tag{40}$$



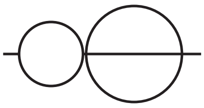
L_1, ε^2

$$\begin{aligned} &= \frac{1}{3\,\varepsilon^3} + \frac{1}{3\,\varepsilon^2} + \frac{1}{3\,\varepsilon} - \frac{7}{3} + \frac{14\,\zeta_3}{3} + \varepsilon\left(-\frac{67}{3} + \frac{14\,\zeta_3}{3} + 7\,\zeta_4\right) \\ &+ \varepsilon^2\left(-\frac{403}{3} + \frac{86\,\zeta_3}{3} + 7\,\zeta_4 + 126\,\zeta_5\right) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{41}$$



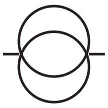
P_1, ε^3

$$\begin{aligned} &= -\frac{1}{3\,\varepsilon^2} - \frac{4}{3\,\varepsilon} - \frac{16}{3} + \varepsilon\left(-\frac{64}{3} + \frac{16\,\zeta_3}{3}\right) + \varepsilon^2\left(-\frac{256}{3} + \frac{64\,\zeta_3}{3} + 8\,\zeta_4\right) \\ &+ \varepsilon^3\left(-\frac{1024}{3} + \frac{256\,\zeta_3}{3} + 32\,\zeta_4 + 64\,\zeta_5\right) + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{42}$$



P_2, ε^3

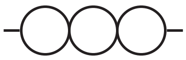
$$\begin{aligned} &= -\frac{1}{4\,\varepsilon^2} - \frac{5}{8\,\varepsilon} - \frac{27}{16} + \varepsilon\left(-\frac{153}{32} + \frac{3\,\zeta_3}{2}\right) + \varepsilon^2\left(-\frac{891}{64} + \frac{15\,\zeta_3}{4} + \frac{9\,\zeta_4}{4}\right) \\ &+ \varepsilon^3\left(\boxed{-\frac{5265}{128} + \frac{81\,\zeta_3}{8} + \frac{45\,\zeta_4}{8} + \frac{21\,\zeta_5}{2}}\right) + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{43}$$



P_3, ε^4

$$\begin{aligned} &= \boxed{\frac{1}{36}}\,\frac{1}{\varepsilon} + \boxed{\frac{35}{216}} + \boxed{\frac{991}{1296}}\,\varepsilon + \varepsilon^2\left(\boxed{\frac{26207}{7776} - \frac{11\,\zeta_3}{18}}\right) \\ &+ \varepsilon^3\left(\boxed{\frac{670951}{46656} - \frac{385\,\zeta_3}{108} - \frac{11\,\zeta_4}{12}}\right) \end{aligned} \tag{44}$$

$$+\,\varepsilon^4\left(\boxed{\frac{16852031}{279936}-\frac{10901\,\zeta_3}{648}-\frac{385\,\zeta_4}{72}-\frac{13\,\zeta_5}{2}}\right)+\mathcal{O}(\varepsilon^5), \tag{45}$$



$$=\quad\frac{\boxed{1}}{\varepsilon^3}+\mathcal{O}(\varepsilon^3), \tag{46}$$

P_4, ε^2

where we have boxed the eight input coefficients. The comparison with eqs. (14,15) and (25) demonstrates the all unboxed coefficients have been correctly determined through the gluing procedure.

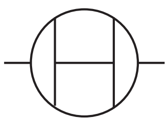
A remarkable feature of the above discussed, glue-and-cut based, determination of the three-loop masters is that the both non-primitive (read non-trivial) master integrals N_0 and L_1 have been expressed through essentially trivial (read primitive) FI's. Even more, as many as eleven coefficients of the primitive MI's P_1 and P_2 (see eqs. (42,43)) have also been fixed through only eight coefficients listed in (39). Thus, we see that integration by parts identities together with the glue-and-cut symmetry severely constrain the *values* of master integrals.

5. Four-loop Integrals

5.1. Four-loop master integrals from glueing

Following the same procedure in the case of four-loop propagator massless integrals, one should consider all possible cuttings of a set of five-loop vacuum massless diagrams with integrand of mass dimension *twenty* and without subdivergences (or, equivalently, superficially and logarithmically divergent).

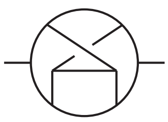
Again, as in three-loop case, GaC relations provide us with enough information to express all the necessary coefficients of the ε-expansions of all MI's through some trivial integrals. The number of the input five-loop tadpoles and the resulting relations (around a hundred and a thousand respectively) are too large to be presented here, so in the equations to follow we give only the final results.



$$=-\frac{10\zeta_5}{\varepsilon}+50\zeta_5-10\,\zeta_3^2-25\zeta_6$$

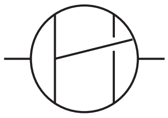
$M_{61},\ \varepsilon^1$

$$+\,\varepsilon\left(90\zeta_5+50\,\zeta_3^2+125\zeta_6-30\,\zeta_3\,\zeta_4+\frac{19\,\zeta_7}{2}\right)+\,\mathcal{O}(\varepsilon^2), \tag{47}$$



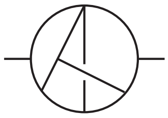
$$=-\frac{10\zeta_5}{\varepsilon}+130\zeta_5-10\,\zeta_3^2-25\zeta_6-70\,\zeta_7+\,\mathcal{O}(\varepsilon), \tag{48}$$

$M_{62},\ \varepsilon^0$



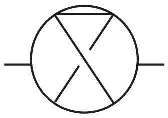
$M_{63},\, \varepsilon^0$

$$= -\frac{5\zeta_5}{\varepsilon} + 45\zeta_5 - 41\,\zeta_3^2 - \frac{25\zeta_6}{2} + \frac{161\,\zeta_7}{2} + \mathcal{O}(\varepsilon), \tag{49}$$



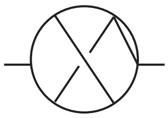
$M_{51},\, \varepsilon^1$

$$= -\frac{5\zeta_5}{\varepsilon} + 45\zeta_5 - 17\,\zeta_3^2 - \frac{25\zeta_6}{2} + \varepsilon\left(-195\zeta_5 + 153\,\zeta_3^2 + \frac{225\zeta_6}{2} - 51\,\zeta_3\,\zeta_4 - \frac{85\,\zeta_7}{2}\right) + \mathcal{O}(\varepsilon^2), \tag{50}$$



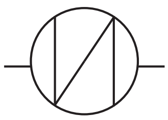
$M_{41},\, \varepsilon^1$

$$= \frac{20\zeta_5}{\varepsilon} - 80\zeta_5 - 22\,\zeta_3^2 + 50\zeta_6 + \varepsilon\left(80\zeta_5 + 88\,\zeta_3^2 - 200\zeta_6 - 66\,\zeta_3\,\zeta_4 + \frac{4685\,\zeta_7}{8}\right) + \mathcal{O}(\varepsilon^2), \tag{51}$$



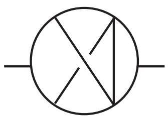
$M_{42},\, \varepsilon^1$

$$= \frac{20\zeta_5}{\varepsilon} - 80\zeta_5 + 8\,\zeta_3^2 + 50\zeta_6 + \varepsilon\left(80\zeta_5 - 32\,\zeta_3^2 - 200\zeta_6 + 24\,\zeta_3\,\zeta_4 + 520\,\zeta_7\right) + \mathcal{O}(\varepsilon^2), \tag{52}$$



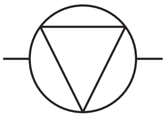
$M_{44},\, \varepsilon^0$

$$= \frac{441\,\zeta_7}{8} + \mathcal{O}(\varepsilon), \tag{53}$$



$M_{45},\, \varepsilon^1$

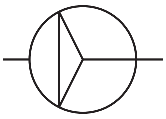
$$= 36\,\zeta_3^2 + \varepsilon\left(108\,\zeta_3\,\zeta_4 - 378\,\zeta_7\right) + \mathcal{O}(\varepsilon^2), \tag{54}$$



$$= \frac{1}{12\,\varepsilon^4} + \frac{1}{4\,\varepsilon^3} + \frac{7}{12\,\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{17}{12} + \frac{25\zeta_3}{6}\right) - \frac{377}{12} + \frac{25\zeta_3}{2} + \frac{25\zeta_4}{4}$$

M_{34}, ε^3

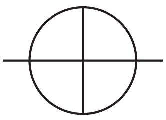
$$\begin{aligned}
&+ \varepsilon\left(-\frac{3401}{12} + \frac{463\zeta_3}{6} + \frac{75\zeta_4}{4} + \frac{465\zeta_5}{2}\right) \\
&+ \varepsilon^2\left(-\frac{24497}{12} + \frac{3031\zeta_3}{6} + \frac{463\zeta_4}{4} + \frac{1395\zeta_5}{2} - \frac{1247\,\zeta_3^2}{6} + \frac{3425\zeta_6}{6}\right) \\
&+ \varepsilon^3\left(-\frac{158273}{12} + \frac{19663\zeta_3}{6} + \frac{3031\zeta_4}{4} + \frac{6807\zeta_5}{2}\right. \\
&\qquad \left. - \frac{1247\,\zeta_3^2}{2} + \frac{3425\zeta_6}{2} - \frac{1247\,\zeta_3\,\zeta_4}{2} + \frac{12503\,\zeta_7}{2}\right) + \mathcal{O}(\varepsilon^4),
\end{aligned} \tag{55}$$



$$= \frac{\zeta_3}{2\,\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{3\zeta_3}{2} + \frac{3\zeta_4}{4}\right) + \frac{19\zeta_3}{2} + \frac{9\zeta_4}{4} - \frac{23\zeta_5}{2}$$

M_{35}, ε^2

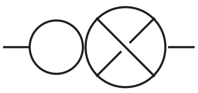
$$\begin{aligned}
&+ \varepsilon\left(\frac{103\zeta_3}{2} + \frac{57\zeta_4}{4} - \frac{69\zeta_5}{2} + \frac{29\,\zeta_3^2}{2} - 30\zeta_6\right) \\
&+ \varepsilon^2\left(\frac{547\zeta_3}{2} + \frac{309\zeta_4}{4} - \frac{437\zeta_5}{2} + \frac{87\,\zeta_3^2}{2} - 90\zeta_6 + \frac{87\,\zeta_3\,\zeta_4}{2} - \frac{1105\,\zeta_7}{4}\right) + \mathcal{O}(\varepsilon^3),
\end{aligned} \tag{56}$$



$$= \frac{5\zeta_5}{\varepsilon} - 5\zeta_5 - 7\,\zeta_3^2 + \frac{25\zeta_6}{2} + \varepsilon\left(35\zeta_5 + 7\,\zeta_3^2 - \frac{25\zeta_6}{2} - 21\,\zeta_3\,\zeta_4 + \frac{127\,\zeta_7}{2}\right) + \mathcal{O}(\varepsilon^2),$$

M_{36}, ε^1

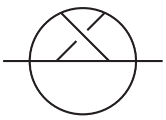
(57)



$$= \frac{20\zeta_5}{\varepsilon} - 80\zeta_5 + 68\,\zeta_3^2 + 50\zeta_6$$

M_{52}, ε^1

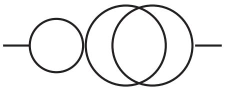
$$+ \varepsilon\left(80\zeta_5 - 272\,\zeta_3^2 - 200\zeta_6 + 204\,\zeta_3\,\zeta_4 + 450\,\zeta_7\right) + \mathcal{O}(\varepsilon^2), \tag{58}$$



$$= -\frac{5\zeta_5}{\varepsilon} + 45\zeta_5 - 17\,\zeta_3^2 - \frac{25\zeta_6}{2}$$

M_{43}, ε^1

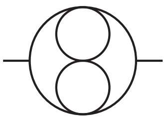
$$+\ \varepsilon\left(-195\zeta_5+153\,\zeta_3^2+\frac{225\zeta_6}{2}-51\,\zeta_3\,\zeta_4-\frac{225\,\zeta_7}{2}\right)+\ \mathcal{O}(\varepsilon^2), \tag{59}$$



$$=\frac{1}{3\,\varepsilon^4}+\frac{1}{3\,\varepsilon^3}+\frac{1}{3\,\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{7}{3}+\frac{14\zeta_3}{3}\right)-\frac{67}{3}+\frac{14\zeta_3}{3}+7\zeta_4$$

$M_{32},\ \varepsilon^3$

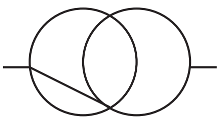
$$\begin{aligned}
&+\ \varepsilon\left(-\frac{403}{3}+\frac{86\zeta_3}{3}+7\zeta_4+126\zeta_5\right)\\
&+\ \varepsilon^2\left(-\frac{2071}{3}+\frac{478\zeta_3}{3}+43\zeta_4+126\zeta_5-\frac{226\,\zeta_3^2}{3}+\frac{910\zeta_6}{3}\right)\\
&+\ \varepsilon^3\left(-\frac{9823}{3}+\frac{2446\zeta_3}{3}+239\zeta_4+534\zeta_5-\frac{226\,\zeta_3^2}{3}+\frac{910\zeta_6}{3}-226\,\zeta_3\,\zeta_4+1960\,\zeta_7\right)+\ \mathcal{O}(\varepsilon^4),
\end{aligned} \tag{60}$$



$$=\frac{1}{6\,\varepsilon^4}+\frac{1}{3\,\varepsilon^3}+\frac{1}{3\,\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{17}{3}+\frac{31\zeta_3}{3}\right)-\frac{197}{3}+\frac{62\zeta_3}{3}+\frac{31\zeta_4}{2}$$

$M_{33},\ \varepsilon^3$

$$\begin{aligned}
&+\ \varepsilon\left(-\frac{1529}{3}+\frac{386\zeta_3}{3}+31\zeta_4+449\zeta_5\right)\\
&+\ \varepsilon^2\left(-\frac{10205}{3}+\frac{2510\zeta_3}{3}+193\zeta_4+898\zeta_5-\frac{983\,\zeta_3^2}{3}+\frac{3290\zeta_6}{3}\right)\\
&+\ \varepsilon^3\left(-\frac{62801}{3}+\frac{15974\zeta_3}{3}+1255\zeta_4+4354\zeta_5\right.\\
&\qquad\left.-\frac{1966\,\zeta_3^2}{3}+\frac{6580\zeta_6}{3}-983\,\zeta_3\,\zeta_4+11338\,\zeta_7\right)+\ \mathcal{O}(\varepsilon^4),
\end{aligned} \tag{61}$$

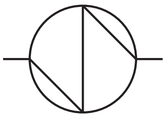


$$=-\frac{5}{48\,\varepsilon^3}-\frac{31}{96\,\varepsilon^2}-\frac{95}{192\,\varepsilon}+\frac{1133}{384}-\frac{19\zeta_3}{12}$$

$M_{21},\ \varepsilon^4$

$$\begin{aligned}
&+\ \varepsilon\left(\frac{30097}{768}-\frac{233\zeta_3}{24}-\frac{19\zeta_4}{8}\right)+\ \varepsilon^2\left(\frac{463349}{1536}-\frac{3385\zeta_3}{48}-\frac{233\zeta_4}{16}-\frac{341\zeta_5}{4}\right)\\
&+\ \varepsilon^3\left(\frac{6004105}{3072}-\frac{46469\zeta_3}{96}-\frac{3385\zeta_4}{32}-\frac{3187\zeta_5}{8}+\frac{493\,\zeta_3^2}{6}-\frac{1255\zeta_6}{6}\right)
\end{aligned}$$

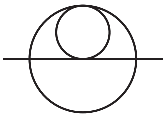
$$+\ \varepsilon^4\left(\frac{71426093}{6144}-\frac{590281\zeta_3}{192}-\frac{46469\zeta_4}{64}-\frac{33875\zeta_5}{16}\right.$$
$$\left.+\frac{4673\,\zeta_3^2}{12}-\frac{2915\zeta_6}{3}+\frac{493\,\zeta_3\,\zeta_4}{2}-\frac{16619\,\zeta_7}{8}\right)+\ \mathcal{O}(\varepsilon^5), \tag{62}$$



$$=-\frac{1}{4\,\varepsilon^3}-\frac{3}{2\,\varepsilon^2}-\frac{33}{4\,\varepsilon}-\frac{175}{4}+10\zeta_3+\ \varepsilon\left(-\frac{1825}{8}+\frac{113\zeta_3}{2}+15\zeta_4\right)$$

M_{22}, ε^4

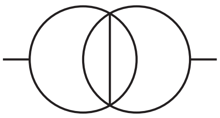
$$+\ \varepsilon^2\left(-\frac{18867}{16}+\frac{1241\zeta_3}{4}+\frac{339\zeta_4}{4}+185\zeta_5\right)$$
$$+\ \varepsilon^3\left(-\frac{194015}{32}+\frac{13425\zeta_3}{8}+\frac{3723\zeta_4}{8}+1028\zeta_5-204\,\zeta_3^2+\frac{875\zeta_6}{2}\right)$$
$$+\ \varepsilon^4\left(-\frac{1987331}{64}+\frac{143605\zeta_3}{16}+\frac{40275\zeta_4}{16}+5588\zeta_5\right.$$
$$\left.-1131\,\zeta_3^2+\frac{9715\zeta_6}{4}-612\,\zeta_3\,\zeta_4+\frac{13157\,\zeta_7}{4}\right)+\ \mathcal{O}(\varepsilon^5), \tag{63}$$



$$=-\frac{1}{8\,\varepsilon^3}-\frac{13}{16\,\varepsilon^2}-\frac{141}{32\,\varepsilon}-\frac{1393}{64}+2\zeta_3+\ \varepsilon\left(-\frac{12997}{128}+13\zeta_3+3\zeta_4\right)$$

M_{26}, ε^4

$$+\ \varepsilon^2\left(-\frac{116697}{256}+\frac{123\zeta_3}{2}+\frac{39\zeta_4}{2}+24\zeta_5\right)$$
$$+\ \varepsilon^3\left(-\frac{1019645}{512}+\frac{907\zeta_3}{4}+\frac{369\zeta_4}{4}+156\zeta_5+\frac{49\,\zeta_3^2}{2}+55\zeta_6\right)$$
$$+\ \varepsilon^4\left(-\frac{8732657}{1024}+\frac{4375\zeta_3}{8}+\frac{2721\zeta_4}{8}+693\zeta_5+\frac{637\,\zeta_3^2}{4}+\frac{715\zeta_6}{2}+\frac{147\,\zeta_3\,\zeta_4}{2}+\frac{2475\,\zeta_7}{4}\right)+\ \mathcal{O}(\varepsilon^5), \tag{64}$$

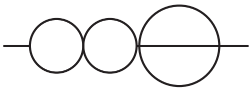


$$=\frac{1}{48\,\varepsilon^3}+\frac{7}{96\,\varepsilon^2}+\frac{11}{192\,\varepsilon}-\frac{605}{384}+\frac{7\zeta_3}{6}$$

M_{27}, ε^4

$$+\ \varepsilon\left(-\frac{13525}{768}+\frac{49\zeta_3}{12}+\frac{7\zeta_4}{4}\right)+\ \varepsilon^2\left(-\frac{208037}{1536}+\frac{161\zeta_3}{6}+\frac{49\zeta_4}{8}+\frac{221\zeta_5}{4}\right)$$
$$+\ \varepsilon^3\left(-\frac{2760397}{3072}+\frac{9535\zeta_3}{48}+\frac{161\zeta_4}{4}+\frac{1547\zeta_5}{8}-\frac{145\,\zeta_3^2}{3}+\frac{3245\zeta_6}{24}\right)$$

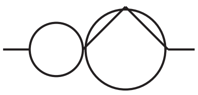
$$+\,\varepsilon^4\left(-\frac{33789053}{6144}+\frac{8273\zeta_3}{6}+\frac{9535\zeta_4}{32}+\frac{14527\zeta_5}{16}\right.$$
$$\left.-\frac{1015\,\zeta_3^2}{6}+\frac{22715\zeta_6}{48}-145\,\zeta_3\,\zeta_4+\frac{11289\,\zeta_7}{8}\right)+\,\mathcal{O}(\varepsilon^5), \tag{65}$$



$$=-\frac{1}{4\,\varepsilon^3}-\frac{5}{8\,\varepsilon^2}-\frac{27}{16\,\varepsilon}-\frac{153}{32}+\frac{3\zeta_3}{2}+\varepsilon\left(-\frac{891}{64}+\frac{15\zeta_3}{4}+\frac{9\zeta_4}{4}\right)$$

M_{23}, ε^4

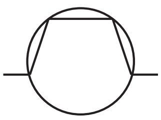
$$+\,\varepsilon^2\left(-\frac{5265}{128}+\frac{81\zeta_3}{8}+\frac{45\zeta_4}{8}+\frac{21\zeta_5}{2}\right)$$
$$+\,\varepsilon^3\left(-\frac{31347}{256}+\frac{459\zeta_3}{16}+\frac{243\zeta_4}{16}+\frac{105\zeta_5}{4}-\frac{9\,\zeta_3^2}{2}+\frac{45\zeta_6}{2}\right)$$
$$+\,\varepsilon^4\left(\boxed{-\frac{187353}{512}+\frac{2673\zeta_3}{32}+\frac{1377\zeta_4}{32}+\frac{567\zeta_5}{8}-\frac{45\,\zeta_3^2}{4}+\frac{225\zeta_6}{4}-\frac{27\,\zeta_3\,\zeta_4}{2}+\frac{147\,\zeta_7}{2}}\right)+\,\mathcal{O}(\varepsilon^5), \tag{66}$$



$$=-\frac{1}{3\,\varepsilon^3}-\frac{4}{3\,\varepsilon^2}-\frac{16}{3\,\varepsilon}-\frac{64}{3}+\frac{16\zeta_3}{3}+\varepsilon\left(-\frac{256}{3}+\frac{64\zeta_3}{3}+8\zeta_4\right)$$

M_{24}, ε^4

$$+\,\varepsilon^2\left(-\frac{1024}{3}+\frac{256\zeta_3}{3}+32\zeta_4+64\zeta_5\right)$$
$$+\,\varepsilon^3\left(-\frac{4096}{3}+\frac{1024\zeta_3}{3}+128\zeta_4+256\zeta_5-\frac{128\,\zeta_3^2}{3}+\frac{440\zeta_6}{3}\right)$$
$$+\,\varepsilon^4\left(-\frac{16384}{3}+\frac{4096\zeta_3}{3}+512\zeta_4+1024\zeta_5-\frac{512\,\zeta_3^2}{3}+\frac{1760\zeta_6}{3}-128\,\zeta_3\,\zeta_4+768\,\zeta_7\right)+\,\mathcal{O}(\varepsilon^5), \tag{67}$$

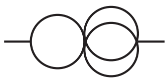


$$=-\frac{3}{8\,\varepsilon^3}-\frac{33}{16\,\varepsilon^2}-\frac{345}{32\,\varepsilon}-\frac{3525}{64}+\frac{45\zeta_3}{4}+\varepsilon\left(-\frac{35625}{128}+\frac{495\zeta_3}{8}+\frac{135\zeta_4}{8}\right)$$

M_{25}, ε^4

$$+\,\varepsilon^2\left(-\frac{358125}{256}+\frac{5175\zeta_3}{16}+\frac{1485\zeta_4}{16}+\frac{855\zeta_5}{4}\right)$$
$$+\,\varepsilon^3\left(-\frac{3590625}{512}+\frac{52875\zeta_3}{32}+\frac{15525\zeta_4}{32}+\frac{9405\zeta_5}{8}-\frac{675\,\zeta_3^2}{4}+\frac{2025\zeta_6}{4}\right)$$

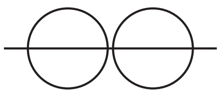
$$+\,\varepsilon^4\left(-\frac{35953125}{1024}+\frac{534375\zeta_3}{64}+\frac{158625\zeta_4}{64}+\frac{98325\zeta_5}{16}\right.$$
$$\left.-\frac{7425\,\zeta_3^2}{8}+\frac{22275\zeta_6}{8}-\frac{2025\,\zeta_3\,\zeta_4}{4}+\frac{16245\,\zeta_7}{4}\right)+\,\mathcal{O}(\varepsilon^5), \tag{68}$$



$$=\frac{1}{36\,\varepsilon^2}+\frac{35}{216\,\varepsilon}+\frac{991}{1296}+\varepsilon\left(\frac{26207}{7776}-\frac{11\zeta_3}{18}\right)+\varepsilon^2\left(\frac{670951}{46656}-\frac{385\zeta_3}{108}-\frac{11\zeta_4}{12}\right)$$

M_{11}, ε^5

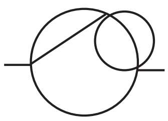
$$+\,\varepsilon^3\left(\boxed{\frac{16852031}{279936}-\frac{10901\zeta_3}{648}-\frac{385\zeta_4}{72}-\frac{13\zeta_5}{2}}\right)$$
$$+\,\varepsilon^4\left(\boxed{\frac{417941623}{1679616}-\frac{288277\zeta_3}{3888}-\frac{10901\zeta_4}{432}-\frac{455\zeta_5}{12}+\frac{121\,\zeta_3^2}{18}-\frac{265\zeta_6}{18}}\right)$$
$$+\,\varepsilon^5\left(\boxed{\frac{10274059439}{10077696}-\frac{7380461\zeta_3}{23328}-\frac{288277\zeta_4}{2592}-\frac{12883\zeta_5}{72}}\right.$$
$$\left.\boxed{+\frac{4235\,\zeta_3^2}{108}-\frac{9275\zeta_6}{108}+\frac{121\,\zeta_3\,\zeta_4}{6}-\frac{433\,\zeta_7}{6}}\right)+\,\mathcal{O}(\varepsilon^6), \tag{69}$$



$$=\frac{1}{16\,\varepsilon^2}+\frac{5}{16\,\varepsilon}+\frac{79}{64}+\varepsilon\left(\frac{9}{2}-\frac{3\zeta_3}{4}\right)+\varepsilon^2\left(\frac{4041}{256}-\frac{15\zeta_3}{4}-\frac{9\zeta_4}{8}\right)$$

M_{12}, ε^5

$$+\,\varepsilon^3\left(\frac{13851}{256}-\frac{237\zeta_3}{16}-\frac{45\zeta_4}{8}-\frac{21\zeta_5}{4}\right)+\,\varepsilon^4\left(\boxed{\frac{186867}{1024}-54\zeta_3-\frac{711\zeta_4}{32}-\frac{105\zeta_5}{4}+\frac{9\,\zeta_3^2}{2}-\frac{45\zeta_6}{4}}\right)$$
$$+\,\varepsilon^5\left(\boxed{\frac{311283}{512}-\frac{12123\zeta_3}{64}-81\zeta_4-\frac{1659\zeta_5}{16}+\frac{45\,\zeta_3^2}{2}-\frac{225\zeta_6}{4}+\frac{27\,\zeta_3\,\zeta_4}{2}-\frac{147\,\zeta_7}{4}}\right)+\,\mathcal{O}(\varepsilon^6), \tag{70}$$

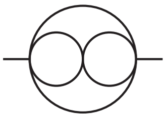


$$=\frac{1}{32\,\varepsilon^2}+\frac{5}{24\,\varepsilon}+\frac{1309}{1152}+\varepsilon\left(\frac{317}{54}-\frac{9\zeta_3}{8}\right)+\varepsilon^2\left(\frac{1234309}{41472}-\frac{15\zeta_3}{2}-\frac{27\zeta_4}{16}\right)$$

M_{13}, ε^5

$$+\,\varepsilon^3\left(\frac{4658207}{31104}-\frac{1309\zeta_3}{32}-\frac{45\zeta_4}{4}-\frac{153\zeta_5}{8}\right)$$
$$+\,\varepsilon^4\left(\frac{1121384029}{1492992}-\frac{634\zeta_3}{3}-\frac{3927\zeta_4}{64}-\frac{255\zeta_5}{2}+\frac{81\,\zeta_3^2}{4}-45\zeta_6\right)$$

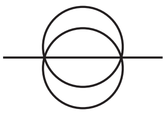
$$+\,\varepsilon^5\left(\frac{2105747071}{559872}-\frac{1234309\zeta_3}{1152}-317\zeta_4-\frac{22253\zeta_5}{32}\right.$$
$$\left.+135\,\zeta_3^2-300\zeta_6+\frac{243\,\zeta_3\,\zeta_4}{4}-\frac{2781\,\zeta_7}{8}\right)+\mathcal{O}(\varepsilon^6), \tag{71}$$



$$=\frac{1}{24\,\varepsilon^2}+\frac{49}{144\,\varepsilon}+\frac{1867}{864}+\varepsilon\left(\frac{64813}{5184}-\frac{23\zeta_3}{12}\right)$$

M_{14}, ε^5

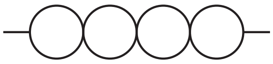
$$+\,\varepsilon^2\left(\frac{2146387}{31104}-\frac{1127\zeta_3}{72}-\frac{23\zeta_4}{8}\right)+\varepsilon^3\left(\frac{69116413}{186624}-\frac{42941\zeta_3}{432}-\frac{1127\zeta_4}{48}-\frac{127\zeta_5}{4}\right)$$
$$+\,\varepsilon^4\left(\frac{2185200787}{1119744}-\frac{1490699\zeta_3}{2592}-\frac{42941\zeta_4}{288}-\frac{6223\zeta_5}{24}+\frac{529\,\zeta_3^2}{12}-\frac{895\zeta_6}{12}\right)$$
$$+\,\varepsilon^5\left(\frac{68213322013}{6718464}-\frac{49366901\zeta_3}{15552}-\frac{1490699\zeta_4}{1728}-\frac{237109\zeta_5}{144}\right.$$
$$\left.+\frac{25921\,\zeta_3^2}{72}-\frac{43855\zeta_6}{72}+\frac{529\,\zeta_3\,\zeta_4}{4}-\frac{2189\,\zeta_7}{4}\right)+\mathcal{O}(\varepsilon^6), \tag{72}$$



$$=-\frac{1}{576\,\varepsilon}-\frac{13}{768}-\frac{9823}{82944}\,\varepsilon+\varepsilon^2\left(\boxed{-\frac{80513}{110592}+\frac{13\zeta_3}{144}}\right)$$

M_{01}, ε^6

$$+\,\varepsilon^3\left(\boxed{-\frac{49995799}{11943936}+\frac{169\zeta_3}{192}+\frac{13\zeta_4}{96}}\right)+\varepsilon^4\left(\boxed{-\frac{122739515}{5308416}+\frac{127699\zeta_3}{20736}+\frac{169\zeta_4}{128}+\frac{67\zeta_5}{48}}\right)$$
$$+\,\varepsilon^5\left(\boxed{-\frac{213973312663}{1719926784}+\frac{1046669\zeta_3}{27648}+\frac{127699\zeta_4}{13824}+\frac{871\zeta_5}{64}-\frac{169\,\zeta_3^2}{72}+\frac{235\zeta_6}{72}}\right)$$
$$+\,\varepsilon^6\left(\boxed{-\frac{1507417628113}{2293235712}+\frac{649945387\zeta_3}{2985984}+\frac{1046669\zeta_4}{18432}+\frac{658141\zeta_5}{6912}}\right.$$
$$\left.\boxed{-\frac{2197\,\zeta_3^2}{96}+\frac{3055\zeta_6}{96}-\frac{169\,\zeta_3\,\zeta_4}{24}+\frac{373\,\zeta_7}{16}}\right)+\mathcal{O}(\varepsilon^7), \tag{73}$$



$$=\boxed{\frac{1}{\varepsilon^4}}+\mathcal{O}(\varepsilon^4). \tag{74}$$

M_{31}, ε^3

Thus, we observe that at the four-loop level the GaC method works as good as the three-loop ones: **all** required terms of the ε-expansion of every four-loop MI have been expressed in terms of only **twelve** coefficients (boxed in eqs. (47-74))

$$M_{23,4}, M_{11,3}, M_{11,4}, M_{11,5}, M_{12,4}, M_{12,5}, M_{01,2}, M_{01,3}, M_{01,4}, M_{01,5}, M_{01,6}, M_{31,-4}, \tag{75}$$

of **primitive watermelon-like** massless propagator integrals.

An inspection of the above results for MI's reveals a few remarkable features.

1. In agreement with common expectations (based on the known solutions of the two- and three-loop B-problem) the transcendental terms up to (and including) weight 7 appear in eqs. (47-73). That is all results depend on only five irrational constants: ζ_3, ζ_4, ζ_5, ζ_6 and ζ_7.
2. For a given MI M_i the term ε^{p_i} (that is one with maximal power in ε) *always*[20] includes ζ_7.
3. A term proportional to ε^{p_i-j} could contain ζ_n with n not exceeding $7-j$; if $7-j<3$ then the term is free from irrational numbers.
4. There is another restriction on the singular part of any MI's (in fact, it is valid for an arbitrary p-intergral). It states that the term ε^{-n} (with $n=1,2,3,4$) may not contain zetas with the transcendentally weight exceeding $(7-2n)$. This property explains a very peculiar feature of MI's M_{62} and M_{63}: the absence of ζ_6 in the corresponding $\mathcal{O}(\varepsilon^{p_i-1})$ terms.
5. The only two *finite* MI's, namely, M_{44} and M_{45} contain only terms of one and same weight in every (available) coefficient of their ε-expansions.
6. The same property of "transcendental homogeneity" is true for the MI M_{52} (which is up to a factor of $1/\varepsilon$ is the three-loop *finite* MI N_0) *if* one divides an extra factor $(1-2\varepsilon)^2$ out of it. (See in this connection work [72], where some general arguments were given in favour of the hypothesis that the property is valid in all orders in ε.)

We want to stress that any statement on the structure of ζ's appearing in an integral does depend on the global normalization which is rather arbitrary. Our normalization condition is a natural one but, certainly, not unique. If we would choose

$$M_{31} = \frac{1}{\varepsilon^4}(1 + \sum_{1\le i\le 7} a_i \varepsilon^i) \tag{76}$$

then all MI's would depend on a_i, and all statements just discussed above could be, obviously, made invalid *if* a coefficient a_i were allowed to contain ζ_i (for $i>1$) and γ_E for $i=1$.

On the other hand, if we restrict ourselves to a natural choice of the normalization of MI M_{31} such as

$$M_{31} = \frac{1}{\varepsilon^4}(1 + \sum_{3\le i\le 7} b_i \zeta_i \varepsilon^i), \tag{77}$$

with b_i being rational numbers, then the properties 1-6 would in general stay untouched.

5.2. Tests of the results

In this subsection we discuss various checks which we have made to test our results expressed in eqs. (47-74). The set of 28 master integrals is naturally divided in three subsets: primitive (M_{23}, M_{24}, M_{25}, M_{11}, M_{12}, M_{13}, M_{14}, M_{01}, M_{31}), simple (M_{32}, M_{33}, M_{21}, M_{22}, M_{26}, M_{27}) and, finally, complicated ones (M_{61}, M_{62}, M_{63}, M_{51}, M_{41}, M_{42}, M_{44}, M_{45}, M_{34}, M_{35}, M_{36}, M_{52}, M_{43}). We will consider these subsets separately.

5.2.1. Primitive integrals

A primitive FI is by definition expressible in terms of the Γ-function. A straightforward use of formulas of section 3 gives:

[20] MI M_{31} is the only exception from this rule since its sub-leading in ε terms are fixed to be zero essentially by hands, that is by choosing the G-scheme.

$$
\begin{aligned}
M_{23} &= \frac{1}{\varepsilon^3}\,G(\varepsilon,1), \quad M_{24} = \frac{1}{\varepsilon^3}\,G(2\varepsilon,1), \quad M_{25} = \frac{1}{\varepsilon^3}\,G(3\varepsilon,1),\\
M_{11} &= \frac{1}{\varepsilon^3}\,G(\varepsilon,\varepsilon), \quad M_{12} = \frac{1}{\varepsilon^2}\,G(\varepsilon,1)\,G(\varepsilon,1), \quad M_{13} = \frac{1}{\varepsilon^2}\,G(\varepsilon,1)\,G(-1+3\varepsilon,1),\\
M_{14} &= \frac{1}{\varepsilon^3}\,G(\varepsilon,2\varepsilon), \quad M_{01} = \frac{1}{\varepsilon^2}\,G(\varepsilon,\varepsilon)\,G(-2+3\varepsilon,1), \quad M_{31} = \frac{1}{\varepsilon^4}. \qquad (78)
\end{aligned}
$$

After the expansion in ε eqs. (78) produce altogether 89 coefficients. As was discussed in subsection 5.1 as many as twelve coefficients listed in eq. (75) have been used in the process of the solution of the system of GaC equations, while the remaining 77 coefficients have been *predicted* from the equation and listed unboxed in eqs. (66–73).

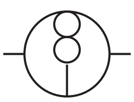
The reader is advised to check that all these 77 coefficients are in full agreement to eqs. (66–73).

5.2.2. Simple integrals

These all could be expressed in terms of G-functions and the generalized two-loop diagram:

$$
\begin{aligned}
M_{32} &= \frac{1}{\varepsilon^2}\,F(1,1,1,1,\varepsilon), &(79)\\
M_{33} &= \frac{1}{\varepsilon^2}\,F(1,1,1,1,2\varepsilon) &(80)\\
M_{21} &= \frac{1}{\varepsilon^2}\,F(1,1,1,\varepsilon,\varepsilon), &(81)\\
M_{22} &= \frac{1}{\varepsilon^2}\,F(1,\varepsilon,1,\varepsilon,1), &(82)\\
M_{26} &= \frac{1}{\varepsilon}\,G(3\varepsilon,1)\,F(1,1,1,1,\varepsilon), &(83)\\
M_{27} &= \frac{1}{\varepsilon}\,G(\varepsilon,1)\,F(1,1,1,1,2\varepsilon-1). &(84)
\end{aligned}
$$

To be specific, let us consider the direct evaluation of M_{33} in some details. First, we define a related FI M'_{33} pictured in eq. (85) below. A simple reduction of FI M'_{33} to MI's gives the (exact) equation



$$
= \quad \frac{4(7D-26)(5D-14)(5D-16)(D-3)}{9(3D-10)(D-4)^3}\,M_{14} + \frac{-2(2D-7)}{3(D-4)}\,M_{33}. \qquad (85)
$$

M'_{33}

On the other hand,

$$
M'_{33} = \frac{1}{\varepsilon^2}\,F(1,1,1,1,1+2\varepsilon) \qquad (86)
$$

and (see [73] as well as [74, 75, 67])

$$
\begin{aligned}
\frac{1}{(1-2\varepsilon)}F(1,1,1,1,1+2\varepsilon) &= 6\zeta_3 + 9\zeta_4\varepsilon + 192\zeta_5\varepsilon^2 + \left(465\zeta_6 - 168\zeta_3^2\right)\varepsilon^3\\
&+ (4509\zeta_7 - 504\zeta_4\zeta_3)\,\varepsilon^4 + \left(\frac{16377}{2}\zeta_8 - 1620\zeta_{6,2} - 3252\zeta_5\zeta_3\right)\varepsilon^5\\
&+ \left(98490\zeta_9 - 14598\zeta_5\zeta_4 - 15390\zeta_6\zeta_3 + 2676\zeta_3^3\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7), \qquad (87)
\end{aligned}
$$

where

$$
\zeta_{6,2} \equiv \sum_{n_1>n_2>0} \frac{1}{n_1^6\,n_2^2}.
$$

Finally, eqs. (85-87) together with eq. (72) lead to the following (*independent* from our calculations) result for MI M_{33} which is not only in full agreement to eq. (61) but also includes two more terms in ε:

$$
\begin{aligned}
M_{33,4} &= -\frac{367253}{3} + \frac{97982}{3}\zeta_3 - \frac{11038}{3}\zeta_3^2 + 7987\zeta_4 - 1966\zeta_3\zeta_4 \\
&+ 22750\zeta_5 - 3914\zeta_3\zeta_5 + \frac{31690}{3}\zeta_6 - 4860\zeta_{6,2} + 22676\zeta_7 + \frac{147181}{8}\zeta_8, \qquad (88) \\
M_{33,5} &= -\frac{2073833}{3} + \frac{580022}{3}\zeta_3 - \frac{66370}{3}\zeta_3^2 + \frac{47918}{9}\zeta_3^3 + 48991\zeta_4 - 11038\zeta_3\zeta_4 \\
&+ 123766\zeta_5 - 7828\zeta_3\zeta_5 - 35031\zeta_4\zeta_5 + \frac{164350}{3}\zeta_6 \\
&- \frac{97340}{3}\zeta_3\zeta_6 - 9720\zeta_{6,2} + 103838\zeta_7 + \frac{147181}{4}\zeta_8 + \frac{2293555}{9}\zeta_9. \qquad (89)
\end{aligned}
$$

In the same way we have successfully checked all other simple MI's. In addition, for all of them we get two extra terms of the ε-expansion. They look similar to the ones listed in (88,89) and include, in addition to $\zeta_3 - \zeta_9$, only $\zeta_{6,2}$.

5.2.3. Complicated integrals

We start from diagrams M_{52} and M_{43} which are relatively simple as they could be expressed through the ε^1 and ε^2 extra terms of the basic three-loop non-planar integral N_0, namely:

$$
\begin{aligned}
M_{52} &= \frac{1}{\varepsilon}N_0(\varepsilon) = \frac{20\zeta_5}{\varepsilon} + N_{0,1} + N_{0,2}\,\varepsilon + \mathcal{O}(\varepsilon^2), \qquad (90) \\
M_{43} &= G(1, 2+3\varepsilon)\,N_0(\varepsilon) = -\frac{5\zeta_5}{\varepsilon} + +25\zeta_5 - \frac{N_{0,1}}{4} + (\frac{6N_{0,1}}{4} - \frac{N_{0,2}}{4} - 75\zeta_5)\varepsilon + \mathcal{O}(\varepsilon^2). (91)
\end{aligned}
$$

Thus,

$$N_{0,1} = -80\zeta_5 + 68\,\zeta_3^2 + 50\zeta_6, \quad N_{0,2} = -272\,\zeta_3^2 - 200\zeta_6 + 204\,\zeta_3\,\zeta_4 + 450\,\zeta_7. \qquad (92)$$

The coefficient $N_{0,1}$ was known since long from calculations of the five-loop β-function in the ϕ^4-model in [76]. The second coefficient $N_{0,2}$ was first computed with the GaC method and presented in [77]. Its completely independent calculation (through fitting a high-precision numerical result with an appropriate analytical ansatz) was performed in [78]. Needless to say that the results of [76] and [78] are in agreement with eq. (92).

All other complicated integrals (except for convergent integral M_{44}, whose value, $\frac{447}{8}\zeta_7$, at $D = 4$ was also analytically found in [76]) have not been known with sufficient accuracy before our calculations. Note, that for a given (four-loop) master integral M_i one needs to know only the $5 + p_i$ first terms in its ε-expansion. (For accounting purposes we assume that every expansion starts from $\frac{1}{\varepsilon^4}$ even if the corresponding term drops out from a specific MI.) Among them first $5 + p^i - 1$ (that is all except for the last one) are in a sense easy as they all could be analytically found by well-known methods based on Infrared Rearrangement (see, e.g. [19] where the issue was spelled out on the example of massive four-loop tadpoles). At any case they all are very well checked in the course of the renormalization procedure.

6. Perspectives

6.1. Five-loop master integrals

As we have seen from the discussion in the previous section all complicated and even simple four-loop MI's have been completely expressed in terms of very simple watermelon-like primitive p-integrals. In fact, the reduction method based on the $1/D$ expansion of coefficient functions of

MI's is in general applicable for any number of loops in the A-Problem. The GaC symmetry is also not limited by number of loops of p-integrals.

Thus, if the five-loop A-problem is solved, then the five-loop B-problem can also be solved in the following sense: the identities stemming from the GaC symmetry will express all five-loop MI's in terms of significantly smaller set of p-integrals. But which exactly set? At present nobody knows for sure. But one could certainly expect that:

- in general the five-loop master p-integrals will contain irrational terms of weight not higher than 9;
- the "small set" of five-loop integrals will include ones primitive as well as those expressible in terms of the generalized F-function.

As both types of the integrals could certainly be analytically evaluated up to the weight 9 [21, 75] we conclude that the five-loop B-problem should be analytically doable. Moreover, we believe that the GaC symmetry + reduction provides the simplest way of analytical solution of the five-loop B-problem.

6.2. General case

At first sight the applicability scope of the GaC method is rather limited and amounts exclusively to the massless propagators. Indeed, the heart of the method is the existence of relations between integrals of *different* topologies *beyond* those provided by the very integration by parts. We are not aware about existence of such relations in general case except for a one: *finiteness*.

Indeed, two finite integrals are in certainly equal to each other with accuracy $\mathcal{O}(\varepsilon^0)$ irrespectively on their topologies. As a result IBP relations will provide some partial information about the values of corresponding master integrals. Unfortunately, the information proves to be rather limited[21].

Indeed, let us consider, as a simple example, eqs. (24). Without any use of GaC symmetry one could, obviously, write

$$N_0 = L_0 + \mathcal{O}(\varepsilon^0), \quad N_0 = N_1 + \mathcal{O}(\varepsilon^0), \quad N_0 = N_2 + \mathcal{O}(\varepsilon^0), \quad N_0 = \mathcal{O}(\varepsilon^0). \tag{93}$$

After reduction to masters and solution of the resulting equations we arrive to the same results (25,26-45) but with the ε-accuracy *downgraded* by one for every master integral. This is certainly *not enough* to solve the three-loop problem: no new information is obtained for the most complicated non-planar master integral N_0.

The reason for the failure is quite clear: the equations (24) do not provide, in fact, any constraints on the value of N_0 as they could be equivalently rewritten as follows:

$$L_0 = \mathcal{O}(\varepsilon^0), \quad N_1 = \mathcal{O}(\varepsilon^0), \quad N_2 = \mathcal{O}(\varepsilon^0), \quad N_0 = \mathcal{O}(\varepsilon^0). \tag{94}$$

Repeating the same exercise at four-loop level we will arrive to a similar conclusion: without any use of the GaC symmetry one could find for every master integral M_i all except for the last one (that is $5 + p^i - 1$) of its ε-expansion. Again finiteness only, without the GaC-symmetry, is not enough to solve the four-loop Problem.

On the other hand, any L-loop MI multiplied by a one-loop scalar massless propagator is, obviously, a (L+1)-loop MI (compare, for, example, T_1 and P_2). Within the G-scheme framework

[21] Nevertheless, there are cases for which even this limited information is enough, see e.g. [79, 80].

the values of the integrals are trivially related by a factor ε. Thus, at two- and three-loops we get the following identities

$$T_1 = \varepsilon P_2, \quad T_2 = \varepsilon P_4$$

and

$$N_0 = \varepsilon M_{52}, \quad L_1 = \varepsilon M_{32}, \quad P_1 = \varepsilon M_{24}, \quad P_2 = \varepsilon M_{23}, \quad P_3 = \varepsilon M_{11}$$

respectively.

Using "downgraded" eqs. (58) and (60) we find the same results for both non-primitive MI's N_0 and L_1 as in eqs. (40) and (41) but with the deepness of the ε-expansion *increased* by one. Thus we arrive at a truly remarkable conclusion: by merely reducing finite four-loop propagators to the master integrals and *without* any use of GaC symmetry we could not only completely solve the three-loop Problem, we even can get one more term in ε-expansion of every non-trivial master integral!

It remains to see how predictive is this trick of finding master integrals for cases with other patterns of external momenta and masses. But its is absolutely clear that at least some useful nontrivial information can be obtained along these lines.

7. Even zetas

7.1. Four and five loops

In addition to six remarkable features of four-loop master p-integrals listed in subsection 5.1 there exist the seventh one, probably most remarkable. Indeed, a scrupulous inspection of eqs. (47-73)) demonstrate that all their right hand sides do depend on only the following **three** combinations of zetas:

$$\hat{\zeta}_3 = \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \; \hat{\zeta}_5 = \zeta_5 + \frac{5\varepsilon}{2}\zeta_6 \quad \text{and} \quad \zeta_7. \tag{95}$$

This simple fact has far reaching consequences. Indeed, a little meditation on (95) leads to the following statement[22]:

Theorem 4.

1. Any *finite* at $\varepsilon \to 0$ p-integral does not contain even zetas $\{\zeta_{2n} \mid n \geq 2\}$ in the limit of $\varepsilon \to 0$.
2. Any *finite* at $\varepsilon \to 0$ combination of p-integrals like

$$\sum C_i(\varepsilon) p_i, \quad C_i = \sum_j C_{ij} \varepsilon^j,$$

   with the coefficient functions being functions (not necessarily finite at $\varepsilon \to 0$) with purely rational coefficients C_{ij}, will *not* contain even zetas in the limit of $\varepsilon \to 0$ (while *odd* zetas $\{\zeta_{2n+1} \mid n \geq 1\}$ are expected and indeed appear in general).
3. Let $F(\varepsilon)$ be any renormalized (and, thus, finite in the limit of $\varepsilon \to 0$) combination of any p-integrals. The *sole* source of possible even zetas in $F(0)$ is the appearance of zetas (not necessarily even) in the renormalization factors involved in carrying out the renormalization of F.

The third point suggests a clean explanation of an old puzzle of pQCD: the absence of even zetas in the Adler function of pQCD at order α_s^3, $D_{(3)}(q^2)$. Indeed, the function at this order is

(i) a *finite* combination of four-loop p-integrals;

(ii) the corresponding renormalization is done with the help of charge coupling renormalization which does not depend on any zetas at the order required.

[22] By any p-integral below we understand any one with number of loops less or equal to four.

As a direct consequence of (ii) the function $D_{(3)}(q^2)$ **should not** depend on ζ_4, ζ_6 and $\zeta_3\zeta_4$. This is indeed the case [61, 62]! At first glance, this explanation of this old puzzle generates another one: why $D_{(3)}(q^2)$ does not include terms proportional odd zetas with weight larger than five (which is expected in general for any combination of four-loop p-integrals)? No, it is not a puzzle since long. It is a well-known fact[23] that the (L+1) loop Adler function (in massless QCD) could be completely expressed through L-loop p-integrals.

Exactly the same reason explains the absence of even zetas in the four-loop contribution to the Bjorken sum rule [30].

The problem of why the five-loop $\mathcal{O}(\alpha_s^4)$ Adler function is also free from even zetas should be possible to solve by extending the above reasoning by one loop higher. The "only" missing ingredients — a property of five-loop master p-integrals analogous to (95). We hope to come back to the subject in future.

7.2. Three loops

In fact, Theorem 4 was proven for the three-loop p-integrals in an early work by Broadhurst [72] with the help of essentially equivalent (though, to our opinion, somewhat more complicated) considerations. This is certainly enough to explain the absence of ζ_4 in the α_s^2 contribution to Adler function [81] and in the α_s^3 ones to the deep inelastic sum rules found in [82, 45]. This is because all these quantities are naturally expressed through some combinations of *three-loop* p-integrals with purely rational coefficients.

Note that, the three-loop version of Theorem 4 [72] is not enough to explain the absence of even zetas from the α_s^3 contribution to Adler function and from the four-loop result for the QCD β-function [83, 79]. The problem is that, to the best of our knowledge, it is not known whether one could find a representation, say, the four-loop contribution to the QCD β-function in term of a finite combination of the three-loop p-integrals with coefficients *free* from any zetas. The same is true for the Adler function.

On the other hand, we do agree with [72] that the four-loop QED β-function should be free from ζ_4 (in agreement with explicit calculations of [84]) as it can be expressed via a *finite* combination (with purely rational coefficients) of three-loop p-integrals [85].

8. Discussion

There are various points deserving further discussion in connection with the algorithm of evaluation of MI's elaborated in sections 4 and 5.

- The results discussed in the present paper have been indispensable for the long-term project of computing the cross section of e^+e^- annihilation into hadrons at order α_s^4 in QCD [28, 30]. While they were first obtained in 2003, their publication had been postponed in favour of the faster completion of the main project.
- Recently, a definite class of massless p-integrals was proven to be expressible in terms of the multiple zeta values for all orders of expansions in $D-4$, and a direct method of their evaluation was suggested ([86, 87]).

  In our case (four-loop p-integrals) it predicts that the result is expressible through ζ_n up to $n = 7$, as confirmed by our calculations. Unfortunately this method in its present form is applicable only for integrals with number of internal lines equal to doubled number of loops plus one, so the most complicated four-loop MI's seem to be unreachable.

[23] First, probably understood on the example of the three-loop $\mathcal{O}(\alpha_s^2)$ Adler function [81].

- The heavy use of identities between Feynman integrals coming, eventually, from IBP relations is not a unique feature of our approach to the evaluation of MI's. It is of interest, that three other, quite different and in a sense more general approaches would also be impossible without intensive use of the reduction of Feynman integrals to masters. We mean (i) the method of differential equations[24], (ii) the use of difference equations [5, 98] and, at last, very new method [99–101] based on recurrence equations with respect to the space-time dimension D [102].

  Finally, we want to mention two popular and powerful methods of evaluation of MI's which do not use directly the IBP reduction. The first approach is based on the Mellin-Barn representation. The early applications of Mellin integrals to evaluation of FI's were performed in pioneering works [103, 104]. Currently it is an actively developing field, for a review see, e.g. [2, 3].

  The second method — the so-called sector decomposition — was originally used as a convenient theoretical tool for the analysis of convergence of FI's [105, 106]. First applications of sector decomposition for evaluation of FI's were considered in [107–109]. The current status of the method can be found in a review [110].

- While the tests of our results described in subsections 5.2.1 and 5.2.2 leave no room for doubt as for the cases of *trivial* and *simple* groups of MI's, it is *not* true for the most difficult group of *complicated* integrals: for this family of thirteen MI's only three had been directly checked in an independent way (see subsection 5.2.3). It means that if a master integral from the remaining ten integrals were assigned a wrong value, it would *change* in all probability *all* physical results obtained with the use of these MI's since 2004.

  Let us, therefore, discuss a little bit further the important issue of the correctness of these MI's. The method of computing master p-integrals described in sections 4 and 5 heavily uses both the GaC symmetry *and* the procedure of reduction of four-loop p-integrals. The latter is the most complicated part of all the calculation as it requires, first, careful computer algebra programming and, second, large-scale calculations. Thus, an independent check of the ten remaining most complicated MI's would also provide us with a quite strong, though non-direct, evidence for the correctness of the reduction algorithm we use and its FORM implementation.

  Fortunately, such an independent check of *all* complicated integrals has been very recently performed in [111] with the use of the sector decomposition, where not only all results have been (numerically) confirmed with better than $1\permil$ accuracy[25] but also one extra term in these ε-expansions have been computed.

9. Summary and Conclusions

In this work we have presented an algorithm for the analytical evaluation of all master integrals which appear in the process of reduction of massless dimensionally regulated Feynman integrals with one external momentum (p-integrals). The algorithm is based on the glue-and-cut symmetry

[24] Starting from early works [88–96] the method has developed into quite a powerful technique. For its modern status and further references, see the review [97].

[25] We mean the accuracy for the most complicated last $\mathcal{O}(\varepsilon^{p_i})$ term in comparison with the exact results listed in eqs. (50-59), the accuracy of simpler terms of order ε^i with $-4 \le i < p_i$) is significantly higher. The typical accuracy of the $\mathcal{O}(\varepsilon^{p_i+1})$ term (which is necessary only in evaluation of *five*-loop master p-integrals) was also about $1\permil$. One should keep in mind that the latter accuracy is an estimate given by the MC-integrator and as such it is not always reliable.

[1] which is an unique and very specific property of such integrals valid irrespectively of their complexity (number of loops). In addition to the symmetry the algorithm heavily uses the reduction procedure.

It has been demonstrated that the algorithm works flawlessly for the case of the three-loop p-integrals (successfully reproducing well-known thirty years old results of [21]) and four-loop p-integrals. In the latter case it produces explicit analytical results for all master integrals, major part of which are new.

Together with Theorem 2 and the $1/D$ method of reduction of p-integrals [14, 23] the algorithm guaranties that the UV counterterm of any five-loop diagram can be calculated within the MS-scheme in terms of rational numbers, ζ_3, ζ_4, ζ_5, ζ_6 and ζ_7. This implies the analytical calculability of the β-functions and anomalous dimensions of fields and composite operators in an arbitrary model at the five-loop level.

Acknowledgments.

This work was supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 “Computational Particle Physics” and by RFBR grants 08-02-01451, 10-02-00525.

We are very grateful to J.H. Kühn for his long-term encouragement, support and patience.

We thank D. Broadhurst, M. Czakon, A. Grozin, S. Moch, V.A. Smirnov and M. Tentyukov for fruitful discussions, attentive reading of the manuscript and good advice. K.G. Chetyrkin is grateful to A. Czarnecki for a discussion of hystorical aspects of the work.

K.G. Chetyrkin should also thank A.L. Kataev and F.V. Tkachov for a fruitful collaboration during their Diploma studies which started the exploration of the wonderful land of massless propagator like FI's [81]. The present work is dedicated to the memory of his late friend and colleague S.G. Gorishny, whose contribution was decisive in another important application of the idea of glueing — the derivation of the Wilson short-distance expansion in the minimally subtracted scheme [112] and, as a logical consequence, the creation of the method of projectors [43].

References

[1] K. G. Chetyrkin and F. V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys.* **B192** (1981) 159–204.

[2] V. A. Smirnov, *Evaluating Feynman integrals.* Berlin, Germany: Springer (2004) 244 p.

[3] V. A. Smirnov, *Feynman integral calculus.* Berlin, Germany: Springer (2006) 283 p.

[4] S. Laporta and E. Remiddi, *The analytical value of the electron (g-2) at order α^3 in QED*, *Phys. Lett.* **B379** (1996) 283–291, [hep-ph/9602417].

[5] S. Laporta, *High-precision calculation of multi-loop feynman integrals by difference equations*, *Int. J. Mod. Phys.* **A15** (2000) 5087–5159, [hep-ph/0102033].

[6] T. Gehrmann and E. Remiddi, *Differential equations for two-loop four-point functions*, *Nucl. Phys.* **B580** (2000) 485–518, [hep-ph/9912329].

[7] P. A. Baikov, *Explicit solutions of the 3–loop vacuum integral recurrence relations*, *Phys. Lett.* **B385** (1996) 404–410, [hep-ph/9603267].

[8] P. A. Baikov, *Explicit solutions of the n–loop vacuum integral recurrence relations*, hep-ph/9604254.

[9] P. A. Baikov, *Explicit solutions of the multi-loop integral recurrence relations and its application*, *Nucl. Instrum. Meth.* **A389** (1997) 347–349, [hep-ph/9611449].

[10] O. V. Tarasov, *Reduction of feynman graph amplitudes to a minimal set of basic integrals*, *Acta Phys. Pol.* **B29** (1998) 2655, [hep-ph/9812250].

[11] A. V. Smirnov and V. A. Smirnov, *Applying Groebner bases to solve reduction problems for Feynman integrals*, *JHEP* **01** (2006) 001, [hep-lat/0509187].

[12] R. N. Lee, *Group structure of the integration-by-part identities and its application to the reduction of multiloop integrals*, *JHEP* **07** (2008) 031, [arXiv:0804.3008].

[13] P. A. Baikov, *The criterion of irreducibility of multi-loop feynman integrals*, *Phys. Lett.* **B474** (2000) 385–388, [hep-ph/9912421].

[14] P. A. Baikov, *A practical criterion of irreducibility of multi-loop feynman integrals*, *Phys. Lett.* **B634** (2006) 325–329, [hep-ph/0507053].

[15] V. A. Smirnov and M. Steinhauser, *Solving recurrence relations for multi-loop feynman integrals*, *Nucl. Phys.* **B672** (2003) 199–221, [hep-ph/0307088].
[16] A. V. Smirnov and V. A. Smirnov, *S-bases as a tool to solve reduction problems for Feynman integrals*, *Nucl. Phys. Proc. Suppl.* **160** (2006) 80–84, [hep-ph/0606247].
[17] A. V. Smirnov and V. A. Smirnov, *On the reduction of Feynman integrals to master integrals*, *PoS* **ACAT2007** (2007) 085, [arXiv:0707.3993].
[18] A. V. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, *JHEP* **10** (2008) 107, [arXiv:0807.3243].
[19] K. G. Chetyrkin, M. Faisst, C. Sturm, and M. Tentyukov, *ε-finite basis of master integrals for the integration-by- parts method*, *Nucl. Phys.* **B742** (2006) 208–229, [hep-ph/0601165].
[20] Y. Schroder, *Automatic reduction of four-loop bubbles*, *Nucl. Phys. Proc. Suppl.* **116** (2003) 402–406, [hep-ph/0211288].
[21] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique*, *Nucl. Phys.* **B174** (1980) 345–377.
[22] F. V. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, *Phys. Lett.* **B100** (1981) 65–68.
[23] P. A. Baikov, *Recurrence relations in the large space-time dimension limit*, *PoS* **RADCOR2007** (2007) 022.
[24] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *The cross section of e+ e- annihilation into hadrons of order $\alpha_s^4 n_f^2$ in perturbative QCD*, *Phys. Rev. Lett.* **88** (2002) 012001, [hep-ph/0108197].
[25] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Strange quark mass from tau lepton decays with $\mathcal{O}(\alpha_s^3)$ accuracy*, *Phys. Rev. Lett.* **95** (2005) 012003, [hep-ph/0412350].
[26] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Scalar correlator at $\mathcal{O}(\alpha_s^4)$, Higgs decay into b- quarks and bounds on the light quark masses*, *Phys. Rev. Lett.* **96** (2006) 012003, [hep-ph/0511063].
[27] P. A. Baikov and K. G. Chetyrkin, *Higgs decay into hadrons to order α_s^5*, *Phys. Rev. Lett.* **97** (2006) 061803, [hep-ph/0604194].
[28] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Order α_s^4 QCD Corrections to Z and τ Decays*, *Phys. Rev. Lett.* **101** (2008) 012002, [arXiv:0801.1821].
[29] P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, *Quark and gluon form factors to three loops*, arXiv:0902.3519.
[30] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory*, *Phys. Rev. Lett.* **104** (Mar, 2010) 132004.
[31] J. A. M. Vermaseren, *New features of form*, math-ph/0010025.
[32] D. Fliegner, A. Retey, and J. A. M. Vermaseren, *Parallelizing the symbolic manipulation program FORM. I: Workstation clusters and message passing*, hep-ph/0007221.
[33] M. Tentyukov *et. al.*, *ParFORM: Parallel Version of the Symbolic Manipulation Program FORM*, cs/0407066.
[34] M. Tentyukov, H. M. Staudenmaier, and J. A. M. Vermaseren, *ParFORM: Recent development*, *Nucl. Instrum. Meth.* **A559** (2006) 224–228.
[35] M. Tentyukov and J. A. M. Vermaseren, *The multithreaded version of FORM*, hep-ph/0702279.
[36] K. G. Chetyrkin, J. H. Kühn, and A. Kwiatkowski, *QCD corrections to the e^+e^- cross-section and the Z boson decay rate: Concepts and results*, *Phys. Rept.* **277** (1996) 189–281.
[37] M. Steinhauser, *Results and techniques of multi-loop calculations*, *Phys. Rept.* **364** (2002) 247–357, [hep-ph/0201075].
[38] K. G. Chetyrkin and J. H. Kühn, *Mass corrections to the Z decay rate*, *Phys. Lett.* **B248** (1990) 359–364.
[39] K. G. Chetyrkin, R. Harlander, J. H. Kühn, and M. Steinhauser, *Mass corrections to the vector current correlator*, *Nucl. Phys.* **B503** (1997) 339–353, [hep-ph/9704222].
[40] K. G. Chetyrkin, R. V. Harlander, and J. H. Kühn, *Quartic mass corrections to R_{had} at $\mathcal{O}(\alpha_s^3)$* , *Nucl. Phys.* **B586** (2000) 56–72, [hep-ph/0005139].
[41] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Vacuum polarization in pQCD: First complete $\mathcal{O}(\alpha_s^4)$ result*, *Nucl. Phys. Proc. Suppl.* **135** (2004) 243–246.
[42] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *R(s) and hadronic tau-Decays in Order α_s^4: technical aspects*, arXiv:0906.2987.
[43] S. G. Gorishny, S. A. Larin, and F. V. Tkachov, *The Algorithm For OPE Coefficient Functions In The MS Scheme*, *Phys. Lett.* **B124** (1983) 217–220.
[44] S. G. Gorishny and S. A. Larin, *Coefficient Functions Of Asymptotic Operator Expansions In Minimal Subtraction Scheme*, *Nucl. Phys.* **B283** (1987) 452.
[45] S. A. Larin and J. A. M. Vermaseren, *The α_s^3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule*, *Phys. Lett.* **B259** (1991) 345–352.
[46] J. F. Ashmore, *A method of gauge invariant regularization*, *Lett. Nuovo Cim.* **4** (1972) 289–290.
[47] G. M. Cicuta and E. Montaldi, *Analytic renormalization via continuous space dimension*, *Nuovo Cim. Lett.* **4** (1972) 329–332.

[48] G. 't Hooft and M. J. G. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl. Phys.* **B44** (1972) 189–213.

[49] G. 't Hooft, *Dimensional regularization and the renormalization group*, *Nucl. Phys.* **B61** (1973) 455–468.

[50] J. C. Collins, *Normal Products in Dimensional Regularization*, *Nucl. Phys.* **B92** (1975) 477.

[51] A. A. Vladimirov, *Method For Computing Renormalization Group Functions In Dimensional Renormalization Scheme*, *Theor. Math. Phys.* **43** (1980) 417.

[52] K. G. Chetyrkin and F. V. Tkachov, *Infrared R-Operation and Ultraviolet Counterterms in the MS Scheme*, *Phys. Lett.* **B114** (1982) 340–344.

[53] K. G. Chetyrkin and V. A. Smirnov, *R* Operation Corrected*, *Phys. Lett.* **B144** (1984) 419–424.

[54] D. J. Broadhurst and D. Kreimer, *Knots and numbers in phi**4 theory to 7 loops and beyond*, *Int. J. Mod. Phys.* **C6** (1995) 519–524, [hep-ph/9504352].

[55] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, *Anomalous dimension with wrapping at four loops in N=4 SYM*, *Nucl. Phys.* **B805** (2008) 231–266, [arXiv:0806.2095].

[56] F. Fiamberti, A. Santambrogio, and C. Sieg, *Five-loop anomalous dimension at critical wrapping order in N=4 SYM*, *JHEP* **03** (2010) 103, [arXiv:0908.0234].

[57] A. V. Kotikov, *Some methods for the evaluation of complicated Feynman integrals*, hep-ph/0112347.

[58] A program by M . Veltman, see Strubbe, H., *Manual for Schoonschip: A CDC 6000 / 7000 program for symbolic evaluation of algebraic expressions*, *Comput. Phys. Commun.* **8** (1974) 1–30.

[59] S. G. Gorishny, S. A. Larin, L. R. Surguladze, and F. V. Tkachov, *MINCER: Program for multiloop calculations in quantum field theory for the schoonschip system*, *Comput. Phys. Commun.* **55** (1989) 381–408.

[60] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren, *The FORM version of MINCER*, . NIKHEF-H-91-18.

[61] S. G. Gorishny, A. L. Kataev, and S. A. Larin, *The* $\mathcal{O}(\alpha_s^3)$ *corrections to* $\sigma_{tot}(e^+e^- \to hadrons)$ *and* $\Gamma(\tau \to \nu_\tau + hadrons)$ *in QCD*, *Phys. Lett.* **B259** (1991) 144–150.

[62] L. R. Surguladze and M. A. Samuel, *Total hadronic cross-section in* e^+e^- *annihilation at the four loop level of perturbative QCD*, *Phys. Rev. Lett.* **66** (1991) 560–563.

[63] S. Moch, J. A. M. Vermaseren, and A. Vogt, *The three-loop splitting functions in QCD: The non-singlet case*, *Nucl. Phys.* **B688** (2004) 101–134, [hep-ph/0403192].

[64] A. Vogt, S. Moch, and J. A. M. Vermaseren, *The three-loop splitting functions in QCD: The singlet case*, *Nucl. Phys.* **B691** (2004) 129–181, [hep-ph/0404111].

[65] J. A. M. Vermaseren, A. Vogt, and S. Moch, *The third-order QCD corrections to deep-inelastic scattering by photon exchange*, *Nucl. Phys.* **B724** (2005) 3–182, [hep-ph/0504242].

[66] J. Rosner, *Higher-order contributions to the divergent part of* Z_3 *in a model quantum electrodynamics*, *Annals of Physics* **44** (1967) 11.

[67] I. Bierenbaum and S. Weinzierl, *The massless two-loop two-point function*, *Eur. Phys. J.* **C32** (2003) 67–78, [hep-ph/0308311].

[68] S. G. Gorishny and A. P. Isaev, *On an Approach to the Calculation of Multiloop Massless Feynman Integrals*, *Theor. Math. Phys.* **62** (1985) 232.

[69] J. S. Kang, *Uniqueness of Renormalized Quantities in Dimensional Regularization*, *Phys. Rev.* **D13** (1976) 851.

[70] D. I. Kazakov, O. V. Tarasov, and A. A. Vladimirov, *Calculation of critical exponents by quantum field theory methods*, *Sov. Phys. JETP* **50** (1979) 521.

[71] K. G. Chetyrkin and V. A. Smirnov, *Dimensional Regularization and Infrared Divergences*, *Theor. Math. Phys.* **56** (1984) 770–776.

[72] D. J. Broadhurst, *Dimensionally continued multi-loop gauge theory*, hep-th/9909185.

[73] D. I. Kazakov, *Multiloop Calculations: Method Of Uniqueness And Functional Equations, Theor. Math. Phys.* **62** (1985) 84–89.

[74] D. J. Broadhurst, *Exploiting the 1440 Fold Symmetry of The Master two Loop Diagram*, *Z. Phys.* **C32** (1986) 249–253.

[75] D. T. Barfoot and D. J. Broadhurst, $Z(2) \times S(6)$ *Symmetry of the Two Loop Diagram*, *Z. Phys.* **C41** (1988) 81.

[76] D. I. Kazakov, *The method of uniqueness, a new powerful technique for multiloop calculations*, *Phys. Lett.* **B133** (1983) 406–410.

[77] K. G. Chetyrkin, *Massless Vacuum Polarization: first complete* $\mathcal{O}(\alpha_s^4)$ *results* . 2004. Talk given at "Loops and Legs 2004", Zinnowitz, available at http://www-zeuthen.desy.de/theory/LL2004.

[78] S. Bekavac, *Calculation of massless Feynman integrals using harmonic sums*, *Comput. Phys. Commun.* **175** (2006) 180–195, [hep-ph/0505174].

[79] M. Czakon, *The four-loop QCD beta-function and anomalous dimensions*, *Nucl. Phys.* **B710** (2005) 485–498, [hep-ph/0411261].

[80] B. Eden, *Operator mixing in N = 4 SYM: The Konishi anomaly re-re- visited*, arXiv:0911.4884.

[81] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, *Higher Order Corrections to* $\sigma_{tot}(e^+e^- \to hadrons)$ *in*

Quantum Chromodynamics, *Phys. Lett.* **B85** (1979) 277.

[82] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren, *The α_s^3 correction to the Bjorken sum rule*, *Phys. Rev. Lett.* **66** (1991) 862–863.

[83] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, *The four-loop beta function in quantum chromodynamics*, *Phys. Lett.* **B400** (1997) 379–384, [hep-ph/9701390].

[84] S. G. Gorishny, A. L. Kataev, and S. A. Larin, *The three loop QED contributions to the photon vacuum polarization function in the MS scheme and the four loop corrections to the QED beta function in the on-shell scheme*, *Phys. Lett.* **B273** (1991) 141–144.

[85] K. Johnson and M. Baker, *Some Speculations on High-Energy Quantum Electrodynamics*, *Phys. Rev.* **D8** (1973) 1110–1122.

[86] F. Brown, *The massless higher-loop two-point function*, *Commun. Math. Phys.* **287** (2009) 925–958, [arXiv:0804.1660].

[87] F. C. S. Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114.

[88] A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158–164.

[89] A. V. Kotikov, *Differential equations method: The Calculation of vertex type Feynman diagrams*, *Phys. Lett.* **B259** (1991) 314–322.

[90] A. V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett.* **B267** (1991) 123–127.

[91] A. V. Kotikov, *New method of massive Feynman diagrams calculation*, *Mod. Phys. Lett.* **A6** (1991) 677–692.

[92] A. V. Kotikov, *New method of massive N point Feynman diagrams calculation*, *Mod. Phys. Lett.* **A6** (1991) 3133–3141.

[93] A. V. Kotikov, *New method of massive Feynman diagrams calculation. Vertex type functions*, *Int. J. Mod. Phys.* **A7** (1992) 1977–1992.

[94] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim.* **A110** (1997) 1435–1452, [hep-th/9711188].

[95] M. Caffo, H. Czyz, S. Laporta, and E. Remiddi, *The master differential equations for the 2-loop sunrise selfmass amplitudes*, *Nuovo Cim.* **A111** (1998) 365–389, [hep-th/9805118].

[96] M. Caffo, H. Czyz, S. Laporta, and E. Remiddi, *Master equations for master amplitudes*, *Acta Phys. Polon.* **B29** (1998) 2627–2635, [hep-th/9807119].

[97] M. Argeri and P. Mastrolia, *Feynman Diagrams and Differential Equations*, *Int. J. Mod. Phys.* **A22** (2007) 4375–4436, [arXiv:0707.4037].

[98] S. Laporta, *Calculation of master integrals by difference equations*, *Phys. Lett.* **B504** (2001) 188–194, [hep-ph/0102032].

[99] G. G. Kirilin and R. N. Lee, *$(Z\alpha)^4$ order of the polarization operator in Coulomb field at low energy*, *Nucl. Phys.* **B807** (2009) 73–88, [arXiv:0807.2335].

[100] R. N. Lee, *Space-time dimensionality D as complex variable: calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D*, *Nucl. Phys.* **B830** (2010) 474–492, [arXiv:0911.0252].

[101] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Analytic Results for Massless Three-Loop Form Factors*, arXiv:1001.2887.

[102] O. V. Tarasov, *Application and explicit solution of recurrence relations with respect to space-time dimension*, *Nucl. Phys. Proc. Suppl.* **89** (2000) 237–245, [hep-ph/0102271].

[103] N. I. Usyukina, *Algorithm for calculating massless Feynman diagrams*, *Teor. Mat. Fiz.* **79** (1989) 385–391.

[104] E. E. Boos and A. I. Davydychev, *A Method of evaluating massive Feynman integrals*, *Theor. Math. Phys.* **89** (1991) 1052–1063.

[105] K. Hepp, *Proof of the Bogolyubov-Parasiuk theorem on renormalization*, *Commun. Math. Phys.* **2** (1966) 301–326.

[106] E. R. Speer, *Ultraviolet and Infrared Singularity Structure of Generic Feynman Amplitudes*, *Annales Poincare Phys. Theor.* **23** (1975) 1–21.

[107] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multi-loop integrals*, *Nucl. Phys.* **B585** (2000) 741–759, [hep-ph/0004013].

[108] T. Binoth and G. Heinrich, *Numerical evaluation of multi-loop integrals by sector decomposition*, *Nucl. Phys.* **B680** (2004) 375–388, [hep-ph/0305234].

[109] T. Binoth and G. Heinrich, *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl. Phys.* **B693** (2004) 134–148, [hep-ph/0402265].

[110] G. Heinrich, *Sector Decomposition*, *Int. J. Mod. Phys.* **A23** (2008) 1457–1486, [arXiv:0803.4177].

[111] A. V. Smirnov and A. Tentyukov, *Four Loop Massless Propagators: a Numerical Evaluation of All Master Integrals*, [arXiv:1004.1149[hep-ph]].

[112] K. G. Chetyrkin, F. V. Tkachov, and S. G. Gorishny, *Operator Product Expansion in the Minimal Subtraction Scheme*, *Phys. Lett.* **B119** (1982) 407–411.