PROCEEDINGS OF SCIENCE

$R(s)$, Bjorken sum rule and the Crewther Relation in order α_s^4

K. G. Chetyrkin[∗†]
Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany
E-mail: konstantin.chetyrkin@kit.edu

We report on our calculation of the order α_s^4 contributions to the Bjorken sum rule for polarized electron-nucleon scattering and to the (non-singlet) part of the cross section for electron-positron annihilation into hadrons for the case of a generic colour gauge group. We confirm at the same order a (generalized) Crewther relation between the Bjorken sum rule and the corresponding Adler function which provides a strong test of the correctness of our previously obtained results: the ratio $R(s)$ in QCD and the five-loop β-function in quenched QED.

RADCOR 2009 - 9th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)
October 25-30 2009
Ascona, Switzerland

[∗]Speaker.
[†]The talk is based on recent results obtained in a collaboration to P.A. Baikov and J.H. Kühn [1].

1. Introduction

At the previous RADCOR Symposium we presented [2] two new results. First, corrections of order α_s^4 the (non-singlet) part of the cross section for electron-positron annihilation into hadrons in QCD with $n_f = 3$. Since then the first result has been extended to QCD with generic value of the quark flavours n_f [3].

Second, the five-loop contribution to the β function of quenched QED which has revealed an unexpected[1] This fact has raised some doubts on the correctness of the QCD result for the order α_s^4 and resulted in a call for an independent testing and/or reevaluation of our results [6, 7].

At the present Symposium we are happy to report about two new results [1] which are not only of interest by themselves but also provide us with a highly non-trivial confirmation of the validity of calculations of [2, 3]. Namely, we discuss calculations the order α_s^4 contributions to the Bjorken sum rule for polarized electron-nucleon scattering and to the (non-singlet) Adler function for the case of a generic colour gauge group. These two seemingly disconnected quantities are related in a non-obvious way by the (generalized) Crewther relation [8, 9]. The results are checked to be in full agreement with the constraints imposed by the Crewther relation.

2. Adler Function and R(s) to order α_s^4 in a general gauge theory

The Adler function is defined through the correlator of the vector current j_μ

$$3Q^2\Pi(Q^2) = i\int \mathrm{d}^4 x e^{iq\cdot x}\langle 0|\mathrm{T} j_\mu(x) j^\mu(0)|0\rangle, \tag{2.1}$$

as follows

$$D(Q^2) = -12\pi^2 Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) = d_R\left(1+\frac{3}{4}C_F\, a_s + \sum_{i=2}^{\infty} d_i\, a_s^i(Q^2)\right), \tag{2.2}$$

where $Q^2 = -q^2$, d_R is the dimension of the quark colour representation (for QCD $d_R = 3$), $a_s \equiv \alpha_s/\pi$ and the normalization scale μ^2 is set $\mu^2 = Q^2$. In fact, the Adler function is the main theoretical object required to study such important physical observables as the cross section for electron-positron annihilation into hadrons and the hadronic decay rates of both the Z-boson and the τ-lepton (see, e.g. [10]). The relation between the Adler function and the ratio $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is (in the next expression μ^2 is set to s)

$$R(s) = D(s) - \pi^2\beta_0^2\{\frac{d_1}{3}a_s^3 + (d_2 + \frac{5}{6\beta_0}d_1\,\beta_1)\,a_s^4\} + \cdots = d_R\left(1+\frac{3}{4}C_F\,a_s + \sum_{i=2}^{\infty} r_i\,a_s^i(s)\right). \tag{2.3}$$

Here $\beta(a_s) = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}a_s(\mu) = -\sum_{i\geq 0}\beta_i a_s^{i+2}$ is the QCD β-function with its first term $\beta_0 = \frac{11}{12}C_A - \frac{T}{3}n_f$ being responsible for asymptotic freedom of QCD. Note that we consider only the so-called "non-singlet" contribution to the Adler function and do not write explicitly a common factor $\sum_i Q_i^2$ (with Q_i being an electric charge of the i-th quark flavour) for $R(s)$.

[1] Unexpected, because there existed a wide-spread belief that the rationality property is not accidental but holds also in higher orders [4, 5] appearance of the irrational constant ζ_3 at five loops.

Our results for the coefficients d_4 and r_4 for a general gauge group read[2]

$$
\begin{aligned}
d_4 \;=\; & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16}-\frac{1}{4}\zeta_3-\frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048}+\frac{3}{8}\zeta_3\right] \\
&+ C_F^3T_f\left[\frac{1001}{384}+\frac{99}{32}\zeta_3-\frac{125}{4}\zeta_5+\frac{105}{4}\zeta_7\right] + C_F^2T_f^2\left[\frac{5713}{1728}-\frac{581}{24}\zeta_3+\frac{125}{6}\zeta_5+3\zeta_3^2\right] \\
&+ C_FT_f^3\left[-\frac{6131}{972}+\frac{203}{54}\zeta_3+\frac{5}{3}\zeta_5\right] + C_F^3C_A\left[-\frac{253}{32}-\frac{139}{128}\zeta_3+\frac{2255}{32}\zeta_5-\frac{1155}{16}\zeta_7\right] \\
&+ C_F^2T_fC_A\left[\frac{32357}{13824}+\frac{10661}{96}\zeta_3-\frac{5155}{48}\zeta_5-\frac{33}{4}\zeta_3^2-\frac{105}{8}\zeta_7\right] \\
&+ C_FT_f^2C_A\left[\frac{340843}{5184}-\frac{10453}{288}\zeta_3-\frac{170}{9}\zeta_5-\frac{1}{2}\zeta_3^2\right] \\
&+ C_F^2C_A^2\left[-\frac{592141}{18432}-\frac{43925}{384}\zeta_3+\frac{6505}{48}\zeta_5+\frac{1155}{32}\zeta_7\right] \\
&+ C_FT_fC_A^2\left[-\frac{4379861}{20736}+\frac{8609}{72}\zeta_3+\frac{18805}{288}\zeta_5-\frac{11}{2}\zeta_3^2+\frac{35}{16}\zeta_7\right] \\
&+ C_FC_A^3\left[\frac{52207039}{248832}-\frac{456223}{3456}\zeta_3-\frac{77995}{1152}\zeta_5+\frac{605}{32}\zeta_3^2-\frac{385}{64}\zeta_7\right],
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
r_4 = & \; C_F^4\left[\frac{4157}{2048}+\frac{3}{8}\zeta_3\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right] + \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16}-\frac{1}{4}\zeta_3-\frac{5}{4}\zeta_5\right] \\
&+ C_FT_f^3\left[-\frac{6131}{972}+\frac{11}{72}\pi^2+\frac{203}{54}\zeta_3-\frac{1}{9}\pi^2\zeta_3+\frac{5}{3}\zeta_5\right] + C_F^2T_f^2\left[\frac{5713}{1728}-\frac{1}{24}\pi^2-\frac{581}{24}\zeta_3+3\,\zeta_3^2+\frac{125}{6}\zeta_5\right] \\
&+ C_FT_f^2C_A\left[\frac{340843}{5184}-\frac{65}{48}\pi^2-\frac{10453}{288}\zeta_3+\frac{11}{12}\pi^2\zeta_3-\frac{1}{2}\,\zeta_3^2-\frac{170}{9}\zeta_5\right] \\
&+ C_F^3T_f\left[\frac{1001}{384}+\frac{99}{32}\zeta_3-\frac{125}{4}\zeta_5+\frac{105}{4}\zeta_7\right] \\
&+ C_F^2T_fC_A\left[\frac{32357}{13824}+\frac{11}{128}\pi^2+\frac{10661}{96}\zeta_3-\frac{33}{4}\,\zeta_3^2-\frac{5155}{48}\zeta_5-\frac{105}{8}\zeta_7\right] \\
&+ C_FT_fC_A^2\left[-\frac{4379861}{20736}+\frac{249}{64}\pi^2+\frac{8609}{72}\zeta_3-\frac{121}{48}\pi^2\zeta_3-\frac{11}{2}\,\zeta_3^2+\frac{18805}{288}\zeta_5+\frac{35}{16}\zeta_7\right] \\
&+ C_F^3C_A\left[-\frac{253}{32}-\frac{139}{128}\zeta_3+\frac{2255}{32}\zeta_5-\frac{1155}{16}\zeta_7\right] \\
&+ C_F^2C_A^2\left[-\frac{592141}{18432}+\frac{121}{1536}\pi^2-\frac{43925}{384}\zeta_3+\frac{6505}{48}\zeta_5+\frac{1155}{32}\zeta_7\right] \\
&+ C_FC_A^3\left[\frac{52207039}{248832}-\frac{16753}{4608}\pi^2-\frac{456223}{3456}\zeta_3+\frac{1331}{576}\pi^2\zeta_3+\frac{605}{32}\,\zeta_3^2-\frac{77995}{1152}\zeta_5-\frac{385}{64}\zeta_7\right].
\end{aligned}
\tag{2.5}
$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, T is the trace normalization of the fundamental representation, $T_f \equiv T\,n_f$,

[2] The full expressions for lower order terms can be found in the original publications [11, 12].

with n_f being the number of quark flavors. The exact definitions of $d_F^{abcd}d_A^{abcd}$ and $d_F^{abcd}d_F^{abcd}$ are given in [13]. For QCD (colour gauge group SU(3)):

$$C_F = 4/3\,, C_A = 3\,, T = 1/2\,, d_R = 3\,, d_F^{abcd}d_A^{abcd} = \frac{15}{2}\,, d_F^{abcd}d_F^{abcd} = \frac{5}{12}. \tag{2.6}$$

It is of interest to note that the coefficient d_4 is free from ζ_4, ζ_3^2 and ζ_6. Separate diagrams do contain such contributions, however, they *all cancel each other out* in a non-trivial way in the final result. The reasons for such mysterious cancellations are unclear (see for some more details [14]).

3. Bjorken sum rule to order α_s^4 in a general gauge theory

The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function C^{Bjp}:

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{g_A}{6}C^{Bjp}(a_s) + \sum_{i=2}^{\infty}\frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, \tag{3.1}$$

where g_1^{ep} and g_1^{en} are the spin-dependent proton and neutron structure functions, g_A is the nucleon axial charge as measured in neutron β-decay. The coefficient function $C^{Bjp}(a_s) = 1 + \mathscr{O}(a_s)$ is proportional to the flavour-nonsinglet axial vector current $\bar{\psi}\gamma^\mu\gamma_5\psi$ in the corresponding short distance Wilson expansion. The sum in the second line of (3.1) describes for the nonperturbative power corrections (higher twist) which are inaccessible for pQCD. The function $C^{Bjp}(a_s)$ is known to order α_s^3 from [15]. It can be conveniently represented as

$$C^{Bjp}(Q^2) = 1 - \frac{3}{4}C_F\, a_s + \sum_{i=2}^{\infty} c_i\, a_s^i(Q^2), \;\; 1/C^{Bjp}(Q^2) = 1 + \frac{3}{4}C_F\, a_s + \sum_{i=2}^{\infty} b_i\, a_s^i(Q^2). \tag{3.2}$$

We have computed the next α_s^4 term in $C^{Bjp}(a_s)$ using the same methods as in [3] as well as the "method of projectors" [16]. The result for b_4 reads:

$$\begin{aligned}
b_4 \;=\; & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048} + \frac{3}{8}\zeta_3\right] \\
+\; & C_F^3T_f\left[-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5\right] + C_F^2T_f^2\left[\frac{869}{576} - \frac{29}{24}\zeta_3\right] + C_FT_f^3\left[-\frac{605}{972}\right] \\
+\; & C_F^3C_A\left[-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5\right] + C_F^2T_fC_A\left[-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7\right] \\
+\; & C_FT_f^2C_A\left[\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2\right] + C_F^2C_A^2\left[-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7\right] \\
+\; & C_FT_fC_A^2\left[-\frac{1238827}{41472} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}\zeta_7\right] \\
+\; & +C_FC_A^3\left[\frac{8004277}{248832} - \frac{1069}{576}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 - \frac{385}{64}\zeta_7\right].
\end{aligned} \tag{3.3}$$

4. Crewther relation to order α_s^4

The Crewther relation has the form[3]

$$D(a_s)C^{Bjp}(a_s) = d_R\left[1+\frac{\beta(a_s)}{a_s}K(a_s)\right], \quad K(a_s) = K_0 + a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \dots \tag{4.1}$$

The term proportional the β-function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2 (hence, $K_0 \equiv 0$), and was suggested [9] on the basis of $\mathscr{O}(\alpha_s^3)$ calculations of $D(a_s)$ [11, 12] and $C^{Bjp}(a_s)$ [15]. A formal proof was carried out in [18, 19]. The original relation without this term was first proposed in [8] (see, also, [20]).

The Crewther relation is trivially met at order α_s. It was demonstrated in [9] that fulfillment of (4.1) at orders α_s^2 and α_s^3 puts as many as 2 and 3 constraints on the differences $d_2 - b_2$ and $d_3 - b_3$ respectively. At order α_s^4 the number of constraints on the difference $d_4 - b_4$ is increased to 6 and our results do meet all of them (see [1] for details).

Here we mention only three most simple relations, namely those between coefficients in front of colour structures ($C_F^4, n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}$ and $\frac{d_F^{abcd}d_A^{abcd}}{d_R}$) appearing in d_4 and b_4. As it follows directly from eq. (4.1) the coefficients must be (pairwise) identical. The prediction is indeed in agreement with our results as expressed by eqs. (2.4) and (3.3)!

In particular the equality of the coefficients of C_F^4 in eqs. (2.4) and (3.3) provides us with a strong check of the correctness of the result for the five-loop β function of quenched QED published in [2]. Note that the test was first suggested in [6, 7].

5. Conclusion

The calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel MPI-based [21, 22] as well as thread-based [23, 24] versions of FORM [25]. For evaluation of color factors the FORM program *COLOR* [26] has been used. The diagrams have been generated with QGRAF [27].

This work was supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungs-bereich/Transregio SFB/TR-9 "Computational Particle Physics" and by RFBR grant 08-02-01451. We thank V.M. Braun for useful discussions.

References

[1] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, (2010), 1001.3606v1.

[2] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, PoS RADCOR2007 (2007) 023, 0810.4048.

[3] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002, 0801.1821.

[4] C.M. Bender, R.W. Keener and R.E. Zippel, Phys. Rev. D15 (1977) 1572.

[5] D.J. Broadhurst, (1999), hep-th/9909185.

[6] A.L. Kataev, Phys. Lett. B668 (2008) 350, 0808.3121.

[3] Very recently an extension of the (generalized) Crewther relation (4.1) has been suggested in [17].

[7] A.L. Kataev, (2008), 0805.0621.

[8] R.J. Crewther, Phys. Rev. Lett. 28 (1972) 1421.

[9] D.J. Broadhurst and A.L. Kataev, Phys. Lett. B315 (1993) 179, hep-ph/9308274.

[10] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Rept. 277 (1996) 189.

[11] S.G. Gorishnii, A.L. Kataev and S.A. Larin, Phys. Lett. B259 (1991) 144.

[12] L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. 66 (1991) 560.

[13] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, Phys. Lett. B405 (1997) 327, hep-ph/9703284.

[14] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Nucl. Phys. Proc. Suppl. 189 (2009) 49.

[15] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345.

[16] S.G. Gorishnii and S.A. Larin, Nucl. Phys. B283 (1987) 452.

[17] A.L. Kataev and S.V. Mikhailov, (2010), 1001.0728.

[18] R.J. Crewther, Phys. Lett. B397 (1997) 137, hep-ph/9701321.

[19] V.M. Braun, G.P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. 51 (2003) 311, hep-ph/0306057.

[20] S.L. Adler et al., Phys. Rev. D6 (1972) 2982.

[21] M. Tentyukov et al., (2004), cs/0407066.

[22] M. Tentyukov, H.M. Staudenmaier and J.A.M. Vermaseren, Nucl. Instrum. Meth. A559 (2006) 224.

[23] J.A.M. Vermaseren and M. Tentyukov, Nucl. Phys. Proc. Suppl. 160 (2006) 38.

[24] M. Tentyukov and J.A.M. Vermaseren, (2007), hep-ph/0702279.

[25] J.A.M. Vermaseren, (2000), math-ph/0010025.

[26] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 41, hep-ph/9802376.

[27] P. Nogueira, J. Comput. Phys. 105 (1993) 279.