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From low-energy moments of $\Pi(q^2)$ to $R(s)$

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We construct approximate formulas for the $\mathcal{O}(\alpha_s^3)$ QCD contributions to the vector current correlator, which is valid for arbitrary values of momenta and masses. The derivation is based on conformal mapping and the Padé approximation procedure and incorporates known expansions in the low energy, threshold and high energy regions. We use our results to estimate additional terms in these expansions.

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1. Introduction

Correlators of heavy quark currents in different kinematical regions are of interest for a number of phenomenological applications. These two-point functions only depend on two scales, namely the square of the external four-momentum q^2 and the heavy quark mass m. Many of the applications focus on one of three distinct kinematical regions: The low energy region with $q^2 \approx 0$, the quark pair production threshold at $q^2 = 4m^2$ and the (euclidean) high energy region $-q^2 \to \infty$.

Moments in the low energy expansion can be used for precise extractions of charm and bottom quark masses via sum rules [1] (for reviews see [2]), whereas threshold and high energy expansions are directly related to production cross sections for $t\bar{t}$, charmed hadrons or bottom hadrons in the respective energy regions.

The aforementioned expansions for the correlators in the three regions are known relatively well: In the low energy region at $\mathcal{O}(\alpha_s^2)$ the leading eight coefficients were computed more than ten years ago [3], and as of today as many as 30 moments are known [4]. At $\mathcal{O}(\alpha_s^3)$, however, only the first three physical moments of the vector correlator are available [5].

Threshold expansions for correlators are expansions in the small heavy quark velocity $v = \sqrt{1-4m^2/q^2} \ll 1$. Currently all of the necessary machinery for NNLO threshold expansion is known (see for instance Ref. [6] and references therein). This means, that all terms of order $(\alpha_s^n/v^{n-1}) \cdot \{1, v, v^2\}$ are in principle known. Explicit expansions for vector correlators can be derived from Refs. [7].

For high energies the leading seven coefficients in the expansion are known at $\mathcal{O}(\alpha_s^2)$ for the vector current [8]. At order α_s^3 the first two terms in the high energy expansion of the vector correlator have been published in Refs. [9]. More information is available for the absorptive parts of the correlators, which correspond to the logarithmic terms in the high energy expansions. Here the first three coefficients are known for the vector correlator [10].

Still, it would be desirable to have results for the correlators which are valid for arbitrary energies in addition to these expansions. One obvious benefit would be the possibility to expand such a result in a kinematical region of interest in order to obtain even more coefficients in the expansion. It would also be possible to predict values of cross sections for intermediate regions between threshold and high energies, where the mere expansions may not be very accurate anymore. Last but not least, the full energy dependence is essential for those QCD sum rule approaches to quark mass determination which use either the Borel transformation of the correlator [1] or so-called contour-improved perturbation theory [11].

Unfortunately, analytical results which are valid for arbitrary energies only exist up to $\mathcal{O}(\alpha_s)$ [12]. Still, it is possible to reconstruct the full energy dependence approximately from the known expansions at higher orders. Using a seminumerical method based on Padé approximations [13], the correlator of the vector current was reconstructed at $\mathcal{O}(\alpha_s^2)$ [3]. Moreover, it was demonstrated in [14] that in spite of the rather low amount of information available at $\mathcal{O}(\alpha_s^3)$ it is still viable to reconstruct the vector correlator and predict expansion coefficients with decent accuracy.

In this work we use the Padé approximation method to reconstruct the vector heavy quark current correlators at $\mathcal{O}(\alpha_s^3)$ and derive approximations to previously unknown expansion coefficients in the low energy, threshold and high energy regions.

2. Calculation

2.1 Polarisation functions

It is convenient to explicitly extract the Lorentz structure of the heavy quark vector current correlator and define the polarisation function $\Pi^v(q^2)$:

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi^v(q^2) = i\int dx e^{iqx}\langle 0|T\, j^v_\mu(x) j^v_\nu(0)|0\rangle, \quad j^v_\mu = \bar{\psi}\gamma_\mu\psi\,. \tag{2.1}$$

We do not take into account singlet contributions originating from diagrams with massless cuts and choose the normalisation $\Pi^v(0) = 0$. The perturbative expansions of the polarisation function reads

$$\Pi^v = \Pi^{(0),v} + C_F\Pi^{(1),v}\frac{\alpha_s}{\pi} + \Pi^{(2),v}\left(\frac{\alpha_s}{\pi}\right)^2 + \Pi^{(3),v}\left(\frac{\alpha_s}{\pi}\right)^3 + \ldots\,, \tag{2.2}$$

where $C_F = \frac{4}{3}$ is the quadratic Casimir operator for the adjoint representation. A natural variable to describe the behaviour of Π^v is given by $z = \frac{q^2}{4m^2}$ where m denotes the heavy quark mass defined in the on-shell scheme.

2.2 Padé approximation

The Padé approximant $p_{n,m}(x)$ to a function f is defined as

$$p_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{i=1}^{m} b_i x^i}\,. \tag{2.3}$$

A naïve application of the Padé approximation method will, however, fail for the functions $\Pi^{(i),v}(z)$ because contrary to the Padé approximants (Eq. (2.3)) they are not meromorphic everywhere in the complex plane. There are two major aspects of this problem, which have to be considered: The functions $\Pi^{(i),v}(z)$ diverge logarithmically for $z \to -\infty$. There are also logarithmic contributions at threshold. There is a branch cut along the real axis starting from $z = 1$. This behaviour can obviously not be reproduced accurately by a Padé approximation.

The first problem related to the appearance of logarithms can be cured by splitting $\Pi^{(i),v}(z)$ into two parts,

$$\Pi^{(i),v}(z) = \Pi^{(i),v}_{reg}(z) + \Pi^{(i),v}_{log}(z)\,, \tag{2.4}$$

so that $\Pi^{(i),v}_{log}(z)$ is a suitable function containing all known logarithmic contributions to $\Pi^{(i),v}(z)$. In this way the problem reduces to finding an approximation to $\Pi^{(i),v}_{reg}(z)$.

The second problem related to the branch cut is treated in a different way: We map the complex plane (including its cut) onto the unit circle in such a way that the branch cut is mapped onto the perimeter. This can be achieved via the conformal transformation $z \to \frac{4\omega}{(1+\omega)^2}$. The functions $\Pi^{(i),v}_{reg}(\omega)$ are now suitable for the Padé approximation procedure.

2.3 Subtractions

As explained in Section 2.2, the first step consists of absorbing the logarithmic contributions in the high energy and threshold expansions into a function $\Pi^{(3),v}_{log}(z)$. This function must be chosen

carefully in order not to introduce undesired additional singularities in $\Pi^{(3),v}_{reg}(z)$. It is very convenient to use lower order analytical results $\Pi^{(0),v}$ and $\Pi^{(1),v}$ as auxiliary functions [14]. $\Pi^{(0),v}$ is given by

$$\Pi^{(0),v}(z) = \frac{3}{16\pi^2}\left(\frac{20}{9}+\frac{4}{3z}-\frac{4(1-z)(1+2z)}{3z}G(z)\right), \tag{2.5}$$

with

$$G(z) = \frac{1}{2z}\log(u)/\sqrt{1-\frac{1}{z}}, \qquad u = (\sqrt{1-\frac{1}{z}}-1)/(\sqrt{1-\frac{1}{z}}+1). \tag{2.6}$$

The expansions of the vector correlator (with $n_l = 3$) around threshold and for $q^2 \to \infty$ are given by

$$\begin{aligned}\Pi^{(3),v}(z) =& 2.63641/(1-z) + (-25.2331 - 7.75157\log(1-z))/\sqrt{1-z} - 11.0654\log(1-z)\\ &+ 1.42833\log^2(1-z) - 0.421875\log^3(1-z) + K_0 + \mathcal{O}(\sqrt{1-z})\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Pi^{(3),v}(z) =& -6.172 - 0.070\log(-4z) + 0.121\log^2(-4z) - 0.037\log^3(-4z) + 1/z(-4.333\\ &- 3.756\log(-4z) + 2.118\log^2(-4z) - 0.319\log^3(-4z)) + 1/z^2(D_2 - 5.130\log(-4z) +\\ &0.318\log^2(-4z) + 0.401\log^3(-4z) - 0.079\log^4(-4z)) + \mathcal{O}(1/z^3),\end{aligned} \tag{2.8}$$

respectively. The construction of $\Pi^{(3),v}_{log}(z)$ is now based on the expansions of $\Pi^{(1),v}$ and $G(z)$ in the threshold and high-energy region:

$$G(z) = \frac{\pi}{2}\frac{1}{\sqrt{1-z}} + \mathcal{O}((1-z)^0), \quad \Pi^{(1),v}(z) = -\frac{3}{16}\log(1-z) + \text{const} + \mathcal{O}(\sqrt{1-z}), \tag{2.9}$$

$$G(z) = \frac{-\log(-4z)}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{2.10}$$

Since these expansions have the desired behavior in the interesting regions we make the ansatz

$$\Pi^{(3),v}_{log}(z) = \sum_{i>0,j} k_{ij}\Pi^{(1),v}(z)^i G(z)^j + \sum_{m,n} d_{mn}\big(zG(z)\big)^m\left(1-\frac{1}{z}\right)^{\lceil \frac{m}{2}\rceil}\frac{1}{z^n}, \tag{2.11}$$

where $\lceil \ldots \rceil$ means rounding up to the next integer number. The coefficients k_{ij} and d_{mn} are chosen for $\Pi^{(3),v}_{log}(z)$ to reproduce the known behavior in the threshold and high-energy region given by Eqs. (2.7) and (2.8).

The choice of $\Pi^{(3),v}_{log}(z)$ is of course not unique. A perfect reconstruction of the polarisation function would clearly not depend on the specific choice, so variations of $\Pi^{(3),v}_{log}(z)$ can be used to estimate the quality of the approximation procedure. Following Ref. [14], we introduce additional parameters a_i and b_i for this purpose. We modify the ansatz Eq. (2.11) for $\Pi^{(3),v}_{log}(z)$ in the following way: In the first sum, we multiply the summands with $i = 3$, $j = 0$ and $i = j = 1$ (which roughly correspond to terms proportional to $\log^3(1-z)$ and $\log(1-z)/\sqrt{1-z}$) by factors $a_1 + 1/z$ and $a_2 + 1/z$, respectively. In the second sum all summands corresponding to the two highest powers of logarithms are multiplied by factors $1 + 1/(b_1 z)$ for $m = 3$ and $1 + 1/(b_2 z)$ for $m = 4$.

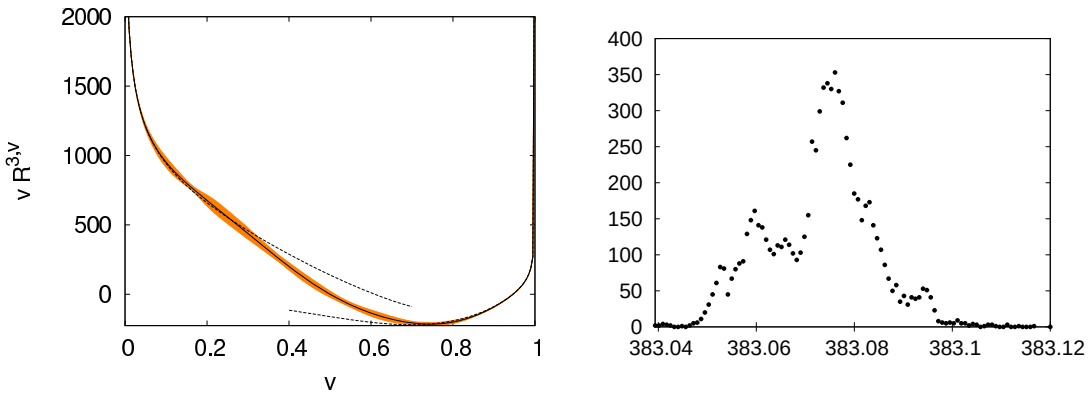

Figure 1: Left: Imaginary part of the four loop contributions to the polarisation function above the charm threshold. The plots show $vR^{(3),v} = v12\pi\mathrm{Im}(\Pi^{(3),v})$ as functions of $v = \sqrt{1-1/z}$. The solid black line is the mean from all approximants, the area covered by three standard deviations is shown by bands. The dashed lines show the expansions in the threshold and high energy regions. Right: Distribution of the values of $C_4^{(3),v}$ in on-shell scheme from different Padé approximants to the charm vector correlator.

Except for the conditions $a_i \neq -1$ and $b_i \neq 0$ the values of the parameters are in principle arbitrary. We vary them independently with $a_i \in \{-1 \pm 1, -1 \pm 4, -1 \pm 16, -1 \pm 64\}$, $b_i \in \{\pm 1, \pm 4, \pm 16, \pm 64\}$.

In the next step we determine the coefficients of the Padé approximants from the expansions in the low energy, threshold, and euclidean high energy region. The low energy expansion for the vector current and $n_l = 3$ reads

$$\Pi^{(3),v}(z) = 6.95649z + 7.2478z^2 + 7.31855z^3 + \mathcal{O}(z^4)\,. \tag{2.12}$$

The expansions around threshold and for high energies are listed in Eqs. (2.7) and (2.8), respectively.

Following Ref. [14], we additionally require that terms proportional to $z^{-\frac{3}{2}}$ and $z^{-\frac{5}{2}}$ are absent in the high energy expansion.

2.4 Results

From the Padé approximants we reconstruct the polarisation function. Its imaginary part corresponding to hadron production cross section is plotted above the charm threshold (i.e. with $n_l = 3$) in Fig. 1. The reconstructed functions can be expanded again to obtain additional low energy, threshold and high energy coefficients. We find that the values of the coefficients are strongly peaked around the mean value (for an example, see Fig. 1). As a consequence we give our errors in terms of standard deviations. As expected the error is a lot smaller for the low energy coefficients in comparison to the coefficients in the threshold and high energy regions.

3. Conclusion

We have used the Padé approximation method to reconstruct the full energy dependence of the heavy quark correlator of the vector current at order α_s^3. As input we have used information from

	$C_4^{(3),v}$	$C_5^{(3),v}$	$C_6^{(3),v}$	$C_7^{(3),v}$	$C_8^{(3),v}$	$K_0^{(3),v}$	$D_2^{(3),v}$
$n_l = 3$	383.073(11)	378.688(32)	373.536(61)	368.23(9)	363.03(13)	17(11)	2.0(42)
$n_l = 4$	339.913(10)	338.233(32)	335.320(63)	331.90(10)	328.33(14)	17(29)	1.2(83)
$n_l = 5$	298.576(9)	299.433(27)	298.622(54)	296.99(9)	294.94(12)	16(10)	1.4(21)

Table 1: Expansion coefficients from the reconstructed polarisation functions for different numbers of light quarks in the on shell scheme. $C_{1-3}^{(3),v}$ are known exactly. The errors always apply to the last digits, i.e. $2.0(42)$ means an error of 4.2.

expansions in the low energy, threshold and high energy regions. Expanding the reconstructed correlators, we have obtained predictions for additional coefficients in these expansions. We find that these predictions are fairly accurate for low energy coefficients but less precise for the coefficients in the threshold and high energy expansions.

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