Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory

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We compute, for the first time, the order α_s^4 contributions to the Bjorken sum rule for polarized electron-nucleon scattering and to the (non-singlet) Adler function for the case of a generic colour gauge group. We confirm at the same order a (generalized) Crewther relation which provides a strong test of the correctness of our previously obtained results: the QCD Adler function and the five-loop β-function in quenched QED. In particular, the appearance of an irrational contribution proportional to ζ_3 in the latter quantity is confirmed. We obtain the *commensurate* scale equation relating the effective strong coupling constants as inferred from the Bjorken sum rule and from the Adler function at order α_s^4.

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INTRODUCTION

The Crewther relation [1, 2] relates in a non-trivial way two seemingly disconnected quantities, namely, the (non-singlet) Adler function [3] D and the coefficient function C^{Bjp}, describing the deviation of the Bjorken sum rule [4, 5] for polarized DIS from its naive-parton model value. The Adler function is defined through the correlator of the vector current j_μ

$$3Q^2\Pi(Q^2) = i\int \mathrm{d}^4x e^{iq\cdot x}\langle 0|\mathrm{T} j_\mu(x) j^\mu(0)|0\rangle, \qquad (1)$$

as follows

$$D(Q^2) = -12\,\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2), \qquad (2)$$

with $Q^2 = -q^2$. In fact, the Adler function is the main theoretical object required to study such important physical observables as the cross section for electron-positron annihilation into hadrons and the hadronic decay rates of both the Z-boson and the τ-lepton (see, e.g. [6]). The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function C^{Bjp}:

$$\begin{aligned}\Gamma_1^{p-n}(Q^2) &= \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx \\ &= \frac{g_A}{6}C^{Bjp}(a_s) + \sum_{i=2}^{\infty}\frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, \end{aligned} \qquad (3)$$

where g_1^{ep} and g_1^{en} are the spin-dependent proton and neutron structure functions, g_A is the nucleon axial charge as measured in neutron β-decay. The coefficient function $C^{Bjp}(a_s) = 1 + \mathcal{O}(a_s)$ is proportional to the flavour-nonsinglet axial vector current $\bar\psi\gamma^\mu\gamma_5\psi$ in the corresponding short distance Wilson expansion. The sum in the second line of (3) describes for the nonperturbative power corrections (higher twist) which are inaccessible for pQCD. Within perturbative QCD we define

$$\begin{aligned} D(Q^2) &= d_R\left(1 + \tfrac{3}{4}C_F\,a_s + \textstyle\sum_{i=2}^{\infty}\ d_i\,a_s^i(Q^2)\right), \\ C^{Bjp}(Q^2) &= 1 - \tfrac{3}{4}C_F\,a_s + \textstyle\sum_{i=2}^{\infty}\ c_i\,a_s^i(Q^2), \\ 1/C^{Bjp}(Q^2) &= 1 + \tfrac{3}{4}C_F\,a_s + \textstyle\sum_{i=2}^{\infty}\ b_i\,a_s^i(Q^2), \end{aligned}$$

where d_R is the dimension of the quark colour representation (for QCD $d_R = 3$), $a_s \equiv \alpha_s/\pi$ and the normalization scale μ is set $\mu^2 = Q^2$. Note that we consider only the so-called "non-singlet" contribition to the Adler function and do not write explictly a common factor $\sum_i Q_i^2$ (with Q_i being the electric charge of the i-th quark flavour) for $R(s)$.

The Crewther relation states that

$$\begin{aligned} D(a_s)\,C^{Bjp}(a_s) &= d_R\left[1 + \tfrac{\beta(a_s)}{a_s}\,K(a_s)\right], \\ K(a_s) &= K_0 + a_s\,K_1 + a_s^2\,K_2 + a_s^3\,K_3 + \ldots \end{aligned} \qquad (4)$$

Here $\beta(a_s) = \mu^2\,\frac{\mathrm{d}}{\mathrm{d}\mu^2}a_s(\mu) = -\sum_{i\geq 0}\beta_i a_s^{i+2}$ is the QCD β-function describing the *running* of the coupling constant a_s with respect to a change of the normalization scale μ and with its first term $\beta_0 = \frac{11}{12}C_A - \frac{T}{3}n_f$ being responsible for asymptotic freedom of QCD. The term proportional the β-function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2, and was suggested [2] on the basis of $\mathcal{O}(\alpha_s^3)$ calculations of $D(a_s)$ [7, 8] and $C^{Bjp}(a_s)$ [9]. A formal proof was carried out in [10, 11]. The original relation without this term was first proposed in [1] (see, also, [12]).

At order α_s the Crewther relation is evidently fulfilled. The colour structures which appear in d_n and c_n (hence also in b_n) for n=1,2,3, and 4 are:

$$
\begin{aligned}
& a_s^1: \; C_F, \quad a_s^2: \; C_F^2, C_F T_f, C_F C_A, \\
& a_s^3: \; C_F^3, C_F^2 T_f, C_F T_f^2, C_F^2 C_A, C_F T_f C_A, C_F C_A^2, \\
& a_s^4: \; \frac{d_F^{abcd} d_A^{abcd}}{d_R}, \frac{n_f d_F^{abcd} d_F^{abcd}}{d_R}, C_F^4, \\
& \quad C_F^3 T_f, C_F^2 T_f^2, C_F T_f^3, C_F^3 C_A, C_F^2 T_f C_A, \\
& \quad C_F T_f^2 C_A, C_F^2 C_A^2, C_F T_f C_A^2, C_F C_A^3. \qquad (5)
\end{aligned}
$$

Here C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, T is the trace normalization of the fundamental representation, $T_f \equiv T\, n_f$, with n_f being the number of quark flavors. The exact definitions of $d_F^{abcd} d_A^{abcd}$ and $d_F^{abcd} d_F^{abcd}$ are given in [13]. For QCD (colour gauge group SU(3)):

$$
\begin{aligned}
& C_F = 4/3, C_A = 3, T = 1/2, d_R = 3, \\
& d_F^{abcd} d_A^{abcd} = \tfrac{15}{2}, d_F^{abcd} d_F^{abcd} = \tfrac{5}{12}.
\end{aligned} \qquad (6)
$$

Note, that all colour structures, apart of the d^2-terms which appear first at order α_s^4, involve at least one factor C_F. As a consequence, K_0 must be set to zero. An inspection of eqs. (4) and (5) clearly shows that the colour structures which may appear in a coefficient K_i are identical to those appearing in the coefficient b_{i-1} and c_{i-1}, listed in eq. (5). Thus, at orders α_s^2, α_s^3 and α_s^4 the Crewther relation puts as many as 2, 3 and, finally, 6 constraints on the differences $d_2 - b_2, d_3 - b_3$ and $d_4 - b_4$ respectively. The fulfillment of these constraints constitutes a powerful check of the correctness of the calculations of $D^{NS}(a_s)$ and $C^{Bjp}(a_s)$.

Indeed, at orders $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ the results for $D^{NS}(a_s)$ and $1/C^{Bjp}(a_s)$

$$
\begin{aligned}
d_2 &= -\frac{3}{32} C_F^2 + C_F T_f \left[\zeta_3 - \frac{11}{8}\right] + C_F C_A \left[\frac{123}{32} - \frac{11\zeta_3}{4}\right], \\
b_2 &= -\frac{3}{32} C_F^2 + C_F T_f \left[-\frac{1}{2}\right] + C_F C_A \left[\frac{23}{16}\right], \\
d_3 &= -\frac{69}{128} C_F^3 + C_F^2 T_f \left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] \\
&+ C_F T_f^2 \left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A \left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right] \\
&\quad + C_F T_f C_A \left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] \\
&\quad + C_F C_A^2 \left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],
\end{aligned}
$$

$$
\begin{aligned}
b_3 &= -\frac{69}{128} C_F^3 + C_F^2 T_f \left[-\frac{299}{576} + \frac{5}{12}\zeta_3\right] \\
&+ C_F T_f^2 \left[\frac{115}{216}\right] + C_F^2 C_A \left[\frac{1}{576} + \frac{11}{12}\zeta_3\right] \\
&+ C_F T_f C_A \left[-\frac{3535}{864} - \frac{3}{4}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2 \left[\frac{5437}{864} - \frac{55}{24}\zeta_5\right]
\end{aligned}
$$

are well consistent [2] with all 5 constraints on the coefficients d_2, d_3, b_2 and b_3 and imply

$$
\begin{aligned}
& K_1 = C_F \left(-\frac{21}{8} + 3\zeta_3\right), \quad K_2 = C_F T_f \left(\frac{163}{24} - \frac{19}{3}\zeta_3\right) \\
& + C_F C_A \left(-\frac{629}{32} + \frac{221}{12}\zeta_3\right) + C_F^2 \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right).
\end{aligned}
$$

The next, $\mathcal{O}(\alpha_s^4)$, contribution to $D(a_s)$ has been recently computed [14] for QCD, i.e. setting the colour structures to their $SU(3)$ numerical values (eq. (6)). The function $C^{Bjp}(a_s)$ is known to order α_s^3 *only.*

The importance of computation of the $\mathcal{O}(\alpha_s^4)$ contribution to the both coefficients d_4 and b_4 for a generic colour gauge group comes from a few reasons.

First, the knowledge of c_4 in the Bjorken sum rule is vital for proper extraction of higher twist contributions. Indeed, in [15] the recent Jefferson Lab data on the spin-dependent proton and neutron structure functions [16–20] were used to extract the leading and subleadinrg higher twist parameters μ_4 and μ_6. It has been demonstrated that, say, the twist four term μ_4 approximately halves its value in transition from LO to NLO, and from NLO to NNLO. This *duality* between perturbative and non-perturbative contributions has been observed before for the structure function F_3 [21] (for a related recent discussion see also [22]).

Second, the Bjorken sum rule provides us with a very convenient definition of the *effective strong coupling constant* (ECC) [20, 23], namely,

$$
6\,\Gamma_1^{p-n}(Q^2) = g_A \left(1 - a_{g_1}(Q^2)\right). \qquad (7)
$$

This quantity is directly measurable down to vanishing values of Q^2 and, due to eq. (3), approaches to the standard $\alpha_s(Q)$ at large Q^2. It is by definition gauge and scheme invariant. Another convenient ECC, a_D, comes from the Adler function [24]:

$$
D(Q^2) = 1 + a_D(Q^2). \qquad (8)
$$

As its perturbative expansion is available to $\mathcal{O}(\alpha_s^4)$ [14] the knowledge of c_4 will allow for the first time to compare two ECC's with the help of a commensurate scale relation [25] at an order unprecedented to date.

Third, the six constraints imposed by eq. (4) provide a highly nontrivial and welcome check of the calculation of d_4 in QCD [14]. In particular, in [26] we computed a part of the full result for d_4, namely, the term proportional to the colour structure C_F^4. As is well-known, an interesting object – the β-function of quenched QED — can be inferred from the part of the Adler function which depends on C_F only by setting $C_F = 1$ and adjusting a global normalization factor. The result ($A \equiv \frac{\alpha}{4\pi}$)

$$\beta^{\rm qQED} = \frac{4}{3}A + 4A^2 - 2A^3 - 46A^4 + \left(\frac{4157}{6} + 128\zeta_3\right)A^5 \tag{9}$$

revealed an unexpected [37] appearance of the irrational constant ζ_3 at five loops and had cast doubt on the correctness of the full QCD result for d_4 [27].

Using the same techniques as in calculations of [14] and [9] we have computed the the Adler function and the function C^{Bjp} for a general gauge group to order α_s^4. Our results read

$$\begin{aligned}
d_4 = {} & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048} + \frac{3}{8}\zeta_3\right] \\
+ {} & C_F^3T_f\left[\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7\right] + C_F^2T_f^2\left[\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2\right] + C_FT_f^3\left[-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5\right] \\
+ {} & C_F^3C_A\left[-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7\right] + C_F^2T_fC_A\left[\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7\right] \\
+ {} & C_FT_f^2C_A\left[\frac{340843}{5184} - \frac{10453}{288}\zeta_3 - \frac{170}{9}\zeta_5 - \frac{1}{2}\zeta_3^2\right] + C_F^2C_A^2\left[-\frac{592141}{18432} - \frac{43925}{384}\zeta_3 + \frac{6505}{48}\zeta_5 + \frac{1155}{32}\zeta_7\right] \\
+ {} & C_FT_fC_A^2\left[-\frac{4379861}{20736} + \frac{8609}{72}\zeta_3 + \frac{18805}{288}\zeta_5 - \frac{11}{2}\zeta_3^2 + \frac{35}{16}\zeta_7\right] \\
+ {} & C_FC_A^3\left[\frac{52207039}{248832} - \frac{456223}{3456}\zeta_3 - \frac{77995}{1152}\zeta_5 + \frac{605}{32}\zeta_3^2 - \frac{385}{64}\zeta_7\right],
\end{aligned} \tag{10}$$

$$\begin{aligned}
b_4 = {} & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048} + \frac{3}{8}\zeta_3\right] \\
+ {} & C_F^3T_f\left[-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5\right] + C_F^2T_f^2\left[\frac{869}{576} - \frac{29}{24}\zeta_3\right] + C_FT_f^3\left[-\frac{605}{972}\right] \\
+ {} & C_F^3C_A\left[-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5\right] + C_F^2T_fC_A\left[-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7\right] \\
+ {} & C_FT_f^2C_A\left[\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2\right] + C_F^2C_A^2\left[-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7\right] \\
& + C_FT_fC_A^2\left[-\frac{1238827}{41472} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}\zeta_7\right] + C_FC_A^3\left[\frac{8004277}{248832} - \frac{1069}{576}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 - \frac{385}{64}\zeta_7\right].
\end{aligned} \tag{11}$$

All six constraints from the generalized Crewther relation are indeed met with

$$\begin{aligned}
K_3 = {} & C_F^3\left(\frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7\right) + C_F^2T_f\left(-\frac{7729}{1152} - \frac{917}{16}\zeta_3 + \frac{125}{2}\zeta_5 + 9\zeta_3^2\right) \\
+ {} & C_FT_f^2\left(-\frac{307}{18} + \frac{203}{18}\zeta_3 + 5\zeta_5\right) + C_F^2C_A\left(\frac{99757}{2304} + \frac{8285}{96}\zeta_3 - \frac{1555}{12}\zeta_5 - \frac{105}{8}\zeta_7\right) \\
+ {} & C_FT_fC_A\left(\frac{1055}{9} - \frac{2521}{36}\zeta_3 - \frac{125}{3}\zeta_5 - 2\zeta_3^2\right) + C_FC_A^2\left(-\frac{406043}{2304} + \frac{18007}{144}\zeta_3 + \frac{2975}{48}\zeta_5 - \frac{77}{4}\zeta_3^2\right).
\end{aligned}$$

Note, that coefficients in front of first three colour structures in eqs. (10,11) ($C_F^4, n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}$ and $\frac{d_F^{abcd}d_A^{abcd}}{d_R}$) are equal, as they should. The C_F^4-term, in particular, provides us with a beautiful confirmation of the correctness of the result (9) for the qQED β-function (the test was originally suggested in [27]).

It is interesting to note that the results do not depend on ζ_n with $n = 2, 4, 6$. Also, unexpected feature of our results is the *separate* proportionality all terms of highest and sub-highest transcendentality in a given loop order (that is ζ_3^2 and ζ_7 at α_s^4, ζ_5 at α_s^3 and, at last, ζ_3 at α_s^2) to β_0. This feature *is not* required by (4), the latter essentially constraints only the *difference* $d_i - b_i$.

In numerical form C^{Bjp} reads (with all colour factors set to their QCD values)

$$\begin{aligned} C^{Bjp} &= 1 - a_s + \left(-4.583 + 0.3333\, n_f\right) a_s^2 \\ &+ a_s^3 \left(-41.44 + 7.607\, n_f - 0.1775\, n_f^2\right) a_s^3 \\ &+ \left(-479.4 + 123.4\, n_f - 7.697\, n_f^2 + 0.1037\, n_f^3\right) a_s^4. \end{aligned} \tag{12}$$

It is of interest to compare the newly found coefficient in front of the α_s^4 term with well-known predictions [28]

$$c_4^{\rm pred}(n_f = 3,4,5,6) = -130,\ -58,\ -18,\ 22$$

and

$$c_4^{\rm exact}(n_f = 3,4,5,6) = -175.7,\ -102.4,\ -41.96,\ 6.2.$$

At last, we derive the commensurate relation connecting two the ECC's a_{g_1} and a_D as defined in eqs. (7,8). Following ref. [29] we get for QCD

$$\left(1 + a_D(Q^{\star 2})\right)\left(1 - a_{g_1}(Q^2)\right) = 1, \tag{13}$$

with ($a_D^\star = a_D(Q^{\star 2})$)

$$\begin{aligned} \ln\left(\frac{Q^{\star 2}}{Q^2}\right) &= -K_1 + a_D^\star \left[\beta_0\, K_1^2 + 2d_2\, K_1 - K_1 - K_2\right] \\ &+ (a_D^\star)^2 \Big[\beta_0\left(-6d_2\, K_1^2 + 2\, K_1^2 + 3\, K_2\, K_1\right) - 2\beta_0^2\, K_1^3 \\ &+ K_1\left(\frac{3}{2}\beta_1 K_1 - 6d_2^2 + 2d_2 + 3d_3\right) + K_2\left(3d_2 - 1\right) - K_3\Big] \\ &= -1.30823 + a_D^\star\left[0.80241 - 0.03933\, n_f\right] \\ &+ (a_D^\star)^2\left[-16.9020 + 2.62311 n_f - 0.10202 n_f^2\right]. \end{aligned}$$

In conclusion we want to mention that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel [30] as well as thread-based [31] versions of FORM [32]. For evaluation of color factors we have used the FORM program *COLOR* [33]. The diagrams have been generated with QGRAF [34].

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[37] Unexpected, because there existed a wide-spread belief that the rationality property is not accidental but holds also in higher orders [35, 36].