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Precise Charm- and Bottom-Quark Masses:
Recent Developments

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Recent theoretical and experimental improvements in the determination of charm- and bottom-quark masses are discussed. The final results, $m_c(3\,\mathrm{GeV}) = 986(13)\,\mathrm{MeV}$ and $m_b(m_b) = 4163(16)\,\mathrm{MeV}$ represent the presently most precise determinations of these two fundamental Standard Model parameters.

The past years have witnessed significant improvement in the determination of charm- and bottom-quark masses as a consequence of improvements in experimental techniques as well as theoretical calculations. Both masses are critical ingredients in the evaluation of various observables, e.g. the masses of charm and bottom mesons and the corresponding quarkonia, the rates for weak decays of $B$ mesons and the closely related CKM matrix elements $|V_{cb}|^2$ and $|V_{ub}|^2$. Rare kaon decays and the $b \to s\gamma$ transition are critically affected by contributions from virtual charmed quarks. If the Higgs boson is as light as expected from electroweak precision measurements, its decay is dominated by bottom quarks with a rate proportional $m_b^2$. From the more theoretical side it is clear that $m_b$ is a decisive input for any attempt of Yukawa unification, in the framework of $SU(5)$, relating $\tau$ and bottom Yukawa couplings, or $SO(10)$, relating those of $\tau$, bottom and top quarks. Requiring comparable relative precision of top and bottom masses, $\delta m_b/m_b \sim \delta m_t/m_t$, an error $\delta m_b$ around or below 25 MeV is required to fully exploit the current precision of $m_t$ with $\delta m_t \approx 1\mathrm{GeV}$, not to speak of the potential precision of a future linear collider.

As stated above, quark mass determinations can be based on a variety of observations and theoretical calculations. The one presently most precise follows an idea advocated by the ITEP group more than thirty years ago [1], and has gained renewed interest after significant advances in higher order perturbative calculations discussed in this paper have been achieved.

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It exploits the fact that the vacuum polarization function $\Pi(q^2)$ and its derivatives, evaluated at $q^2 = 0$, can be considered short distance quantities with an inverse scale characterized by the distance between the reference point $q^2 = 0$ and the location of the threshold $q^2 = (3\mathrm{GeV})^2$ and $q^2 = (10\mathrm{GeV})^2$ for charm and bottom respectively. This idea has been taken up in [2] after the first three-loop evaluation of the moments became available [3, 4, 5] and has been further improved in [6] using four-loop results [7, 8] for the lowest moment. An analysis which is based on the most recent theoretical [9, 10, 11] and experimental progress has been performed in [12] and will be reviewed in the following.

Let us recall some basic notation and definitions. The vacuum polarization $\Pi_Q(q^2)$ induced by a heavy quark $Q$ with charge $Q_Q$ (ignoring in this short note the so-called singlet contributions), is an analytic function with poles and a branch cut at and above $q^2 = M^2_{J/\psi}$. Its Taylor coefficients $\bar{C}_n$, defined through

$$\Pi_Q(q^2) \equiv Q_Q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n z^n \tag{1}$$

can be evaluated in pQCD

$$\begin{aligned}
\bar{C}_n = & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi} \left( \bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_Q} \right) \\
& + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( \bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_Q} + \bar{C}_n^{(22)} l_{m_Q}^2 \right) \\
& + \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \left( \bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_Q} + \bar{C}_n^{(32)} l_{m_Q}^2 + \bar{C}_n^{(33)} l_{m_Q}^3 \right) + \dots .
\end{aligned} \tag{2}$$

Here $z \equiv q^2/4m_Q^2$, where $m_Q = m_Q(\mu)$ is the running $\overline{\mathrm{MS}}$ mass at scale $\mu$. Using a once-subtracted dispersion relation

$$\Pi_Q(q^2) = \frac{1}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s-q^2)} \tag{3}$$

(with $R_Q$ denoting the familiar $R$-ratio for the production of heavy quarks), the Taylor coefficients can be expressed through moments of $R_Q$. Equating perturbatively calculated and experimentally measured moments,

$$\mathcal{M}_n^{\mathrm{exp}} = \int \frac{ds}{s^{n+1}} R_Q(s) \tag{4}$$

leads to an ($n$-dependent) determination of the quark mass

$$m_Q = \frac{1}{2} \left( \frac{9 Q_Q^2}{4} \frac{C_n}{\mathcal{M}_n^{\mathrm{exp}}} \right) . \tag{5}$$

| | $n_l = 3$ | $n_l = 4$ | $n_l = 5$ |
|---|---|---|---|
| $\bar{C}_1^{(3),v}$ | $-5.6404$ | $-7.7624$ | $-9.6923$ |
| $\bar{C}_2^{(3),v}$ | $-3.4937$ | $-2.6438$ | $-1.8258$ |
| $\bar{C}_3^{(3),v}$ | $-2.8395$ | $-1.1745$ | $0.4113$ |
| $\bar{C}_4^{(3),v}$ | $-3.349(11)$ | $-1.386(10)$ | $0.471(9)$ |
| $\bar{C}_5^{(3),v}$ | $-3.737(32)$ | $-1.754(32)$ | $0.104(27)$ |
| $\bar{C}_6^{(3),v}$ | $-3.735(61)$ | $-1.910(63)$ | $-0.228(54)$ |
| $\bar{C}_7^{(3),v}$ | $-3.39(10)$ | $-1.85(10)$ | $-0.46(9)$ |
| $\bar{C}_8^{(3),v}$ | $-2.85(13)$ | $-1.67(14)$ | $-0.66(12)$ |
| $\bar{C}_9^{(3),v}$ | $-2.22(17)$ | $-1.47(18)$ | $-0.91(16)$ |
| $\bar{C}_{10}^{(3),v}$ | $-1.65(20)$ | $-1.37(22)$ | $-1.30(19)$ |

Table 1. Expansion coefficients from the reconstructed vector polarization function for different numbers of light quarks in the $\overline{\mathrm{MS}}$ scheme. $C_{1-3}^{(3),v}$ are exact.

The consistency of the results for different $n$ and their stabilization with increasing orders in perturbation theory gives confidence in their reliability.

Significant progress has been made in the perturbative evaluation of the moments since the first analysis of the ITEP group. The $\mathcal{O}(\alpha_s^2)$ contribution (three loops) has been evaluated more than 13 years ago [3, 4, 5], as far as the terms up to $n = 8$ are concerned, recently even up to $n = 30$ [13, 14]. About ten years later the lowest two moments ($n = 0, 1$) of the vector correlator were evaluated in $\mathcal{O}(\alpha_s^3)$, i. e. in four-loop approximation [7, 8]. The corresponding two lowest moments for the pseudoscalar correlator were obtained in [15] in order to derive the charmed quark mass from lattice simulations [16]. In [9, 10] the second and third moments were evaluated for vector, axial and pseudoscalar correlators. Combining, finally, these results with information about the threshold and high-energy behaviour in the form of a Padé approximation, the full $q^2$/dependence of all four correlators was reconstructed and the next moments, from four up to ten, were obtained with adequate accuracy. A list of the relevant coefficients is shown in Table 1 (for an earlier, less precise analysis, see Ref. [17]).

Most of the experimental input had already been compiled and exploited in [6], where it is described in more detail. However, until recently the only measurement of the cross section above but still close $B$-meson threshold was performed by the CLEO collaboration more than twenty years ago [18]. Its large systematic uncertainty was responsible for a sizable fraction of the final error on $m_b$. This measurement has been recently superseded by a measurement of BABAR [19] with a systematic error between 2 and 3%. In [12] the radiative corrections were unfolded and used to obtain a sig-

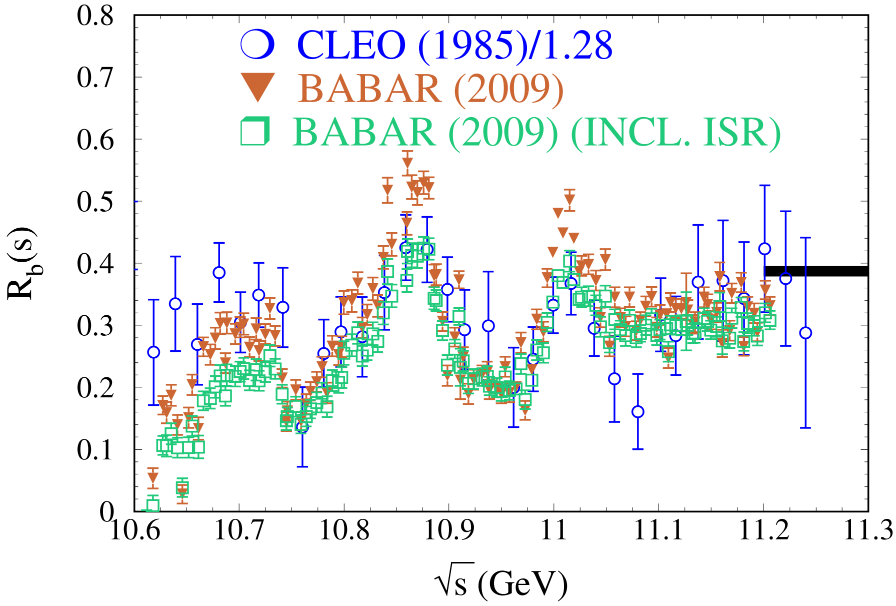


Fig. 1. Comparison of rescaled CLEO data for $R_b$ with BABAR data before and after deconvolution. The black bar on the right corresponds to the theory prediction [20].

| $n$ | $m_c(3\text{ GeV})$ | exp | $\alpha_s$ | $\mu$ | np | total |
|---|---|---|---|---|---|---|
| 1 | 986 | 9 | 9 | 2 | 1 | 13 |
| 2 | 976 | 6 | 14 | 5 | 0 | 16 |
| 3 | 978 | 5 | 15 | 7 | 2 | 17 |
| 4 | 1004 | 3 | 9 | 31 | 7 | 33 |

Table 2. Results for $m_c(3\text{ GeV})$ in MeV obtained from Eq. (5). The errors are from experiment, $\alpha_s$, variation of $\mu$ and the gluon condensate.

nificantly improved determination of the moments. The final results for $m_c(3\text{GeV})$ and $m_b(10\text{GeV})$ are listed in Tables 2 and 3. Despite the significant differences in the composition of the errors, the results for different values of $n$ are perfectly consistent. For charm the result from $n = 1$ has the smallest dependence on the strong coupling and the smallest total error, which we take as our final value

$$m_c(3\text{ GeV}) = 986(13)\text{ MeV}\,, \tag{6}$$

and consider its consistency with $n = 2$, 3 and 4 as additional confirmation.

| $n$ | $m_b(10\text{ GeV})$ | exp | $\alpha_s$ | $\mu$ | total | $m_b(m_b)$ |
|---|---|---|---|---|---|---|
| 1 | 3597 | 14 | 7 | 2 | 16 | 4151 |
| 2 | 3610 | 10 | 12 | 3 | 16 | 4163 |
| 3 | 3619 | 8 | 14 | 6 | 18 | 4172 |
| 4 | 3631 | 6 | 15 | 20 | 26 | 4183 |

Table 3. Results for $m_b(10\text{ GeV})$ and $m_b(m_b)$ in MeV obtained from Eq. (5). The errors are from experiment, $\alpha_s$ and the variation of $\mu$.

Transforming this to the scale-invariant mass $m_c(m_c)$ [21], including the four-loop coefficients of the renormalization group functions one finds [12] $m_c(m_c) = 1279(13)$ MeV. Let us recall at this point that a recent study [16], combining a lattice simulation for the data for the pseudoscalar correlator with the perturbative three- and four-loop result [5, 15, 10] has led to $m_c(3\text{ GeV}) = 986(10)$ MeV in remarkable agreement with [6, 12].

The treatment of the bottom quark case proceeds along similar lines. However, in order to suppress the theoretically evaluated input above 11.2 GeV (which corresponds to roughly 60% for the lowest, 40% for the second and 26% for the third moment), the result from the second moment has been adopted as our final result,

$$m_b(10\text{ GeV}) = 3610(16)\text{ MeV}, \tag{7}$$

corresponding to $m_b(m_b) = 4163(16)$MeV. The explicit $\alpha_s$ dependence of $m_c$ and $m_b$ can be found in [12]. When considering the ratio of charm and bottom quark masses, part of the $\alpha_s$ and of the $\mu$ dependence cancels

$$\frac{m_c(3\text{ GeV})}{m_b(10\text{ GeV})} = 0.2732 - \frac{\alpha_s - 0.1189}{0.002} \cdot 0.0014 \pm 0.0028\,, \tag{8}$$

which might be a useful input in ongoing analyses of bottom decays.

The results presented in [12] constitute the most precise values for the charm- and bottom-quark masses available to date. Nevertheless it is tempting to point to the dominant errors and thus identify potential improvements. In the case of the charmed quark the error is dominated by the parametric uncertainty in the strong coupling $\alpha_s(M_Z) = 0.1185 \pm 0.002$. Experimental and theoretical errors are comparable, the former being dominated by the electronic width of the narrow resonances. In principle this error could be further reduced by the high luminosity measurements at BESS III. A further reduction of the (already tiny) theory error, e. g. through a five-loop calculation looks difficult. Further confidence in our result can be obtained from the comparison with the forementioned lattice evaluation.

Improvements in the bottom quark mass determination could originate from the experimental input, e. g. through an improved determination of the electronic widths of the narrow $\Upsilon$ resonances or through a second, independent measurement of the $R$ ratio in the region from the $\Upsilon(4S)$ up to 11.2 GeV. As shown in Fig. 1, there is a slight mismatch between the theory prediction above 11.2 GeV and the data in the region below with their systematic error of less than 3%.

*To summarize:* Charm and bottom quark mass determinations have made significant progress during the past years. A further reduction of the theoretical and experimental error seems difficult at present. However, independent experimental results on the $R$ ratio would help to further consolidate the present situation. The confirmation by a recent lattice analysis with similarly small uncertainty gives additional confidence in the result for $m_c$.

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