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Three-loop static potential

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We compute the three-loop corrections to the potential of two heavy quarks. In particular we consider in this Letter the purely gluonic contribution which provides in combination with the fermion corrections of Ref. [1] the complete answer at three loops.

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The potential between two heavy quarks constitutes a fundamental quantity in Quantum Chromodynamics. It enters in a variety of physical processes like the threshold production of top quark pairs and the description of charm and bottom quark bound states. Furthermore, it is crucial for the understanding of fundamental quantities of QCD, such as confinement. (See Ref. [2] for a recent review.)

The idea to describe a bound state of heavy coloured objects in analogy to the well-established hydrogen atom, goes back to the middle of the 1970s [3]. Shortly afterwards, about 30 years ago, one-loop radiative corrections have been evaluated in the works [4, 5]. It took almost 20 years until the next order became available [6–8] which, at that time, was a heroic enterprize. The two-loop corrections turned out to be numerically quite important which triggered several investigations to go beyond. End of last year the fermionic corrections to the three-loop static potential have been completed [1, 9, 10]. In this Letter we report about the pure gluonic part which completes the three-loop corrections to the static potential.

We present our results for the static potential in momentum space where it takes the form

$$V(|\vec{q}\,|) = -\frac{4\pi C_F\alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}}\left[1+\frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1+\left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3\left(a_3+8\pi^2C_A^3\ln\frac{\mu^2}{\vec{q}^{\,2}}\right)+\cdots\right]. \qquad (1)$$

Here, $C_A = N_c$ and $C_F = (N_c^2-1)/(2N_c)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N_c)$ colour gauge group, respectively, and α_s denotes the strong coupling in the $\overline{\rm MS}$ scheme. The one- and two-loop coefficients a_1 [4, 5] and a_2 [6–8, 11] are given in Eq. (4) of Ref. [1] where also the higher order terms in ϵ, necessary for the three-loop calculation, are presented. In Eq. (1) we identify the renormalization scale μ^2 and the momentum transfer $\vec{q}^{\,2}$. The complete dependence on μ can easily be restored with the help of Eq. (2) of Ref. [1].

A new feature of the three-loop corrections to $V(|\vec{q}\,|)$ is the appearance of infrared divergences [12] which is represented by the $\ln(\mu^2/\vec{q}^{\,2})$ term in Eq. (1). It has been evaluated for the first time in Refs. [13, 14] (see also Ref. [15]); in Eq. (1) we adopt the $\overline{\rm MS}$ scheme which has been used in Ref. [14]. Let us mention that the infrared divergence cancels in physical quantities after including the contribution where so-called ultrasoft gluons interact with the heavy quark anti-quark bound state. An explicit result can, e.g., be found in Ref. [14] where the cancellation has been demonstrated in order to arrive at the measurable energy levels of the heavy-quark system. We note in passing that higher order logarithmic contributions to the infrared behaviour of the static potential have been computed in Refs. [16, 17].

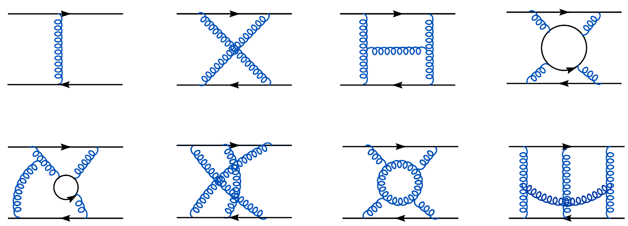

FIG. 1: Sample diagrams contributing to the static potential at tree-level, one-, two- and three-loop order. Solid and curly lines represent quarks and gluons, respectively. In the case of closed loops the quarks are massless; the external quarks are heavy and treated in the static limit.

Before presenting our results for a_3 let us provide some technical details. We generate the four-point quark anti-quark amplitudes with the help of `QGRAF` [18]. Some sample diagrams up to three-loop order are shown in Fig. 1. In a next step they are processed further with `q2e` and `exp` [19, 20] where a mapping to the diagrams of Fig. 2 is achieved. The mapping to two-point functions is possible since the only dimenionful quantity in our problem is given by the momentum transfer between the quark and the anti-quark. Although there is only one mass scale in our problem technical complications arise from the simultaneous presence of static lines (zigzag lines) and relativistic propagators (solid lines) which significantly increases the complexity of the reduction to master integrals. For this task we employ the program package `FIRE` [21] in order to achieve a reduction to about 100 basic integrals, so-called master integrals. The latter have

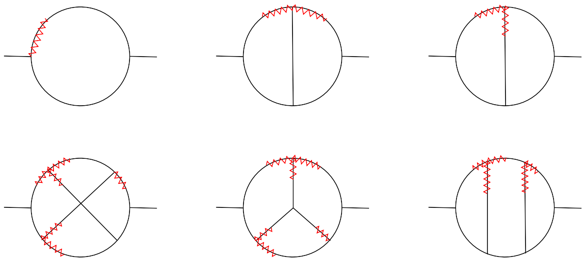

FIG. 2: One-, two- and three-loop diagrams. The solid line stands for massless relativistic propagators and the zigzag line represents static propagators.

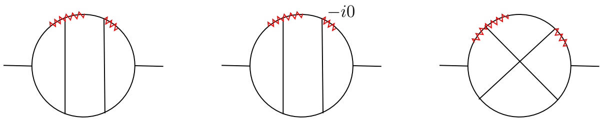

FIG. 3: Three-loop master integrals where the $\mathcal{O}(\epsilon)$ part is only known numerically. The label "$-i0$" indicates that instead of the static propagator $1/(p_0+i0)$ there is the propagator $1/(p_0-i0)$.

to be evaluated in an expansion in ϵ which we achieve with the help of the Mellin–Barnes method (see, e.g., Refs. [22–25]). We managed to compute all the necessary coefficients of the ϵ expansion of the master integrals analytically with the exception of terms of order ϵ^1 of the three diagrams shown in Fig. 3. Results for the master integrals as well as more details on their evaluation will be published elsewhere. As a crucial tool providing very important numerical cross checks of the analytical results we applied the program `FIESTA` [26] which is a convenient and efficient implementation of the sector decomposition algorithm. The colour factors of the individual Feynman diagrams have been computed with program `color` [27].

In our calculation we allowed for a general gauge parameter ξ in the gluon propagator. For individual diagrams we observe the appearance of terms up to ξ^6. We have checked that the coefficients of the ξ^n terms with $n = 2, 3, 4, 5$ and 6 are zero.

In order to present our results we decompose the three-loop coefficient in the form

$$a_3 \;=\; a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}\,, \qquad (2)$$

where n_l is the number of light quarks and the first three coefficients on the right-hand side have been presented in Ref. [1] (see Eq. (6)). Whereas for the fermionic contributions there are seven different colour structures in the case of $a_3^{(0)}$ there are only two. Note that the result of all colour structures containing a factor C_F are generated by iterations of lower-order contributions and thus do not contribute to $a_3^{(0)}$. Diagrammatically such contributions are easily identified since the corresponding Feynman integrals contain so-called pinch contributions of the form $1/(p_0+i0)\times 1/(p_0-i0)$ where p_0 is the zeroth component of a loop momentum. Our result for $a_3^{(0)}$ reads

$$a_3^{(0)} \;=\; 502.24(1)\; C_A^3 - 136.39(12)\; \frac{d_F^{abcd} d_A^{abcd}}{N_A}\,. \qquad (3)$$

Similarly to the fermionic contribution new colour invariants appear which can be traced back to Feynman diagrams as the third one in the second row of Fig. 1. Expressed in terms of N_c one has $d_F^{abcd} d_A^{abcd}/N_A = (N_c^3 + 6N_c)/48$. The coefficient of $d_F^{abcd} d_A^{abcd}$ has already been presented in Refs. [28, 29], the coefficient of the C_A^3 term is new.

Let us now discuss the numerical effect of the three-loop contribution to the static potential. Inserting the numerical results for the coefficients a_i in Eq. (1) we obtain

$$\begin{aligned} V(|\vec{q}\,|) \;=\; -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}} \Bigg[& 1 + \frac{\alpha_s}{\pi}\,(2.5833 - 0.2778 n_l) \\ & + \left(\frac{\alpha_s}{\pi}\right)^2 \left(28.5468 - 4.1471 n_l + 0.0772 n_l^2\right) \\ & + \left(\frac{\alpha_s}{\pi}\right)^3 \big(209.884(1) - 51.4048 n_l \\ & + 2.9061 n_l^2 - 0.0214 n_l^3\big) + \cdots \Bigg]\,, \end{aligned} \qquad (4)$$

where $\mu^2 = \vec{q}^{\,2}$ has been adopted in order to suppress the infrared logarithm and the ellipses denote higher order terms in α_s. The term "209" in the three-loop coefficient receives a large contribution ("211") from the C_A^3 term whereas the new colour structure only contributes with a coefficient "-2". From Eq. (4) we observe at one-, two- and three-loop order a large screening of the non-fermionic contributions by the n_l terms which is most prominent in the case of a_3 for $n_l = 5$. Here the difference between $a_3^{(0)}$ and the fermionic contribution is one order smaller than the individual pieces.

In Tab. I we show the numerical evaluation of the square bracket of Eq. (4) for the charm, bottom and top quark case, i.e. for $n_l = 3, 4$ and 5, adopting the appropriate values of α_s. For charm the three-loop corrections are almost as big as the one- and two-loop contributions whereas for bottom the three-loop contribution is already a factor of four smaller than the two-loop one. In the case of the top quark one observes a good convergence: the three-loop term is already a factor ten smaller than the two-loop counterpart.

As already mentioned above, $V(|\vec{q}\,|)$ itself is not a physical quantity. Hence let us consider the ground state energy E_1 of a heavy quarkonium system which has been evaluated to the third order in perturbation theory in Ref. [30] where the contribution from a_3 has been kept unevaluated. We are now in the position to complete the numerical analysis. It is convenient to write the perturbative contribution to E_1 in the form

$$E_1^{\rm p.t.} \;=\; E_1^C + \delta E_1^{(1)} + \delta E_1^{(2)} + \delta E_1^{(3)} + \ldots\,, \qquad (5)$$

n_l	$\alpha_s^{(n_l)}$	1 loop	2 loop	3 loop
3	0.40	0.2228	0.2723	0.1677
4	0.25	0.1172	0.08354	0.02489
5	0.15	0.05703	0.02220	0.002485

TABLE I: Radiative corrections to the potential $V(|\vec{q}|)$ where the tree-level result is normalized to 1 (cf. Eq. (4)). In the second column we also provide the numerical value of α_s corresponding to the soft scale where $\mu \approx m_q\alpha_s$ (m_q being the heavy quark mass).

with the Coulomb energy $E_1^C = -C_F^2\alpha_s^2 m_q/4$. m_q is the heavy quark mass and the superscript in brackets indicates the order in perturbation theory. Adopting for the renormalization scale the choice $\mu_S = C_F\alpha_s(\mu_S)m_q$ we obtain

$$\begin{aligned} \delta E_1^{(3)}\Big|_{\rm charm} &= \alpha_s^3 E_1^C\left(129.79 + 5.241\Big|_{a_3} + 15.297\ln(\alpha_s)\right), \\ \delta E_1^{(3)}\Big|_{\rm bottom} &= \alpha_s^3 E_1^C\left(104.82 + 3.186\Big|_{a_3} + 15.297\ln(\alpha_s)\right), \\ \delta E_1^{(3)}\Big|_{\rm top} &= \alpha_s^3 E_1^C\left(83.386 + 1.473\Big|_{a_3} + 15.297\ln(\alpha_s)\right), \end{aligned} \tag{6}$$

where the contribution from a_3 has been marked separately. One observes that the numerical effect amounts between 1 and 4% of the non-logarithmic constant.

Finally, it is interesting to compare our results with the predictions obtained on the basis of certain assumptions on the perturbative expansion. In Ref. [31] a Padé approximation in the coupling constant has been performed whereas the findings of Ref. [32] are based on renormalon studies. For $a_3^{(0)}/4^3$ they predict 313 and 292, respectively, which overshoots the exact result by 40 to 50%.

More recently, a detailed comparison of the perturbative result with lattice simulations has been performed [33] with the aim to extract $a_3^{(0)}$. After transforming the result of Ref. [33] to momentum space using the formulae provided in their Appendix one obtains $209 \le a_3^{(0)}/4^3 \le 350$. Thus the lower limit of the (relatively big) interval covers the exact result.

To conclude, in this Letter the three-loop corrections to the static potential have been completed by evaluating the gluonic contribution. Our main result can be found in Eq. (3) where the three-loop coefficients are given for general colour structure. Numerical sizeable corrections are observed for the non-fermionic contributions which are partly canceled by the fermionic corrections evaluated in Ref. [1].

Let us stress that the static potential constitutes a fundamental quantity of QCD. It represents a building block in many physical quantities like the determination of the bottom quark mass from the $\Upsilon(1S)$ bound state or the third-order correction to top quark threshold production cross section at a future electron positron linear collider which would result in the most precise value for the top quark mass. The static energy is also a crucial object when comparing perturbation theory and lattice simulations (see, e.g., Refs. [33–36]). We also want to mention the extraction of the strong coupling constant from lattice simulations where again the static potential and in particular a_3 plays an important role [37, 38].

Note added: While finishing this paper we became aware of the preprint C. Anzai, Y. Kiyo and Y. Sumino, "Static QCD potential at three-loop order," arXiv:0911.4335 [hep-ph], where also $a_3^{(0)}$ has been computed. We agree with their Eq. (10), however, obtain a better precision.

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