Power suppressed effects in $\bar{B} \to X_s\gamma$ at $O(\alpha_s)$

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Abstract

We compute the $O(\alpha_s)$ corrections to the Wilson coefficients of the dimension five operators emerging from the Operator Product Expansion of inclusive radiative B decays. We discuss the impact of the resulting $O(\alpha_s \Lambda^2_{QCD}/m_b^2)$ corrections on the extraction of m_b and μ_π^2 from the moments of the photon spectrum.

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1 Introduction

The inclusive radiative decays of the B meson play a central role in the search for new physics. While the total rate of $\bar{B} \to X_s\gamma$ is sensitive to new physics in flavor-changing transitions, its photon spectrum is almost completely determined by Standard Model physics and can be employed to extract information on the B meson structure and on the b quark mass, see for instance [1, 2]. The latter are useful in a number of B physics applications, like the determination of $|V_{ub}|$ and of $|V_{cb}|$, which are both important inputs for the determination of the CKM unitarity triangle and the study of CP violation in the Standard Model.

After integrating out the heavy degrees of freedom at the electroweak scale and evolving the resulting weak Hamiltonian to the b quark mass scale, inclusive radiative and semileptonic B decays are well described by an Operator Product Expansion (OPE) in inverse powers of the b quark mass. In this way one can factorize the long distance dynamics into the matrix elements of a few local operators [3, 4]. Since the Wilson coefficients of these operators are perturbative, and the matrix elements of the local operators parameterize the non-perturbative physics, the radiative and semileptonic rates and the moments of their distributions are double series in α_s and Λ/m_b, with Λ being the QCD scale. The lowest order of this expansion corresponds to the decay of a free b quark and linear $O(\Lambda/m_b)$ corrections are absent. The relevant parameters are therefore the heavy quark mass m_b (and possibly m_c for the charmed decays), α_s, and the matrix elements of the local operators: μ_π^2 and μ_G^2 at $O(1/m_b^2)$, ρ_D^3 and ρ_{LS}^3 at $O(1/m_b^3)$. In the case of $b \to s\gamma$ transitions, the first and second central moments of the photon spectrum, $\langle E_\gamma\rangle$ and $\langle E_\gamma^2 - \langle E_\gamma\rangle^2\rangle$, are proportional to the b quark mass and to μ_π^2, respectively, up to subleading corrections in the $1/m_b$ expansion, and contribute in an important way to the global fits for the extraction of $|V_{cb}|$, m_b and the other OPE parameters (see [5] for recent results).

The precision of $\bar{B} \to X_s\gamma$ calculations in this framework is known to be limited in two ways: *i*) The dominant contribution to the radiative b decay is associated to the electromagnetic dipole operator $O_7 = (\alpha_{\rm em}/4\pi)\, m_b \bar{s}_L \sigma^{\mu\nu} F_{\mu\nu} b_R$, with additional operators appearing at $O(\alpha_s)$. It turns out that a local OPE can be written down for the O_7 contribution only, while the other operators are expected to induce unknown $O(\alpha_s\,\Lambda/m_b)$ contributions; *ii*) Even for the dominant O_7 component, the experimental cuts employed to isolate the signal introduce a sensitivity to the Fermi motion of the b quark inside the B meson and tend to disrupt the OPE. One can still resum the higher order terms into a non-local distribution function [6] and since the lowest integer moments of this function are known, one can parameterize it assuming different functional forms [7], although alternative approaches are also possible [1].

The perturbative corrections to the leading O_7 contribution are presently known to complete Next-to-Next-to-Leading Order (NNLO), i.e. $O(\alpha_s^2)$, while the NNLO contributions involving other operators, quite important for the rate, are not yet complete [8], see [9] for recent updates. As for the non-perturbative corrections to the O_7 contribution, they are known through $O(1/m_b^3)$ [10, 11]. Among the non-perturbative corrections associated to operators other than O_7, only one class is known: they are of $O(1/m_c^2)$ and $O(1/(m_c^2 m_b))$ and numerically small [11]. Overall, the known power corrections modify the total $\bar{B} \to X_s\gamma$ rate by just $\sim 3\%$, but they are essential in the calculation of the moments.

In this paper we present the first calculation of the $O(\alpha_s)$ corrections to the Λ^2/m_b^2 terms in $\bar{B} \to X_s\gamma$. The $O(\alpha_s)$ perturbative corrections to the μ_π^2 coefficient in the total rate are fixed by Lorentz invariance [3]. Their contribution to the photon moments has been computed in the context of a multiscale OPE [12] which applies however to the end-point region only. We have computed the relevant Wilson coefficients at $O(\alpha_s)$ by expanding off-shell amputated Green functions around the b quark mass shell, and by matching them onto local operators in Heavy Quark Effective Theory (HQET). Our results allow for an improved analysis of the radiative moments. In particular, the inclusion of the $O(\alpha_s)$ perturbative corrections to the variance of the spectrum, permits the extraction of μ_π^2 at Next-to-Leading-Order (NLO).

The outline of this paper is as follows. In section 2 we recalculate the Wilson coefficients of the matrix elements of dimension five operators, μ_π^2 and μ_G^2, at leading order in α_s and introduce our notation. Section 3 is devoted to the calculation of these Wilson coefficients at $O(\alpha_s)$. It also contains our final results and a first numerical estimate of their importance. In section 4 we summarize and conclude.

2 Leading Order Coefficients

At low energy the $b \to s\gamma$ transition is governed by the effective Lagrangian

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \, C_7^{\rm eff}(\mu) O_7 + \dots \, , \tag{2.1}$$

which includes, apart from the QED and QCD interactions of the light quarks u, d, s, c and b, the renormalization scale dependent effective Wilson coefficient $C_7^{\rm eff}$ and the corresponding electromagnetic operator O_7. The latter mediates the $b \to s\gamma$ transition and is given by

$$O_7 = \frac{\alpha_{\rm em}}{4\pi}\, \overline{m}_b(\mu) (\bar{s}\sigma^{\mu\nu} P_R b) F_{\mu\nu} \, , \tag{2.2}$$

where $\overline{m}_b(\mu)$ is the running b quark mass in the $\overline{\rm MS}$-scheme, $\sigma_{\mu\nu} = i\,[\gamma_\mu, \gamma_\nu]/2$ and $P_{R,L} = (1 \pm \gamma_5)/2$. The ellipses in equation (2.1) denote contributions from operators that are not relevant for our calculation.

Using this effective Lagrangian we can derive that part of the differential decay rate that is induced by the self-interference of the electromagnetic dipole operator,

$$d\Gamma_{77}(\bar{B} \to X_s\gamma) = \frac{G_F^2 \alpha_{\rm em} \overline{m}_b^2(\mu)}{16\pi^3 m_B} |V_{tb}V_{ts}^*|^2 |C_7^{\rm eff}(\mu)|^2 \frac{d^3q}{(2\pi)^3 2E_\gamma} \, W_{\mu\nu\alpha\beta} P^{\mu\nu\alpha\beta} \, . \tag{2.3}$$

Here, m_B is the mass of the B meson, q the momentum of the photon, $W_{\mu\nu\alpha\beta}$ the hadronic tensor,

$$\begin{aligned} W_{\mu\nu\alpha\beta} = \sum_X \int & \frac{d^3p_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^{(4)}(p_B - p_X - q) \\ & \times \langle \bar{B}(p_B)|\bar{b}(0)\sigma_{\mu\nu} P_L s(0)|X_s(p_X)\rangle \langle X_s(p_X)|\bar{s}(0)\sigma_{\alpha\beta} P_R b(0)|\bar{B}(p_B)\rangle \, , \end{aligned} \tag{2.4}$$

and $P^{\mu\nu\alpha\beta}$ the photon tensor,

$$P^{\mu\nu\alpha\beta} = \sum_{\lambda=\pm 1} \langle 0|F^{\mu\nu}|\gamma(q,\lambda)\rangle \langle \gamma(q,\lambda)|F^{\alpha\beta}|0\rangle$$

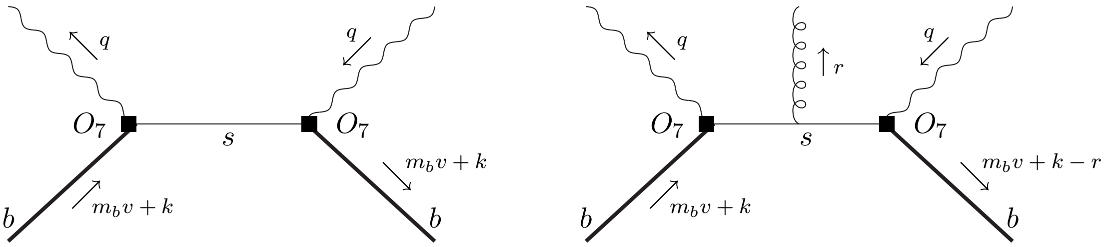

Figure 1: *The imaginary parts of these tree-level diagrams contribute to the Wilson coefficients of the operators with dimension 3, 4 and 5.*

$$= q^\alpha q^\nu g^{\mu\beta} - q^\alpha q^\mu g^{\nu\beta} + q^\mu q^\beta g^{\nu\alpha} - q^\beta q^\nu g^{\mu\alpha} \,. \tag{2.5}$$

The hadronic tensor $W_{\mu\nu\alpha\beta}$ itself can be rewritten as

$$W_{\mu\nu\alpha\beta} = 2\,\mathrm{Im}\left(i\int d^4x\, e^{-iq\cdot x}\langle\bar{B}(p_B)|\, T\left\{\bar{b}(x)\sigma_{\mu\nu}P_L s(x)\bar{s}(0)\sigma_{\alpha\beta}P_R b(0)\right\}|\bar{B}(p_B)\rangle\right) , \tag{2.6}$$

which is useful because the time-ordered product can be expanded into a series of local operators that are suppressed by powers of the b quark mass. Hence we can write

$$W_{\mu\nu\alpha\beta}P^{\mu\nu\alpha\beta} = -16\pi m_b\Big(c_{\dim 3}O_{\dim 3} + \frac{1}{m_b}\,c_{\dim 4}O_{\dim 4} + \frac{1}{m_b^2}\,c_{\dim 5}O_{\dim 5} + \dots\Big) , \tag{2.7}$$

where $O_{\dim n}$ is an operator of dimension n that contains $n-3$ derivatives, and

$$c_{\dim n} = c^{(0)}_{\dim n} + \frac{\alpha_s}{4\pi}\,c^{(1)}_{\dim n} + \dots \tag{2.8}$$

is the corresponding Wilson coefficient that can be determined in perturbation theory. In the reminder of this section we will review the calculation of $c^{(0)}_{\dim n}$ for $n = 3$, 4, 5.

The two Feynman diagrams contributing to these Wilson coefficients are depicted in Fig. 1. The momentum of the incoming b quark is $m_b v + k$, where v is the velocity of the B meson, and k is a residual momentum that accounts for the interaction of the almost on-shell b quark with the light degrees of freedom in the B meson. The components of this residual momentum are of $O(\Lambda)$, i.e. much smaller than the b quark mass. The same is true for the components of the momentum r of the radiated soft gluon that is present in the Feynman diagram on the right-hand side. To achieve the anticipated OPE (2.7) we perform a Taylor expansion of the amputated Green functions corresponding to the two tree-level diagrams up to $O(k^2)$ for the left diagram and up to $O(k)$ and $O(r)$ for the right diagram. This is equivalent to an expansion in inverse powers of m_b, or more precisely, to an expansion in $(iD_\mu - m_b v_\mu)/m_b$, where $D_\mu = \partial_\mu + i g_s G^a_\mu T^a$. In the algebraic manipulations we refrain from using any on-shell relations for the b quark, that is we do not impose any restriction on the residual momenta. The only on-shell condition we apply is that of the B meson, namely $v^2 = 1$. After extracting the imaginary part of the amputated Green functions we replace the residual momenta k and r with derivatives acting on the b and the gluon fields, interpreting the latter as parts of a covariant derivative. We end up with expressions for the two tree-level diagrams in terms of local operators of the form $\bar{b}F(v,q,D)b$, where $F(v,q,D)$ is a matrix-valued function of v_μ, q_μ and D_μ in spinor space. As far as the s-quark is concerned, we will use $m_s = 0$.

At this point it is convenient to introduce HQET [13] which incorporates the expansion in $(iD_\mu - m_b v_\mu)/m_b$ in a natural manner. Its Lagrangian is given by [14, 15]

$$\mathcal{L}_{\rm HQET} = i\bar{b}_v v\cdot D b_v + \frac{1}{2m_b}\bar{b}_v (iD_\perp)^2 b_v - a(\mu)\frac{g_s}{4m_b}\bar{b}_v\sigma_{\mu\nu}G^{\mu\nu}b_v + O\left(\frac{1}{m_b^2}\right)\,, \tag{2.9}$$

where $D_\perp^\mu = D^\mu - v^\mu\, v\cdot D$ and [16]

$$a(\mu) = 1 + \left[C_F + C_A\left(1 + \ln\frac{\mu}{m_b}\right)\right]\frac{\alpha_s}{4\pi} + \ldots\,. \tag{2.10}$$

The numerical values of the color factors are $C_F = 4/3$ and $C_A = 3$. The relation between the b quark fields in QCD and in HQET reads

$$b(x) = e^{-im_b v\cdot x}\left(1 + \frac{i\not{D}_\perp}{2m_b}\right) b_v(x) + O\left(\frac{1}{m_b^2}\right)\,, \tag{2.11}$$

at tree-level as well as at $O(\alpha_s)$.

Using (2.11), together with $\not{v} b_v = b_v$, in all local operators of dimension 4 and 5, we end up with the following set of operators,

$$\begin{aligned}
O_b^\mu &= \bar{b}\gamma^\mu b, & O_2^{\mu\nu} &= \bar{b}_v\frac{1}{2}\{iD^\mu, iD^\nu\}b_v\,,\\
O_1^\mu &= \bar{b}_v iD^\mu b_v, & O_3^{\mu\nu} &= \bar{b}_v\frac{g_s}{2}G^{a\mu}{}_\alpha\sigma^{\alpha\nu}T^a b_v\,,
\end{aligned} \tag{2.12}$$

where $g_s G^a_{\mu\nu}T^a = -i[D_\mu, D_\nu]$. We also find operators that include a γ_5. However, we will eventually calculate matrix elements of operators between B meson states. From parity considerations it follows then that only the operators given in (2.12) give non-vanishing contributions. Hence it is not necessary to calculate the Wilson coefficients of those operators that include a γ_5. We should also mention that up to this point of the calculation we did not use the equation of motion for the b_v field that follows from the HQET Lagrangian.

For the Wilson coefficients of the operators given in (2.12) we find, in $d = 4-2\epsilon$ space-time dimensions,

$$\begin{aligned}
c_{b\mu}^{(0)} &= -\frac{1}{2}(1-\epsilon)\,\hat{q}_\mu\delta(1-z)\,,\\
c_{1\mu}^{(0)} &= -\frac{1}{2}(1-\epsilon)\Big[(2v_\mu - \hat{q}_\mu)\,\delta(1-z) + (v_\mu - \hat{q}_\mu)\,\delta'(1-z)\Big],\\
c_{2\mu\nu}^{(0)} &= -\frac{1}{4}(1-\epsilon)\Big[2\,(g_{\mu\nu} + v_\mu\hat{q}_\nu)\,\delta(1-z) + (g_{\mu\nu} + 6v_\mu v_\nu - 6v_\mu\hat{q}_\nu)\,\delta'(1-z)\\
&\qquad\qquad + 2\,(v_\mu v_\nu - 2v_\mu\hat{q}_\nu + \hat{q}_\mu\hat{q}_\nu)\,\delta''(1-z)\Big],\\
c_{3\mu\nu}^{(0)} &= -\Big[2v_\mu\hat{q}_\nu - \frac{1+\epsilon}{2}g_{\mu\nu}\Big]\delta(1-z) + \Big[(3+\epsilon)\hat{q}_\mu\hat{q}_\nu - 2v_\mu\hat{q}_\nu + \frac{1+\epsilon}{4}g_{\mu\nu}\Big]\delta'(1-z)\,,
\end{aligned} \tag{2.13}$$

where $z = 2\,v\cdot\hat{q} = 2E_\gamma/m_b$ and $\hat{q} = q/m_b$.

In the last step we have to calculate the forward matrix elements of the four operators given in (2.12) between B meson states. Since we expressed the leading operator O_b^μ in terms of the b quark fields of QCD we know its matrix element exactly,

$$\langle \bar{B}(p_B)|O_b^\mu|\bar{B}(p_B)\rangle = 2m_B v^\mu \,. \tag{2.14}$$

The evaluation of the matrix elements of the other operators involves the equation of motion of the effective theory, and leads to two additional matrix elements [17],

$$\lambda_1 = \frac{1}{2m_B}\langle \bar{B}(v)|\bar{b}_v (iD)^2 b_v|\bar{B}(v)\rangle \,, \qquad \lambda_2 = -\frac{1}{6m_B}\langle \bar{B}(v)|\bar{b}_v \frac{g_s}{2} G_{\mu\nu}\sigma^{\mu\nu} b_v|\bar{B}(v)\rangle \,. \tag{2.15}$$

The velocity dependent B meson states used here are related to the momentum dependent ones introduced in (2.4) by $|\bar{B}(p_B)\rangle = \sqrt{m_B}\,|\bar{B}(v)\rangle + O(1/m_b)$. The power correction to this relation is irrelevant for our calculation. While λ_1 and λ_2 are defined in the asymptotic HQET regime, in practical applications one deals with $\mu_\pi^2 = -\lambda_1 + O(1/m_b)$ and $\mu_G^2 = 3\lambda_2 + O(1/m_b)$, defined in terms of the m_b-finite QCD states appearing in (2.4).

Now we are in a position to calculate the contribution to the differential decay rate due to the electromagnetic dipole operator only. We obtain

$$\frac{d\Gamma_{77}}{dz} = \Gamma_{77}^{(0)}\left[c_0^{(0)} + c_{\lambda_1}^{(0)}\frac{\lambda_1}{2m_b^2} + c_{\lambda_2}^{(0)}\frac{\lambda_2(\mu)}{2m_b^2} + \frac{\alpha_s(\mu)}{4\pi}\left(c_0^{(1)} + c_{\lambda_1}^{(1)}\frac{\lambda_1}{2m_b^2} + c_{\lambda_2}^{(1)}\frac{\lambda_2(\mu)}{2m_b^2}\right)\right], \tag{2.16}$$

where

$$\Gamma_{77}^{(0)} = \frac{G_F^2\alpha_{\rm em}\bar{m}_b^2(\mu)m_b^3}{32\pi^4}\,|V_{tb}V_{ts}^*|^2|C_7^{\rm eff}(\mu)|^2 \,, \tag{2.17}$$

and

$$\begin{aligned} c_0^{(0)} &= \delta(1-z)\,, \qquad c_{\lambda_1}^{(0)} = \delta(1-z) - \delta'(1-z) - \frac{1}{3}\,\delta''(1-z)\,, \\ c_{\lambda_2}^{(0)} &= -9\,\delta(1-z) - 3\,\delta'(1-z)\,. \end{aligned} \tag{2.18}$$

This is in agreement with the well-known results given in [10]. The calculation of the Wilson coefficients $c_0^{(1)}$, $c_{\lambda_1}^{(1)}$ and $c_{\lambda_2}^{(1)}$ is the subject of the next section.

3 Next-to-Leading Order Coefficients

In order to determine the Wilson coefficients $c_0^{(1)}$, $c_{\lambda_1}^{(1)}$ and $c_{\lambda_2}^{(1)}$ we calculate the amputated Green functions corresponding to the Feynman diagrams shown in Fig. 2. The momentum assignments of the external lines are exactly the same as in the tree-level calculation performed in the last section. For the Green functions that contain a radiated soft gluon we apply the background-field formalism [18]. Furthermore, we work in the general R_ξ-gauge for the gluon propagator. Again we perform a Taylor expansion up to $O(k^2)$ for the diagrams without a radiated soft gluon and up to $O(k)$ and $O(r)$ for the diagrams with a radiated soft gluon, and refrain from using any on-shell relations for the b quark. We apply integration-by-parts techniques [19] to reduce all integrals to a few so-called master integrals and solve

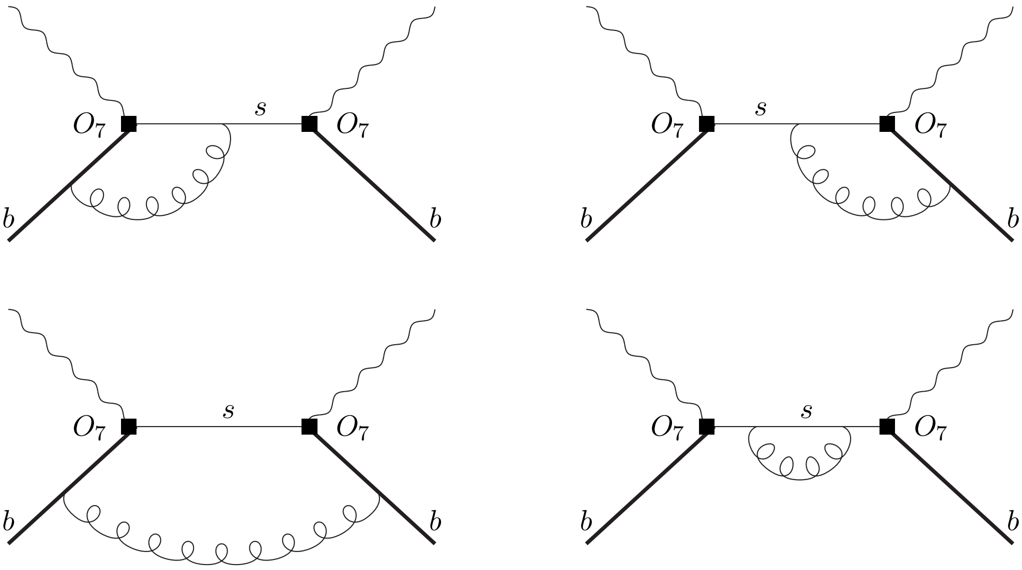

Figure 2: *One-loop diagrams contributing to the Wilson coefficients of the operators with dimension 3, 4 and 5. The momenta assignments of the external lines are the same as in figure 1. Sixteen additional diagrams with a gluon radiated off an internal line can be obtained from the diagrams shown here.*

the latter analytically. Ultraviolet as well as infrared divergences are handled by dimensional regularization. The ultraviolet divergences can be removed by appropriate renormalization. For the self-mixing of the operator O_7 we use the $\overline{\text{MS}}$-scheme,

$$Z_{m_b}^{\overline{\text{MS}}} Z_{77}^{\overline{\text{MS}}} = 1 + \frac{C_F}{\epsilon}\frac{\alpha_s}{4\pi} + \dots\,, \tag{3.1}$$

and for the field renormalization constant of the b quark we apply the on-shell scheme,

$$Z_b^{\text{OS}} = 1 - C_F\left(\frac{3}{\epsilon} + 4 + 6\ln\frac{\mu}{m_b}\right)\frac{\alpha_s}{4\pi} + \dots\,. \tag{3.2}$$

As far as the renormalization of the background gluon field G_μ^a is concerned, we only have to remember that $g_s G_\mu^a$ does not get renormalized. Finally, we extract the imaginary part of the amputated Green functions, replace the residual momenta k and r with covariant derivatives $iD_\mu - m_b v_\mu$ and the b quark spinors with b_v spinors via (2.11), and calculate the forward matrix elements of all operators between B meson states. Our result is then a linear combination of the three matrix elements introduced in (2.14) and (2.15) with coefficients that are ultraviolet finite but still contain infrared divergences,

$$f_0^\mu\left(z,\xi,\mu,\frac{1}{\epsilon_{IR}}\right)v_\mu + f_{\lambda_1}\left(z,\xi,\mu,\frac{1}{\epsilon_{IR}}\right)\frac{\lambda_1}{2m_b} + f_{\lambda_2}\left(z,\xi,\mu,\frac{1}{\epsilon_{IR}}\right)\frac{\lambda_2}{2m_b}\,. \tag{3.3}$$

We have also made explicit the dependence on the gauge parameter ξ. The infrared divergences are removed in the matching procedure.

We now turn our attention to the right-hand side of the matching equation (2.7). At the one-loop level it schematically looks like

$$-16\pi m_b \sum_{n=3}^{\infty} \frac{1}{m_b^{n-3}} \left[c^{(0)}_{\mathrm{dim\,n}} \langle O_{\mathrm{dim\,n}} \rangle_{\text{1-loop}} + \left(\frac{\alpha_s}{4\pi}\, c^{(1)}_{\mathrm{dim\,n}} + \delta Z_{\mathrm{dim\,n}} c^{(0)}_{\mathrm{dim\,n}} \right) \langle O_{\mathrm{dim\,n}} \rangle_{\text{tree}} \right] , \tag{3.4}$$

where $\langle O_{\mathrm{dim\,n}} \rangle_{\text{tree}}$ and $\langle O_{\mathrm{dim\,n}} \rangle_{\text{1-loop}}$ denote the tree-level and one-loop matrix elements of the operator $O_{\mathrm{dim\,n}}$ between B meson states, respectively, and $Z_{\mathrm{dim\,n}} = 1 + \delta Z_{\mathrm{dim\,n}}$ collects the Z-factors to render this expression ultraviolet finite. In the case at hand we only have to consider the one-loop matrix elements of the operators given in (2.12) since these are the only ones that have non-vanishing Wilson coefficients at the tree-level. The same holds for the tree-level matrix elements that are multiplied by Z-factors. Because of the Taylor expansion in the momenta k and r the one-loop matrix elements of the operators O_2^μ and $O_{3,4}^{\mu\nu}$ vanish in dimensional regularization and we only need to compute the one-loop matrix elements of O_b^μ. The operators that include a γ_5 can again be discarded due to parity considerations. For the renormalization constants, we use the on-shell scheme for the b and b_v spinors, and the $\overline{\mathrm{MS}}$ scheme for the operator renormalization,

$$\begin{aligned} [c_{b\mu} O_b^\mu]^{\text{bare}} &= Z_b^{\text{OS}}\, c_{b\mu} O_b^\mu\,, & [c_{2\mu\nu} O_2^{\mu\nu}]^{\text{bare}} &= Z_{b_v}^{\text{OS}} Z_{\text{kin}}^{\overline{\text{MS}},\mu\nu\alpha\beta}\, c_{2\mu\nu} O_{2\alpha\beta}\,, \\ [c_{1\mu} O_1^\mu]^{\text{bare}} &= Z_{b_v}^{\text{OS}}\, c_{1\mu} O_1^\mu\,, & [c_{3\mu\nu} O_3^{\mu\nu}]^{\text{bare}} &= Z_{b_v}^{\text{OS}} Z_{\text{chromo}}^{\overline{\text{MS}},\mu\nu\alpha\beta}\, c_{3\mu\nu} O_{3\alpha\beta}\,. \end{aligned} \tag{3.5}$$

A simple one-loop calculation yields

$$\begin{aligned} Z_{\text{kin}}^{\overline{\text{MS}},\mu\nu\alpha\beta} &= -C_F \frac{3-\xi}{\epsilon} \left(g^{\mu\nu} - 2v^\mu v^\nu\right) v^\alpha v^\beta \frac{\alpha_s}{4\pi} + \ldots \\ Z_{\text{chromo}}^{\overline{\text{MS}},\mu\nu\alpha\beta} &= \frac{C_A}{\epsilon} \left(g^{\mu\alpha} - v^\mu v^\alpha\right) g^{\nu\beta} \frac{\alpha_s}{4\pi} + \ldots\,. \end{aligned} \tag{3.6}$$

The Feynman gauge is obtained by setting $\xi = 1$. The on-shell renormalization constant $Z_{b_v}^{\text{OS}}$ can be set equal to 1 since the $O(\alpha_s)$-corrections to the selfenergy of the b_v field depends on no other scale than the renormalization scale μ.

Requiring equality between (3.3) and (3.4), and solving for $c^{(1)}_{\mathrm{dim\,n}}$, we obtain infrared finite and gauge independent expressions. Writing the coefficients as follows,

$$\begin{aligned} &c_0^{(1)} = C_F c_0^{(1,\mathrm{F})}\,, \qquad c_{\lambda_1}^{(1)} = C_F c_{\lambda_1}^{(1,\mathrm{F})}\,, \\ &c_{\lambda_2}^{(1)} = C_F \left(c_{\lambda_2}^{(1,\mathrm{F})} + \Delta c_{\lambda_2}^{(1,\mathrm{F})} \right) + C_A \left(c_{\lambda_2}^{(1,\mathrm{A})} + \Delta c_{\lambda_2}^{(1,\mathrm{A})} \right)\,, \end{aligned} \tag{3.7}$$

our final results are given by

$$\begin{aligned} c_0^{(1,\mathrm{F})} = &-\left(5 + \frac{4\pi^2}{3}\right) \delta(1-z) - 7 \left[\frac{1}{1-z}\right]_+ \\ &-4 \left[\frac{\ln(1-z)}{1-z}\right]_+ + 7 + z - 2z^2 - 2(1+z)\ln(1-z) - 4\, c_0^{(0)} \ln\frac{\mu}{m_b}\,, \end{aligned} \tag{3.8}$$

$$c_{\lambda_1}^{(1,\mathrm{F})} = -\frac{2}{3}\left(15 + 2\pi^2\right) \delta(1-z) - \frac{4}{3}\left(3 - \pi^2\right) \delta'(1-z)$$

$$
-\left(\frac{7}{6}-\frac{4\pi^2}{9}\right)\delta''(1-z)+\frac{2}{3}\left[\frac{1}{(1-z)^3}\right]_+-\frac{5}{3}\left[\frac{1}{(1-z)^2}\right]_+
$$

$$
-9\left[\frac{1}{1-z}\right]_+ +\frac{8}{3}\left[\frac{\ln(1-z)}{(1-z)^3}\right]_+ -4\left[\frac{\ln(1-z)}{(1-z)^2}\right]_+
$$

$$
-4\left[\frac{\ln(1-z)}{1-z}\right]_+ +10+\frac{7}{3}z-\frac{2}{3}(4+3z)\ln(1-z)-4\,c_{\lambda_1}^{(0)}\ln\frac{\mu}{m_b}\,, \tag{3.9}
$$

$$
c_{\lambda_2}^{(1,\mathrm{F})}=\left(41+\frac{20\pi^2}{3}\right)\delta(1-z)+4\left(1+\pi^2\right)\delta'(1-z)
$$

$$
-9\left[\frac{1}{(1-z)^2}\right]_+ +15\left[\frac{1}{1-z}\right]_+ -12\left[\frac{\ln(1-z)}{(1-z)^2}\right]_+
$$

$$
+20\left[\frac{\ln(1-z)}{1-z}\right]_+ -6+55z+2(22-17z)\ln(1-z)-4\,c_{\lambda_2}^{(0)}\ln\frac{\mu}{m_b}\,, \tag{3.10}
$$

$$
c_{\lambda_2}^{(1,\mathrm{A})}=2\left(7-\frac{8\pi^2}{3}\right)\delta(1-z)+4\,\delta'(1-z)-4\left[\frac{1}{(1-z)^2}\right]_+
$$

$$
+2\left[\frac{1}{1-z}\right]_+ -16\left[\frac{\ln(1-z)}{1-z}\right]_+ +2-6z+4(1+3z)\ln(1-z)\,. \tag{3.11}
$$

The contributions $\Delta c_{\lambda_2}^{(1,\mathrm{F})}$ and $\Delta c_{\lambda_2}^{(1,\mathrm{A})}$ are a consequence of the application of the equation of motion of the effective theory in the evaluation of the matrix elements of the operators given in (2.12). Their explicit expressions read

$$
\Delta c_{\lambda_2}^{(1,\mathrm{F})}=2\,c_{\lambda_2}^{(0)}\,,\qquad \Delta c_{\lambda_2}^{(1,\mathrm{A})}=2\left(1+\ln\frac{\mu}{m_b}\right)c_{\lambda_2}^{(0)}. \tag{3.12}
$$

We note that the coefficient $c_0^{(1)}$ agrees with the well-known result of [20]. The coefficients $c_{\lambda_1}^{(1)}$ and $c_{\lambda_2}^{(1)}$ have been calculated here for the first time. For the first one we confirm the expected relation

$$
\int_0^1 dz\, c_0^{(n)}=-\int_0^1 dz\, c_{\lambda_1}^{(n)},\qquad n=0,1,2,\ldots\,. \tag{3.13}
$$

Not surprisingly, $c_{\lambda_{1,2}}^{(1)}$ diverge more strongly than $c_0^{(1)}$ at the endpoint. The plus-distributions introduced above follow the prescription

$$
\int_0^1 dz\left[\frac{\ln^n(1-z)}{(1-z)^m}\right]_+ f(z)
$$

$$
=\int_0^1 dz\,\frac{\ln^n(1-z)}{(1-z)^m}\left\{f(z)-\sum_{p=0}^{m-1}(-1)^p\frac{(1-z)^p}{p!}\left[\frac{\partial^p f(z)}{\partial z^p}\right]_{z=1}\right\}, \tag{3.14}
$$

where $f(z)$ is an arbitrary test function which is regular at $z=1$, and $n\geq 0$, $m\geq 1$. In case the integration does not include the endpoint, we have ($c<1$)

$$
\int_0^c dz\left[\frac{\ln^n(1-z)}{(1-z)^m}\right]_+ f(z)=\int_0^c dz\,\frac{\ln^n(1-z)}{(1-z)^m}f(z)\,. \tag{3.15}
$$

We remark that the μ-dependence of $c_0^{(1,\mathrm{F})}$ has its origin in the $\overline{\mathrm{MS}}$ renormalization of the electromagnetic dipole operator. The same is true for the μ-dependence of $c_{\lambda_1}^{(1,\mathrm{F})}$ and $c_{\lambda_2}^{(1,\mathrm{F})}$. It reflects the fact that λ_1 is not renormalized to all orders in perturbation theory. On the other hand, for the coefficient of λ_2 there is an additional μ-dependence present in $c_{\lambda_2}^{(1,\mathrm{A})}$. It originates in the $\overline{\mathrm{MS}}$ renormalization of λ_2. Indeed, using the renormalization group equation

$$\mu\frac{d}{d\mu}\,c_{\lambda_2}^{(0)}(\mu) = \frac{\alpha_s(\mu)}{4\pi}\,\gamma_{\lambda_2}^{(0)}c_{\lambda_2}^{(0)}(\mu)\,, \tag{3.16}$$

where $\gamma_{\lambda_2}^{(0)} = 2C_A$ [21], we recover the μ-dependence of $\Delta c_{\lambda_2}^{(1,\mathrm{A})}$.

In the remainder of this section we discuss the numerical impact of the new contributions on the total decay rate of $\bar{B}\to X_s\gamma$ as well as on its first and second moments. For the O_7 contribution to the total decay rate we find (using numerical values for C_F and C_A)

$$\begin{aligned}
\Gamma_{77}|_{E_\gamma>E_0} &= \int_{z0}^1 dz\,\frac{d\Gamma_{77}}{dz} = \Gamma_{77}^{(0)}\bigg[1+\frac{\lambda_1-9\lambda_2(\mu)}{2m_b^2}+\frac{\alpha_s(\mu)}{4\pi}\bigg(\frac{16}{9}\bigg[4-\pi^2-3\ln\frac{\mu}{m_b}\bigg]\\
&-\frac{8}{3}\ln^2(1-z_0)-\frac{4}{3}\left(10-2z_0-z_0^2\right)\ln(1-z_0)-\frac{4}{9}z_0\left(30+3z_0-2z_0^2\right)\\
&+\frac{\lambda_1}{2m_b^2}\bigg\{\frac{16}{9}\bigg[4-\pi^2-3\ln\frac{\mu}{m_b}\bigg]-\frac{8}{3}\ln^2(1-z_0)-\frac{4z_0\left(30-63z_0+31z_0^2+5z_0^3\right)}{9(1-z_0)^2}\\
&-\frac{4\left(30-72z_0+51z_0^2-2z_0^3-3z_0^4\right)}{9(1-z_0)^2}\ln(1-z_0)\bigg\}+\frac{\lambda_2(\mu)}{2m_b^2}\bigg\{-\frac{2}{9}\bigg[87+32\pi^2\\
&+27\ln\frac{\mu}{m_b}\bigg]-\frac{32}{3}\ln^2(1-z_0)+\frac{2\left(162-244z_0+113z_0^2-7z_0^3\right)}{3(1-z_0)}\ln(1-z_0)\\
&+\frac{2z_0\left(54-49z_0+15z_0^2\right)}{1-z_0}\bigg\}\bigg)\bigg]+\dots\,.
\end{aligned} \tag{3.17}$$

The ellipses denote higher order terms in α_s and Λ/m_b and $z_0 = 2E_0/m_b$. In order to get a rough estimate of the size of the power-corrections at $O(\alpha_s)$ we set $\mu = m_b$ and use the numerical values $\alpha_s(m_b) = 0.22$, $m_b = 4.6\,\mathrm{GeV}$, $\lambda_1 = -0.4\,\mathrm{GeV}^2$ and $\lambda_2 = 0.12\,\mathrm{GeV}^2$ to obtain $\Gamma_{77}|_{E_\gamma>1.8\,\mathrm{GeV}}/\Gamma_{77}^{(0)} = 0.763-0.007 = 0.756$, a -0.9% effect. In the intermediate step we have singled out the new $O(\alpha_s\Lambda^2/m_b^2)$ contributions. The effect of the new corrections on the rate varies with the cut, from -0.4% at $E_0 = 0$ to -0.9% at $E_0 = 1.8\,\mathrm{GeV}$. For values of $E_0 > 1.8\,\mathrm{GeV}$ the corrections to the rate are significant, a -3% effect for $E_0 = 2\,\mathrm{GeV}$, however, such high values of E_0 are well outside the range of applicability of the local OPE. Moreover, at high E_0 part of the new effect is implicitly contained in the approach of [7].

The truncated n-th moment is defined through

$$\langle E_\gamma^n\rangle_{E_\gamma>E_0} = \left(\frac{m_b}{2}\right)^n\left(\int_{z_0}^1 dz\,z^n\,\frac{d\Gamma_{77}}{dz}\right)\Big/\left(\int_{z_0}^1 dz\,\frac{d\Gamma_{77}}{dz}\right)\,. \tag{3.18}$$

After expanding in α_s, the first moment reads

$$\langle E_\gamma\rangle_{E_\gamma>E_0} = \frac{m_b}{2}\bigg[1-\frac{\lambda_1+3\lambda_2(\mu)}{2m_b^2}+\frac{\alpha_s(\mu)}{4\pi}\bigg(-\frac{46}{27}+\frac{8}{9}\left(8-9z_0+z_0^3\right)\ln(1-z_0)$$

$$
+\frac{2}{27}\,z_0\left(96-60z_0-22z_0^2+9z_0^3\right)+\frac{\lambda_1}{2m_b^2}\left\{-\frac{2z_0\,(96-120z_0-52z_0^2+25z_0^3-9z_0^4)}{27(1-z_0)}\right.
$$

$$
\left.+\frac{46}{27}-\frac{16\,(4-7z_0+z_0^3)}{9(1-z_0)}\ln(1-z_0)\right\}+\frac{\lambda_2(\mu)}{2m_b^2}\left\{-\frac{1130}{27}-18\ln\frac{\mu}{m_b}+\frac{4}{9}\,(80-33z_0\right.
$$

$$
\left.\left.\left.-81z_0^2+25z_0^3\right)\ln(1-z_0)+\frac{2}{27}\,z_0\left(480+42z_0-425z_0^2+81z_0^3\right)\right\}\right)\Bigg]+\ldots, \tag{3.19}
$$

while the second central moment is given by

$$
\langle E_\gamma^2\rangle-\langle E_\gamma\rangle^2\big|_{E_\gamma>E_0}=-\frac{\lambda_1}{12}+\frac{\alpha_s(\mu)}{4\pi}\left\{\frac{m_b^2}{4}\left[-\frac{2}{9}\,(1-z_0)^2\left(17-2z_0-3z_0^2\right)\ln(1-z_0)\right.\right.
$$

$$
\left.+\frac{1}{270}\,(1-z_0)^2\left(61-898z_0-207z_0^2+144z_0^3\right)\right]+\frac{\lambda_1}{8}\left[\frac{8}{27}\left(2-27z_0-7z_0^3\right)\ln(1-z_0)\right.
$$

$$
\left.+\frac{4}{405}\left(169+60z_0-780z_0^2-385z_0^3+225z_0^4-54z_0^5\right)\right]+\frac{\lambda_2(\mu)}{8}\left[\frac{1}{9}\,(43-84z_0\right.
$$

$$
+258z_0^2-292z_0^3+75z_0^4)\ln(1-z_0)-\frac{1}{540}\left(707-2580z_0+3750z_0^2-13820z_0^3\right.
$$

$$
\left.\left.\left.+14535z_0^4-2592z_0^5\right)\right]\right\}+\ldots. \tag{3.20}
$$

Using the same numerical input as above we obtain, for $E_0=1.8\,\mathrm{GeV}$, $\langle E_\gamma\rangle=(m_b/2)(0.978-0.001)=2.246\,\mathrm{GeV}$ and $\langle E_\gamma^2\rangle-\langle E_\gamma\rangle^2=0.0422-0.0036=0.0386\,\mathrm{GeV}^2$, where we have again singled out the contribution of $O(\alpha_s\Lambda^2/m_b^2)$ in the intermediate steps. Their effect on the truncated first and second central moment is of around -0.1% and -8.5%, respectively. For $E_0\in[0,1.8]\,\mathrm{GeV}$, the effect on the first moment varies between -0.2% and -0.1%, whereas that on the second central moment varies between -3.5% and -8.5%.

Fig. 3 shows the ratios of NLO to leading order coefficients of $\lambda_{1,2}$ in the rate and in the first two moments as a function of the cut E_0 using the same inputs as above. The NLO corrections to λ_2 are close to 20%. Note that in the right panel we have not shown a curve for the second central moment since λ_2 has a vanishing leading order coefficient.

The second moment represents a powerful constraint on the kinetic expectation value $\mu_\pi^2=-\lambda_1+O(1/m_b)$. Since the $O(\alpha_s)$ correction decreases its coefficient by 5 to 9% in the range of cuts between 0 and 1.8 GeV, while the $O(\alpha_s\lambda_2/m_b^2)$ corrections are much smaller, we expect to extract a higher value of μ_π^2 from radiative moments once the new corrections are included. Indeed, using α_s at a more appropriate scale of order 1-2 GeV, the extracted μ_π^2 gets shifted by approximately +10%. Analogously, the small correction to the first moment leads to a roughly 10 MeV positive shift in m_b.

All the above expressions refer to the on-shell scheme for m_b and λ_1, which is inherent in the HQET calculation. In practical applications however one adopts short-distance definitions of these parameters, as in the kinetic scheme [22]. In this scheme the new corrections are identical, but they additionally induce small $O(\alpha_s^2\mu_{\rm kin}^2/m_b^2)$ contributions. Therefore, the above rough estimates hold in the kinetic scheme as well. A complete phenomenological analysis of the moments, including all the available contributions, will be presented elsewhere.

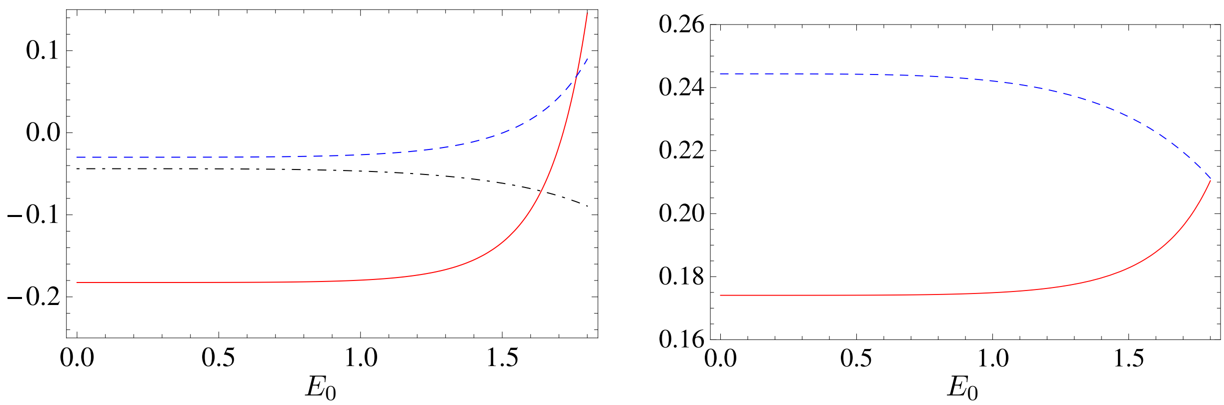

Figure 3: *Ratio of NLO to leading order coefficients of λ_1 (left) and λ_2 (right) in the rate (red solid curves), the first moment (blue dashed curves) and the second moment (black dash-dotted curve) as a function of E_0. See text for more details.*

4 Conclusions

We have computed the NLO contributions to the Wilson coefficients of dimension five operators relevant for inclusive radiative B decays. Our results allow for a more precise evaluation of the moments of the photon distribution and will improve the determination of m_b and of the kinetic expectation value, μ_π^2, from radiative moments. We have estimated that the new contributions shift the value of μ_π^2 extracted from the radiative moments by approximately $+10\%$ and that of m_b by roughly $+10\,\mathrm{MeV}$. The effect on the $\bar{B} \to X_s\gamma$ rate is below 1% for $E_0 < 1.8\,\mathrm{GeV}$.

We have performed the calculation analytically, using an off-shell matching procedure, a method that can be applied to inclusive semileptonic decays as well. The $O(\alpha_s\mu_\pi^2/m_b^2)$ corrections to the moments of $B \to X_c\ell\nu$ have been computed numerically [23], however, the $O(\alpha_s\mu_G^2/m_b^2)$ corrections are not yet known. We also believe that an analytical result might be easier to implement in the fitting codes.

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