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Matching QCD and HQET heavy–light currents at three loops

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Abstract

We consider the currents formed by a heavy and a light quark within Quantum Chromodynamics and compute the matching to Heavy Quark Effective Theory to three-loop accuracy. As an application we obtain the third-order perturbative corrections to ratios of B-meson decay constants.

1 Introduction

Quite often there are problems within Quantum Chromodynamics (QCD) involving a single heavy quark with momentum

$$p = mv + k\,, \tag{1.1}$$

where m is the on-shell heavy-quark mass and $v^2 = 1$. In situations when the characteristic residual momentum is small ($|k^\mu| \ll m$) and light quarks and gluons have small momenta ($|k_i^\mu| \ll m$) it is possible to use a simpler effective field theory — Heavy Quark Effective Theory (HQET) (see, e.g., the review [1] and the textbooks [2,3]). Its Lagrangian is a series in $1/m$. QCD operators are also given by series in $1/m$ in terms of HQET operators. Here we shall consider $\overline{\mathrm{MS}}$ renormalized heavy–light QCD quark currents

$$j(\mu) = Z_j^{-1}(\mu) j_0\,, \quad j_0 = \bar{q}_0 \Gamma Q_0\,, \tag{1.2}$$

where Γ is a Dirac matrix, and the index 0 means bare quantities. They can be expressed via operators in HQET

$$j(\mu) = C_\Gamma(\mu)\tilde{\jmath}(\mu) + \frac{1}{2m}\sum_i B_i(\mu)O_i(\mu) + \mathcal{O}\left(\frac{1}{m^2}\right) , \tag{1.3}$$

where

$$\tilde{\jmath}(\mu) = \tilde{Z}_j^{-1}(\mu)\tilde{\jmath}_0 , \quad \tilde{\jmath}_0 = \bar{q}_0\Gamma Q_{v0} , \tag{1.4}$$

Q_{v0} is the bare HQET heavy-quark field satisfying $\rlap/v Q_{v0} = Q_{v0}$, and O_i are dimension-4 HQET operators with appropriate quantum numbers. We shall not discuss $1/m$ corrections in this paper; our main subject is the matching coefficients $C_\Gamma(\mu)$.

The coefficients $C_\Gamma(\mu)$ have been calculated at one-loop order in the pioneering paper [4]. At two loops, they were calculated in Ref. [5], and corrected in Ref. [6].[1] All-order results in the large-β_0 limit were obtained in [5] (see also [7]). Asymptotics of perturbative series were investigated in a model-independent way in Ref. [8].

In the present paper we calculate the matching coefficients C_Γ up to three loops. These coefficients are useful for obtaining matrix elements of QCD currents (such as f_B) from results of lattice HQET simulations (see, e.g., Refs. [9,10]) or HQET sum rules (see, e.g., Refs. [11,1]).

2 Matching

There are eight Dirac structures giving non-vanishing quark currents in four dimensions:

$$\begin{aligned}\Gamma = 1 , \quad & \rlap/v , \quad \gamma_\perp^\alpha , \quad \gamma_\perp^\alpha \rlap/v , \\ & \gamma_\perp^{[\alpha}\gamma_\perp^{\beta]} , \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta]}\rlap/v , \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta}\gamma_\perp^{\gamma]} , \quad \gamma_\perp^{[\alpha}\gamma_\perp^{\beta}\gamma_\perp^{\gamma]}\rlap/v ,\end{aligned} \tag{2.1}$$

where $\gamma_\perp^\alpha = \gamma^\alpha - \rlap/v v^\alpha$, and square brackets mean antisymmetrization. The last four of them can be obtained from the first four by multiplying by the 't Hooft–Veltman γ_5^{HV}. We are concerned with flavour non-singlet currents only, therefore, we may also use the anticommuting γ_5^{AC} (there is no anomaly). The currents renormalized at a scale μ with different prescriptions for γ_5 are related by [12]

[1] The results incorporating this correction are given by formulae (5.65–5.68) and Table 5.1 in [3]. Note a misprint in this table: in the row for $\Gamma = \gamma^1$, the term with C_F in the square bracket, $-1453/48$, should be $+1453/48$.

$$
\begin{aligned}
\left(\bar{q}\gamma_5^{\mathrm{AC}}Q\right)_\mu &= Z_P(\mu)\left(\bar{q}\gamma_5^{\mathrm{HV}}Q\right)_\mu\,,\\
\left(\bar{q}\gamma_5^{\mathrm{AC}}\gamma^\alpha Q\right)_\mu &= Z_A(\mu)\left(\bar{q}\gamma_5^{\mathrm{HV}}\gamma^\alpha Q\right)_\mu\,,\\
\left(\bar{q}\gamma_5^{\mathrm{AC}}\gamma^{[\alpha}\gamma^{\beta]}Q\right)_\mu &= Z_T(\mu)\left(\bar{q}\gamma_5^{\mathrm{HV}}\gamma^{[\alpha}\gamma^{\beta]}Q\right)_\mu\,,
\end{aligned}
\tag{2.2}
$$

where the finite renormalization constants $Z_{P,A,T}$ can be reconstructed from the differences of the anomalous dimensions of the currents. Multiplying Γ by γ_5^{AC} does not change the anomalous dimension. In the case of $\Gamma = \gamma^{[\alpha}\gamma^{\beta]}$, multiplying it by γ_5^{HV} just permutes its components, and also does not change the anomalous dimension, therefore,

$$
Z_T(\mu) = 1\,;
\tag{2.3}
$$

$Z_{P,A}(\mu)$ are known up to three loops [12].

The anomalous dimension of the HQET current (1.4) does not depend on the Dirac structure Γ. Therefore, there are no factors similar to $Z_{P,A}$ in HQET. Multiplying Γ by γ_5^{AC} does not change the matching coefficient. Therefore, the matching coefficients for the currents in the second row of (2.1) are not independent: they can be obtained from those for the first row. In the v rest frame, where $\rlap/v = \gamma^0$, we have

$$
\begin{aligned}
Z_P(\mu) &= \frac{C_{\gamma_5^{\mathrm{AC}}}(\mu)}{C_{\gamma_5^{\mathrm{HV}}}(\mu)} = \frac{C_1(\mu)}{C_{\gamma^0\gamma^1\gamma^2\gamma^3}(\mu)}\,,\\
Z_A(\mu) &= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0}(\mu)}{C_{\gamma_5^{\mathrm{HV}}\gamma^0}(\mu)} = \frac{C_{\gamma^0}(\mu)}{C_{\gamma^1\gamma^2\gamma^3}(\mu)}\\
&= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^3}(\mu)}{C_{\gamma_5^{\mathrm{HV}}\gamma^3}(\mu)} = \frac{C_{\gamma^3}(\mu)}{C_{\gamma^0\gamma^1\gamma^2}(\mu)}\,,\\
Z_T(\mu) &= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0\gamma^1}(\mu)}{C_{\gamma_5^{\mathrm{HV}}\gamma^0\gamma^1}(\mu)} = \frac{C_{\gamma^0\gamma^1}(\mu)}{C_{\gamma^2\gamma^3}(\mu)}\\
&= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^2\gamma^3}(\mu)}{C_{\gamma_5^{\mathrm{HV}}\gamma^2\gamma^3}(\mu)} = \frac{C_{\gamma^2\gamma^3}(\mu)}{C_{\gamma^0\gamma^1}(\mu)} = 1\,.
\end{aligned}
\tag{2.4}
$$

In particular, two matching coefficients are equal:

$$
C_{\gamma_\perp \rlap/v}(\mu) = C_{\gamma_\perp^{[\alpha}\gamma_\perp^{\beta]}}(\mu)\,.
\tag{2.5}
$$

In the following we shall consider only the matching coefficients for the first four Dirac structures in (2.1).

In order to find the matching coefficients $C_\Gamma(\mu)$, we equate on-shell matrix elements of the left- and right-hand side of Eq. (1.3). They are obtained by considering transitions of the heavy quark with momentum $p = mv + k$ (1.1) to the light quark with momentum k_q:

$$\langle q(k_q)|j(\mu)|Q(mv+k)\rangle = C_\Gamma(\mu)\langle q(k_q)|\tilde{\jmath}(\mu)|Q_v(k)\rangle + \mathcal{O}\left(\frac{k}{m},\frac{k_q}{m}\right) . \tag{2.6}$$

The on-shell matrix elements are [2]

$$\begin{aligned}
\langle q(k_q)|j(\mu)|Q(p)\rangle &= \bar{u}_q(k_q)\Gamma(p,k_q)u(p)\, Z_j^{-1}(\mu) Z_Q^{1/2} Z_q^{1/2} , \\
\langle q(k_q)|\tilde{\jmath}(\mu)|Q_v(k)\rangle &= \bar{u}_q(k_q)\tilde{\Gamma}(k,k_q)u_v(k)\, \tilde{Z}_j^{-1}(\mu) \tilde{Z}_Q^{1/2} \tilde{Z}_q^{1/2} ,
\end{aligned} \tag{2.7}$$

where $\Gamma(p,k_q)$ and $\tilde{\Gamma}(k,k_q)$ are the bare vertex functions, Z_Q and Z_q are the on-shell wave-function renormalization constants of the heavy and the light quark in QCD, $\tilde{Z}_Q$ is the on-shell wave-function renormalization constant of the HQET quark field Q_v, and $\tilde{Z}_q$ differs from Z_q because there are no Q loops in HQET. The difference between $u(mv+k)$ and $u_v(k)$ is of order k/m, and can be neglected. It is most convenient to use $k = k_q = 0$, then the $\mathcal{O}(1/m)$ term is absent. The QCD vertex has two Dirac structures:

$$\Gamma(mv,0) = \Gamma \cdot (A + B\rlap{/}v) .$$

This leads to

$$\bar{u}(0)\Gamma(mv,0)u(mv) = \bar{\Gamma}(mv,0)\,\bar{u}(0)\Gamma u(mv) \quad \text{with} \quad \bar{\Gamma}(mv,0) = A + B .$$

The scalar vertex function $\bar{\Gamma}(mv,0)$ can be obtained by multiplying the diagrams by a projector and taking the trace. For the first two Dirac structures in (2.1), the projector $(\rlap{/}v+1)$ can be used; for the next two structures, $(\rlap{/}v+1)\gamma_\alpha$. The HQET vertex has just one Dirac structure. Therefore,

$$C_\Gamma(\mu) = \frac{\bar{\Gamma}(mv,0) Z_j^{-1}(\mu) Z_Q^{1/2} Z_q^{1/2}}{\tilde{\Gamma}(0,0)\tilde{Z}_j^{-1}(\mu)\tilde{Z}_Q^{1/2}\tilde{Z}_q^{1/2}} . \tag{2.8}$$

Here m is the on-shell mass of the heavy quark (because the external heavy quark with $p^2 = m^2$ should be on its mass shell). Therefore, mass-counterterm vertices have to be taken into account on all Q lines.

If all flavours except Q are massless, all loop corrections to $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ contain no scale and hence vanish: $\tilde{\Gamma}(0,0) = 1$, $\tilde{Z}_Q = 1$, $\tilde{Z}_q = 1$. If there is another massive flavour (charm in b-quark HQET), this is no longer so. The vertex $\tilde{\Gamma}(0,0)$ and $\tilde{Z}_Q$ have been calculated up to three loops in Ref. [13].

[2] Both on-shell matrix elements of the renormalized currents are UV-finite. Both contain IR divergences, which are the same on the left- and right-hand sides of (2.6), yielding a finite $C_\Gamma(\mu)$.

The massless-quark on-shell wave-function renormalization constant in the presence of a massive flavour has been calculated up to three loops in Ref. [14] (an explicit expression can be found in Appendix A of Ref. [13]). The three-loop renormalization of the HQET current $\tilde{Z}_j$ has been calculated in Ref. [15].

If all flavours except Q are massless, $\Gamma(mv,0)$, Z_Q, and Z_q contain a single scale m. The on-shell heavy-quark wave-function renormalization constant Z_Q has been calculated at three loops in [16] (and confirmed in Ref. [17]); Z_q has been found in Ref. [14]. The vertex $\Gamma(mv,0)$ is the subject of the present paper. If there is another flavour with a non-zero mass $m_c < m$, there are two scales, and calculations become more difficult. The renormalization constant Z_Q has been calculated in this case, up to three loops, in Ref. [18] (the master integrals appearing in this case are discussed in Ref. [19]). The vertex $\Gamma(mv,0)$ and Z_q with two masses are considered in Sect. 3 and 4. The three-loop renormalization of the QCD currents with all Dirac structures Γ has been obtained in Ref. [20].

The QCD quantities $\Gamma(mv,0)$, Z_Q, and Z_q in the numerator of (2.8) are calculated in terms of $\alpha_s^0 = g_0^2/(4\pi)^{1-\varepsilon}$, the bare coupling of the n_f-flavour QCD. If there is another massive flavour, say charm, they also contain its bare mass m_{c0}; we re-express it via the on-shell mass m_c. These quantities do not involve μ. The $\overline{\text{MS}}$ renormalization constant $Z_j^{-1}(\mu)$ is expressed in terms of $\alpha_s^{(n_f)}(\mu)$. The HQET quantities $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q$, and $\tilde{Z}_q$ in the denominator of (2.8) are calculated via $\alpha_s^{0\prime}$ and m'_{c0}, the bare coupling and the bare c-quark mass in the n'_f-flavour QCD [3] (we re-express m'_{c0} via the on-shell mass m_c, which is the same in both theories). The $\overline{\text{MS}}$ renormalization constant $\tilde{Z}_j^{-1}(\mu)$ is expressed in terms of $\alpha_s^{(n'_f)}(\mu)$. To combine all these quantities in (2.8), we re-express them via the coupling $\alpha_s^{(n'_f)}(\mu)$ using the decoupling relation [14] (an explicit expression for $\alpha_s^{(n_f)}(\mu)$ via $\alpha_s^{(n'_f)}(\mu)$, including the necessary terms with positive powers of ε, is given in Eq. (12) of Ref. [21]).

There exists an exact relation [5] between the matching coefficients $C_1(\mu)$ and $C_{\not{v}}(\mu)$. Namely, the renormalized vector and scalar currents are related by

$$i\partial_\alpha j^\alpha = m(\mu) j(\mu)\,, \tag{2.9}$$

where $m(\mu)$ is the $\overline{\text{MS}}$ mass of the heavy quark Q. Taking the on-shell matrix element of this equality between the heavy quark with $p = mv$ and the light quark with $k_q = 0$ and re-expressing both QCD matrix elements via the matrix

[3] To keep track of flavours in quark loops, we introduce the number n_h of heavy flavours with mass m and the number n_m of massive flavours with mass m_c, so that $n_f = n_l + n_m + n_h$ and $n'_f = n_l + n_m$. In reality, $n_h = 1$ and $n_m = 1$. If the c quark is considered massless, we can also include it in the number of light flavours n_l, and set $n_m = 0$.

element of the HQET current with $\Gamma = 1$, we obtain

$$mC_{\not{v}}(\mu) = m(\mu)C_1(\mu)\,. \tag{2.10}$$

The ratio $m(\mu)/m$ has been calculated at three loops in Refs. [22,23]. Comparing $C_{\not{v}}(\mu)/C_1(\mu)$ with the analytical result [23] for $m(\mu)/m$ provides a strong check of our calculations. For $m_c \neq 0$, this ratio has been calculated in Ref. [18].

3 Bare vertex functions

The calculation of $\bar{\Gamma}(mv, 0)$ at $m_c = 0$ is a single-scale problem. The calculation is almost completely automated and similar to Refs. [17,21]. The Feynman diagrams were generated with `QGRAF` [24] and classified into various topologies with the help of `q2e` and `exp` [25]. Some sample diagrams are shown in Fig. 1 (the first four diagrams). Scalar Feynman integrals were reduced to master integrals using integration-by-parts identities [26]. This was done in two independent ways: using the `Form` [27] package `SHELL3` [16] and the `C++` program `Crusher` [28] implementing the Laporta algorithm [29]. The master integrals for the case of a massless charm quark are known from Ref. [16] (see also comments in Ref. [17]). As an independent check the automatic setup described in Ref. [30] has been used and the bare vertex functions have been checked numerically with the help of `FIESTA` [31]. The results for the bare vertex functions with all four Dirac structures Γ can be found on the web page [32].

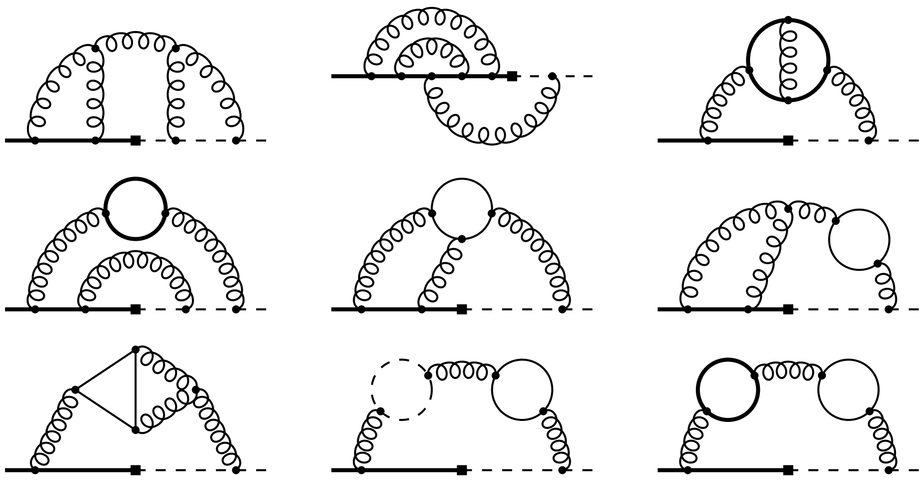

Fig. 1. Sample Feynman diagrams for the matching coefficient. Thick, thin and dashed lines denote heavy, light and massless quarks, respectively; curly lines are gluons. The square denotes the vertex of the heavy–light current.

Next we study the influence of a non-zero c-quark mass, $m_c < m$, on the b-quark matching coefficients C_Γ. Then c-quark loops exist both in the full

QCD and in the b-quark HQET. The full-theory quantities in the numerator of (2.8) depend on the ratio of the on-shell quark masses,

$$x = \frac{m_c}{m} \,. \tag{3.1}$$

Some sample diagrams contributing to $\bar{\Gamma}(mv, 0)$ are shown in Fig. 1 (all diagrams except the first four ones depend on x). The HQET quantities in the denominator of (2.8) contain a single scale m_c.

From the technical point of view the calculation is similar to Refs. [18,33]. We have used `Crusher` [28] for the reduction. The master integrals are known from Refs. [18,19], see also Ref. [34]. Most of the needed terms of their ε expansions are known analytically, in terms of Harmonic Polylogarithms [35,36] (HPLs) of x (the status of these expansions is summarized in Tables 1–4 of Ref. [19]). From the requirement of cancellation of $1/\varepsilon$ poles in the matching coefficients (Sect. 5) we were able to find exact analytical expressions (in terms of HPLs) for the $\mathcal{O}(\varepsilon^0)$ terms of the master integrals 5.2 and 5.2a (Fig. 8 in [19]); formerly, they were known only as a truncated series in x. The corresponding entry in Table 3 of Ref. [19] needs updating. We do not present these long expressions here, they can be found at [32].

We have checked the m_c dependent results by taking the limit $x \to 0$ and reproducing the n_l part of the results given in Sect. 5. Another check is taking the limit $x \to 1$. If we set $n_h = 0$ and re-express the renormalized matching coefficients C_Γ via $\alpha_s^{(n_l)}$, the results of Sect. 5 with the substitution $n_m \to n_h$ are reproduced.

4 Wave-function renormalization of massless quarks

The last ingredient of Eq. (2.8) which we had to calculate is the on-shell wave-function renormalization constant Z_q of a massless quark. The result for $m_c = 0$ can be extracted from Ref. [14], however, we have performed an independent calculation. The m_c-dependent part is a new result. The calculation is similar to that of Ref. [18], where Z_Q has been calculated. The integrals contributing to Z_q reduce to tadpoles when the mass of the incoming particle is set to zero. There is only one new type of diagram which is shown in Fig. 2, all others can be reduced to known results.

For completeness we give here the result for an arbitrary gauge group. It reads

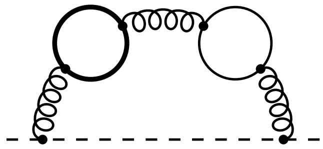

Fig. 2. The diagram which gives the coefficient of $n_m n_h$ in Z_q. For the meaning of the lines see Fig. 1.

$$
\begin{aligned}
Z_q \;=\; & 1 + C_F T_F \left(\frac{\alpha_s^0}{\pi}\Gamma(\varepsilon)\right)^2 \left(n_h m^{-4\varepsilon} + n_m m_c^{-4\varepsilon}\right) \\
& \times \frac{\varepsilon}{16}\left(1 - \frac{5}{6}\varepsilon + \frac{89}{36}\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right) \\
& + C_F T_F \left(\frac{\alpha_s^0}{\pi}\Gamma(\varepsilon)\right)^3 \Big(n_h Z_1(n_h) m^{-6\varepsilon} + n_m Z_1(n_m) m_c^{-6\varepsilon} \\
& \quad + T_F n_h n_m (m m_c)^{-3\varepsilon} Z_2(m_c/m)\Big) + \mathcal{O}(\alpha_s^4)\,, \qquad (4.1)
\end{aligned}
$$

where α_s^0 has the dimensionality $m^{2\varepsilon}$, and the single-scale contributions are [14,13]

$$
\begin{aligned}
Z_1(n) \;=\; & C_F Z_F + C_A Z_A + T_F n_l Z_L + T_F n Z_H\,, \\
Z_F \;=\; & \frac{\varepsilon}{96}\left[1 - \frac{3}{2}\varepsilon + \left(12\zeta_3 + \frac{443}{12}\right)\varepsilon^2\right] + \mathcal{O}(\varepsilon^4)\,, \\
Z_A \;=\; & \frac{1}{192}\Bigg\{1 + \frac{10}{3}\varepsilon + \frac{227}{9}\varepsilon^2 - \left(16\zeta_3 + \frac{1879}{54}\right)\varepsilon^3 \\
& - \xi\left[1 - 3\varepsilon + \frac{35}{3}\varepsilon^2 + \left(8\zeta_3 - \frac{407}{9}\right)\varepsilon^3\right]\Bigg\} + \mathcal{O}(\varepsilon^4)\,, \\
Z_L \;=\; & -\frac{\varepsilon}{72}\left[1 - \frac{5}{6}\varepsilon + \frac{337}{36}\varepsilon^2\right] + \mathcal{O}(\varepsilon^4)\,, \\
Z_H \;=\; & -\frac{\varepsilon}{36}\left[1 - \frac{5}{6}\varepsilon + \frac{151}{36}\varepsilon^2\right] + \mathcal{O}(\varepsilon^4)\,, \qquad (4.2)
\end{aligned}
$$

where $\xi = 1 - a_0$, a_0 is the bare gauge-fixing parameter.[4] The two-scale contribution (Fig. 2) is given by [34]

$$
Z_2(x) = Z_2(x^{-1}) = 2Z_H - \frac{\varepsilon^3}{12}\ln^2 x + \mathcal{O}(\varepsilon^4)\,. \qquad (4.3)
$$

[4] The gauge-fixing term in the Lagrangian is $-\left(\partial_\mu A_0^{a\mu}\right)^2/(2a_0)$, the free gluon propagator is $-(i/k^2)\left(g_{\mu\nu} - \xi k_\mu k_\nu/k^2\right)$.

5 Matching coefficients

In this Section, we present the results for the matching coefficients of the different heavy–light currents for the colour group SU(3) (results for a general colour group can be found at [32]). For this purpose we decompose the coefficients as follows

$$
\begin{aligned}
C_\Gamma(\mu) &= 1 + \frac{\alpha_s^{(n_f')}(m)}{\pi} C_\Gamma^{(1)} + \left(\frac{\alpha_s^{(n_f')}(m)}{\pi}\right)^2 C_\Gamma^{(2)}(x) \\
&\quad + \left(\frac{\alpha_s^{(n_f')}(m)}{\pi}\right)^3 C_\Gamma^{(3)}(x) + \mathcal{O}(\alpha_s^4) \,, \qquad (5.1)\\
C_\Gamma^{(2)}(x) &= C_\Gamma^G + C_\Gamma^H n_h + C_\Gamma^L n_l + C_\Gamma^M(x) n_m \,, \\
C_\Gamma^{(3)}(x) &= C_\Gamma^{GG} + C_\Gamma^{GH} n_h + C_\Gamma^{GL} n_l + C_\Gamma^{HH} n_h^2 + C_\Gamma^{HL} n_h n_l + C_\Gamma^{LL} n_l^2 \\
&\quad + C_\Gamma^{GM}(x) n_m + C_\Gamma^{HM}(x) n_h n_m + C_\Gamma^{LM}(x) n_l n_m + C_\Gamma^{MM}(x) n_m^2 \,,
\end{aligned}
$$

where $\Gamma = 1$, $\not{v}$, $\gamma_\perp$, $\gamma_\perp \not{v}$, and $n_f' = n_l + n_m$ is the number of active flavours in HQET. Furthermore, we use the abbreviation $L = \ln(\mu^2/m^2)$ (m is the on-shell b-quark mass).

We first present the results for $m_c = 0$. The individual contributions read

$$
\begin{aligned}
C_1^{(1)} &= \frac{2}{3} + \frac{1}{2} L \,, \\
C_1^G &= \frac{1843}{192} + \frac{11}{72}\pi^2 + \frac{1}{18}\pi^2 \ln 2 - \frac{17}{36}\zeta_3 + \left(\frac{527}{144} - \frac{7}{108}\pi^2\right) L - \frac{9}{16} L^2 \,, \\
C_1^H &= \frac{149}{216} - \frac{1}{18}\pi^2 - \frac{5}{36} L + \frac{1}{12} L^2 \,, \\
C_1^L &= -\frac{95}{288} - \frac{1}{36}\pi^2 - \frac{5}{72} L + \frac{1}{24} L^2 \,, \\
C_1^{GG} &= \frac{8765231}{62208} + \frac{235073}{46656}\pi^2 + \frac{917}{324}\pi^2 \ln 2 - \frac{1}{81}\pi^2 \ln^2 2 + \frac{3371}{1296}\zeta_3 \\
&\quad + \frac{4733}{3888}\pi^2\zeta_3 - \frac{50039}{116640}\pi^4 - \frac{28975}{2592}\zeta_5 - \frac{4}{27}\ln^4 2 - \frac{32}{9} a_4 \\
&\quad + \left(\frac{46123}{1728} - \frac{25}{108}\pi^2 + \frac{1}{36}\pi^2 \ln 2 + \frac{55}{144}\zeta_3 - \frac{95}{1944}\pi^4\right) L \\
&\quad + \left(-\frac{605}{64} + \frac{7}{48}\pi^2\right) L^2 + \frac{15}{16} L^3 \,,
\end{aligned}
$$

$$
\begin{aligned}
C_1^{GH} &= \frac{3349}{1944} - \frac{30917}{9720}\pi^2 + \frac{443}{81}\pi^2\ln 2 + \frac{1}{9}\pi^2\ln^2 2 - \frac{27845}{5184}\zeta_3 + \frac{29}{96}\pi^2\zeta_3 \\
&\quad - \frac{19}{2430}\pi^4 - \frac{45}{32}\zeta_5 - \frac{1}{9}\ln^4 2 - \frac{8}{3}a_4 \\
&\quad + \left(-\frac{319}{432} - \frac{1}{36}\pi^2 - \frac{5}{6}\zeta_3\right) L + \frac{211}{144}L^2 - \frac{19}{72}L^3 \,, \\
C_1^{GL} &= -\frac{528353}{46656} - \frac{15553}{17496}\pi^2 - \frac{25}{324}\pi^2\ln 2 + \frac{1}{81}\pi^2\ln^2 2 - \frac{1591}{972}\zeta_3 \\
&\quad + \frac{3281}{116640}\pi^4 + \frac{1}{162}\ln^4 2 + \frac{4}{27}a_4 + \left(-\frac{7399}{5184} + \frac{11}{972}\pi^2 - \frac{97}{216}\zeta_3\right) L \\
&\quad + \left(\frac{809}{864} - \frac{7}{648}\pi^2\right) L^2 - \frac{19}{144}L^3 \,, \\
C_1^{HH} &= -\frac{4045}{11664} - \frac{1}{135}\pi^2 + \frac{11}{27}\zeta_3 - \frac{35}{1296}L - \frac{5}{216}L^2 + \frac{1}{108}L^3 \,, \\
C_1^{HL} &= \frac{353}{5832} + \frac{1}{216}\pi^2 - \frac{2}{27}\zeta_3 - \frac{35}{648}L - \frac{5}{108}L^2 + \frac{1}{54}L^3 \,, \\
C_1^{LL} &= \frac{6457}{46656} + \frac{13}{648}\pi^2 + \frac{7}{108}\zeta_3 - \frac{35}{2592}L - \frac{5}{432}L^2 + \frac{1}{216}L^3 \,, \qquad (5.2) \\
C_{\not p}^{(1)} &= -\frac{2}{3} - \frac{1}{2}L \,, \\
C_{\not p}^{G} &= -\frac{177}{64} - \frac{5}{72}\pi^2 - \frac{1}{18}\pi^2\ln 2 - \frac{11}{36}\zeta_3 + \left(-\frac{79}{144} - \frac{7}{108}\pi^2\right) L + \frac{13}{16}L^2 \,, \\
C_{\not p}^{H} &= \frac{727}{432} - \frac{1}{6}\pi^2 \,, \\
C_{\not p}^{L} &= \frac{47}{288} + \frac{1}{36}\pi^2 + \frac{5}{72}L - \frac{1}{24}L^2 \,, \\
C_{\not p}^{GG} &= -\frac{62575}{62208} - \frac{231253}{46656}\pi^2 - \frac{517}{324}\pi^2\ln 2 + \frac{20}{81}\pi^2\ln^2 2 + \frac{5645}{1296}\zeta_3 \\
&\quad + \frac{2089}{486}\pi^2\zeta_3 - \frac{17347}{58320}\pi^4 - \frac{49435}{2592}\zeta_5 + \frac{11}{54}\ln^4 2 + \frac{44}{9}a_4 \\
&\quad + \left(\frac{115}{54} - \frac{121}{648}\pi^2 + \frac{1}{36}\pi^2\ln 2 + \frac{37}{48}\zeta_3 - \frac{95}{1944}\pi^4\right) L \\
&\quad + \left(\frac{2257}{576} + \frac{91}{432}\pi^2\right) L^2 - \frac{13}{8}L^3 \,, \\
C_{\not p}^{GH} &= \frac{2051}{96} - \frac{24583}{2430}\pi^2 + \frac{361}{27}\pi^2\ln 2 + \frac{10}{81}\pi^2\ln^2 2 - \frac{45869}{5184}\zeta_3 + \frac{53}{96}\pi^2\zeta_3 \\
&\quad - \frac{1}{20}\pi^4 - \frac{85}{32}\zeta_5 - \frac{10}{81}\ln^4 2 - \frac{80}{27}a_4 + \left(-\frac{727}{864} + \frac{1}{12}\pi^2\right) L \,, \\
C_{\not p}^{GL} &= \frac{24457}{46656} + \frac{5575}{8748}\pi^2 + \frac{19}{324}\pi^2\ln 2 - \frac{1}{81}\pi^2\ln^2 2 + \frac{3181}{1944}\zeta_3 - \frac{379}{116640}\pi^4 \\
&\quad - \frac{1}{162}\ln^4 2 - \frac{4}{27}a_4 + \left(-\frac{319}{5184} + \frac{11}{972}\pi^2 + \frac{83}{216}\zeta_3\right) L \\
&\quad + \left(-\frac{469}{864} - \frac{7}{648}\pi^2\right) L^2 + \frac{25}{144}L^3 \,,
\end{aligned}
$$

$$
\begin{aligned}
C_{\not{p}}^{HH} &= -\frac{5857}{7776} + \frac{1}{405}\pi^2 + \frac{11}{18}\zeta_3\,, \\
C_{\not{p}}^{HL} &= -\frac{193}{432} + \frac{29}{648}\pi^2\,, \\
C_{\not{p}}^{LL} &= \frac{1751}{46656} - \frac{13}{648}\pi^2 - \frac{7}{108}\zeta_3 + \frac{35}{2592}L + \frac{5}{432}L^2 - \frac{1}{216}L^3\,, \qquad (5.3) \\
C_{\gamma_\perp}^{(1)} &= -\frac{4}{3} - \frac{1}{2}L\,, \\
C_{\gamma_\perp}^{G} &= -\frac{14651}{1728} - \frac{125}{648}\pi^2 - \frac{7}{54}\pi^2\ln 2 - \frac{7}{36}\zeta_3 + \left(-\frac{31}{144} - \frac{7}{108}\pi^2\right)L \\
&\quad + \frac{13}{16}L^2\,, \\
C_{\gamma_\perp}^{H} &= \frac{133}{144} - \frac{5}{54}\pi^2\,, \\
C_{\gamma_\perp}^{L} &= \frac{445}{864} + \frac{1}{36}\pi^2 + \frac{5}{72}L - \frac{1}{24}L^2\,, \\
C_{\gamma_\perp}^{GG} &= -\frac{5046967}{62208} - \frac{361033}{46656}\pi^2 - \frac{1745}{324}\pi^2\ln 2 + \frac{124}{243}\pi^2\ln^2 2 + \frac{3929}{1296}\zeta_3 \\
&\quad + \frac{3463}{972}\pi^2\zeta_3 - \frac{461}{3888}\pi^4 - \frac{43835}{2592}\zeta_5 + \frac{163}{486}\ln^4 2 + \frac{652}{81}a_4 \\
&\quad + \left(\frac{301}{54} - \frac{53}{648}\pi^2 + \frac{7}{108}\pi^2\ln 2 + \frac{103}{144}\zeta_3 - \frac{95}{1944}\pi^4\right)L \\
&\quad + \left(\frac{1945}{576} + \frac{91}{432}\pi^2\right)L^2 - \frac{13}{8}L^3\,, \\
C_{\gamma_\perp}^{GH} &= \frac{4133}{288} - \frac{3385}{486}\pi^2 + \frac{2069}{243}\pi^2\ln 2 + \frac{26}{243}\pi^2\ln^2 2 - \frac{10445}{5184}\zeta_3 \\
&\quad + \frac{35}{288}\pi^2\zeta_3 - \frac{233}{14580}\pi^4 - \frac{35}{96}\zeta_5 - \frac{26}{243}\ln^4 2 - \frac{208}{81}a_4 \\
&\quad + \left(-\frac{133}{288} + \frac{5}{108}\pi^2\right)L\,, \\
C_{\gamma_\perp}^{GL} &= \frac{455461}{46656} + \frac{4937}{4374}\pi^2 + \frac{169}{972}\pi^2\ln 2 - \frac{7}{243}\pi^2\ln^2 2 + \frac{5173}{1944}\zeta_3 \\
&\quad - \frac{2963}{116640}\pi^4 - \frac{7}{486}\ln^4 2 - \frac{28}{81}a_4 + \left(-\frac{1471}{5184} + \frac{11}{972}\pi^2 + \frac{83}{216}\zeta_3\right)L \\
&\quad + \left(-\frac{445}{864} - \frac{7}{648}\pi^2\right)L^2 + \frac{25}{144}L^3\,, \\
C_{\gamma_\perp}^{HH} &= -\frac{3641}{7776} + \frac{11}{1215}\pi^2 + \frac{17}{54}\zeta_3\,, \\
C_{\gamma_\perp}^{HL} &= -\frac{2545}{3888} + \frac{127}{1944}\pi^2\,, \\
C_{\gamma_\perp}^{LL} &= -\frac{7993}{46656} - \frac{7}{216}\pi^2 - \frac{7}{108}\zeta_3 + \frac{35}{2592}L + \frac{5}{432}L^2 - \frac{1}{216}L^3\,, \qquad (5.4) \\
C_{\gamma_\perp\not{p}}^{(1)} &= -\frac{4}{3} - \frac{5}{6}L\,, \\
C_{\gamma_\perp\not{p}}^{G} &= -\frac{20749}{1728} - \frac{7}{24}\pi^2 - \frac{1}{6}\pi^2\ln 2 - \frac{5}{36}\zeta_3 + \left(-\frac{329}{144} - \frac{7}{108}\pi^2\right)L \\
&\quad + \frac{215}{144}L^2\,,
\end{aligned}
$$

$$
\begin{aligned}
C^{H}_{\gamma_\perp \not{v}} &= \frac{809}{648} - \frac{7}{54}\pi^2 + \frac{13}{108}L - \frac{1}{36}L^2\,,\\
C^{L}_{\gamma_\perp \not{v}} &= \frac{1745}{2592} + \frac{5}{108}\pi^2 + \frac{41}{216}L - \frac{5}{72}L^2\,,\\
C^{GG}_{\gamma_\perp \not{v}} &= -\frac{21556403}{186624} - \frac{488167}{46656}\pi^2 - \frac{757}{108}\pi^2\ln 2 + \frac{142}{243}\pi^2\ln^2 2 + \frac{8357}{3888}\zeta_3\\
&\quad + \frac{16153}{3888}\pi^2\zeta_3 - \frac{2447}{23328}\pi^4 - \frac{15925}{864}\zeta_5 + \frac{113}{243}\ln^4 2 + \frac{904}{81}a_4\\
&\quad + \left(\frac{7871}{1728} + \frac{7}{108}\pi^2 + \frac{5}{36}\pi^2\ln 2 + \frac{61}{48}\zeta_3 - \frac{95}{1944}\pi^4\right)L\\
&\quad + \left(\frac{22177}{1728} + \frac{301}{1296}\pi^2\right)L^2 - \frac{4085}{1296}L^3\,,\\
C^{GH}_{\gamma_\perp \not{v}} &= \frac{125005}{5832} - \frac{268333}{29160}\pi^2 + \frac{301}{27}\pi^2\ln 2 + \frac{1}{9}\pi^2\ln^2 2 - \frac{22469}{5184}\zeta_3\\
&\quad + \frac{59}{288}\pi^2\zeta_3 - \frac{73}{2430}\pi^4 - \frac{25}{32}\zeta_5 - \frac{1}{9}\ln^4 2 - \frac{8}{3}a_4\\
&\quad + \left(-\frac{1117}{3888} + \frac{35}{324}\pi^2 + \frac{5}{18}\zeta_3\right)L - \frac{1225}{1296}L^2 + \frac{1}{8}L^3\,,\\
C^{GL}_{\gamma_\perp \not{v}} &= \frac{211705}{15552} + \frac{28133}{17496}\pi^2 + \frac{71}{324}\pi^2\ln 2 - \frac{1}{27}\pi^2\ln^2 2 + \frac{3347}{972}\zeta_3\\
&\quad - \frac{4183}{116640}\pi^4 - \frac{1}{54}\ln^4 2 - \frac{4}{9}a_4 + \left(\frac{4091}{15552} - \frac{13}{972}\pi^2 + \frac{143}{216}\zeta_3\right)L\\
&\quad + \left(-\frac{3845}{2592} - \frac{7}{648}\pi^2\right)L^2 + \frac{5}{16}L^3\,,\\
C^{HH}_{\gamma_\perp \not{v}} &= -\frac{21281}{34992} + \frac{1}{81}\pi^2 + \frac{31}{81}\zeta_3 + \frac{1}{144}L + \frac{13}{648}L^2 - \frac{1}{324}L^3\,,\\
C^{HL}_{\gamma_\perp \not{v}} &= -\frac{14567}{17496} + \frac{17}{216}\pi^2 + \frac{2}{81}\zeta_3 + \frac{1}{72}L + \frac{13}{324}L^2 - \frac{1}{162}L^3\,,\\
C^{LL}_{\gamma_\perp \not{v}} &= -\frac{29309}{139968} - \frac{89}{1944}\pi^2 - \frac{35}{324}\zeta_3 + \frac{53}{2592}L + \frac{41}{1296}L^2 - \frac{5}{648}L^3\,,
\end{aligned}
\tag{5.5}
$$

where $a_4 = \mathrm{Li}_4\left(\frac{1}{2}\right)$. The two-loop results, as well as the coefficients C^{LL}_Γ, are known from Ref. [5]; all remaining three-loop results are new.

It is instructive to re-write the matching coefficients for $m_c = 0$ and $\mu = m$ via the leading β-function coefficient in HQET, $\beta_0' = 11 - \frac{2}{3}n_f'$. We obtain

$$
\begin{aligned}
C^{(2)}_{1} &= 0.91\beta_0' + 1.09 = 7.55 + 1.09 = 8.64\,,\\
C^{(2)}_{\not{v}} &= -0.66\beta_0' + 3.06 = -5.47 + 3.06 = -2.41\,,\\
C^{(2)}_{\gamma_\perp} &= -1.18\beta_0' + 1.53 = -9.87 + 1.53 = -8.34\,,\\
C^{(2)}_{\gamma_\perp \not{v}} &= -1.70\beta_0' + 2.42 = -14.13 + 2.42 = -11.70
\end{aligned}
\tag{5.6}
$$

at two loops, and the following results at three loops:

$$
\begin{aligned}
C_1^{(3)} &= 0.93\beta_0'^2 + 9.04\beta_0' - 38.16 = 64.74 + 75.34 - 38.16 = 101.92\,,\\
C_{\not{p}}^{(3)} &= -0.54\beta_0'^2 - 1.29\beta_0' + 29.74 = -37.25 - 10.72 + 29.74 = -18.23\,,\\
C_{\gamma_\perp}^{(3)} &= -1.28\beta_0'^2 - 5.56\beta_0' + 45.34 = -88.92 - 46.34 + 45.34 = -89.92\,,\\
C_{\gamma_\perp \not{p}}^{(3)} &= -1.78\beta_0'^2 - 7.63\beta_0' + 63.22 = -123.61 - 63.57 + 63.22\\
&= -123.96\,.
\end{aligned} \tag{5.7}
$$

A method to estimate higher loop contributions called naive nonabelianization has been formulated in Ref. [5]. It is based on the fact that each polynomial in n_f can be re-written as a polynomial in β_0. Usually it is relatively simple to calculate the term with the highest power of n_f. This means that we know the coefficient of the leading power of β_0. Neglecting subleading powers of β_0 we obtain an estimate of the full result. The two-loop corrections to the matching coefficients (5.6) were among the examples confirming naive nonabelianization [5]. At three loops we see that this prescription reproduces the correct signs and roughly the correct magnitude of the full results. In all cases we observe a compensation between the $\mathcal{O}(\beta_0')$ and $\mathcal{O}(1)$ terms. In the case of $C_{\gamma_\perp}^{(3)}$ and $C_{\gamma_\perp \not{p}}^{(3)}$ this compensation is almost complete, and naive nonabelianization works surprisingly well.

Since the expressions for the charm-mass dependence are quite involved and not completely expressed in terms of HPLs of x (the $\mathcal{O}(\varepsilon)$ terms of the master integrals 5.2, 5.2a, 5.3, 5.3a are known analytically only as truncated series in x, see Table 3 in Ref. [19]), we refrain from listing them here. Instead, we present an expansion of our results to the second order in x. The results in terms of the master integrals and expansions to higher orders can be obtained from the web page [32].

Our results for the m_c-dependent coefficients read

$$
\begin{aligned}
C_1^M(x) &= C_1^L + \frac{1}{8}\pi^2 x + \left(\ln x + \frac{1}{2}\right) x^2 + \mathcal{O}(x^3)\,,\\
C_1^{GM}(x) &= C_1^{GL} + \left(\frac{4361}{1296}\pi^2 - \frac{119}{108}\pi^2\ln 2 + \frac{13}{216}\pi^3 + \frac{}{16}\pi^2 L - \frac{9}{8}\pi^2\ln x\right) x\\
&\quad + \left[\frac{48493}{5184} + \frac{395}{432}\pi^2 + \frac{7}{36}\pi^2\ln 2 + \frac{27}{16}\zeta_3 - \pi^2\zeta_3 - \frac{49}{720}\pi^4 - \frac{5}{2}\zeta_5\right.\\
&\qquad + \frac{1}{4}L + \left(\frac{6239}{432} + \frac{1}{4}\pi^2 + \frac{15}{8}\zeta_3 - \frac{19}{180}\pi^4 + \frac{1}{2}L\right)\ln x\\
&\qquad \left. - \frac{37}{9}\ln^2 x\right] x^2 + \mathcal{O}(x^3)\,,
\end{aligned}
$$

$$
\begin{aligned}
C_1^{HM}(x) &= C_1^{HL} + \left(-\frac{517}{1350} + \frac{1}{18}\pi^2 - \frac{1}{15}\ln x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_1^{LM}(x) &= 2C_1^{LL} + \left(-\frac{7}{36}\pi^2 + \frac{1}{6}\pi^2 \ln 2 + \frac{1}{12}\pi^2 \ln x\right) x \\
&\quad + \left(-\frac{5}{18} - \frac{1}{18}\pi^2 - \frac{1}{9} 2 \ln x + \frac{1}{3}\ln^2 x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_1^{MM}(x) &= C_1^{LL} + \left(-\frac{1}{45}\pi^2 + \frac{1}{12}\pi^2 \ln x\right) x \\
&\quad + \left(-\frac{19}{36} - \frac{1}{18}\pi^2 - \frac{2}{9}\ln x + \frac{1}{3}\ln^2 x\right) x^2 + \mathcal{O}(x^3)\,, \qquad (5.8) \\
C_{\not{p}}^{M}(x) &= C_{\not{p}}^{L} - \frac{1}{24}\pi^2 x + \left(\frac{3}{2} + \ln x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_{\not{p}}^{GM}(x) &= C_{\not{p}}^{GL} + \Bigg(-\frac{35333}{3888}\pi^2 + \frac{4439}{324}\pi^2 \ln 2 - \frac{13}{648}\pi^3 + \frac{1}{48}\pi^2 L \\
&\qquad + \frac{17}{24}\pi^2 \ln x\Bigg) x \\
&\quad + \Bigg[\frac{86509}{5184} + \frac{863}{432}\pi^2 + \frac{7}{36}\pi^2 \ln 2 + \frac{115}{16}\zeta_3 - \frac{7}{4}\pi^2\zeta_3 - \frac{49}{720}\pi^4 \\
&\qquad - 5\zeta_5 - \frac{3}{4}L + \left(\frac{7391}{432} + \frac{7}{4}\pi^2 + \frac{15}{8}\zeta_3 - \frac{17}{90}\pi^4 - \frac{1}{2}L\right)\ln x \\
&\qquad - \frac{37}{9}\ln^2 x\Bigg] x^2 + \mathcal{O}(x^3)\,, \\
C_{\not{p}}^{HM}(x) &= C_{\not{p}}^{HL} + \left(-\frac{817}{1350} + \frac{1}{18}\pi^2 - \frac{1}{15}\ln x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_{\not{p}}^{LM}(x) &= 2C_{\not{p}}^{LL} + \left(\frac{7}{108}\pi^2 - \frac{1}{18}\pi^2 \ln 2 - \frac{1}{36}\pi^2 \ln x\right) x \\
&\quad + \left(-\frac{1}{2} - \frac{1}{18}\pi^2 - \frac{2}{9}\ln x + \frac{1}{3}\ln^2 x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_{\not{p}}^{MM}(x) &= C_{\not{p}}^{LL} + \left(\frac{1}{135}\pi^2 - \frac{1}{36}\pi^2 \ln x\right) x \\
&\quad + \left(-\frac{3}{4} - \frac{1}{18}\pi^2 - \frac{2}{9}\ln x + \frac{1}{3}\ln^2 x\right) x^2 + \mathcal{O}(x^3)\,, \qquad (5.9) \\
C_{\gamma_\perp}^{M}(x) &= C_{\gamma_\perp}^{L} - \frac{1}{24}\pi^2 x + \left(\frac{3}{2} + \ln x\right) x^2 + \mathcal{O}(x^3)\,, \\
C_{\gamma_\perp}^{GM}(x) &= C_{\gamma_\perp}^{GL} + \Bigg[-\frac{130327}{11664}\pi^2 + \frac{14089}{972}\pi^2 \ln 2 - \frac{143}{1944}\pi^3 + \frac{11}{144}\pi^2 L \\
&\qquad + \frac{409}{216}\pi^2 \ln x\Bigg] x
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \Bigg[\frac{24217}{5184} + \frac{1517}{1296}\pi^2 - \frac{7}{108}\pi^2\ln 2 + \frac{595}{144}\zeta_3 - \frac{7}{12}\pi^2\zeta_3 \\
&\qquad - \frac{19}{2160}\pi^4 - \frac{5}{12}L \\
&\qquad + \left(-\frac{79}{144} + \frac{595}{324}\pi^2 - \frac{1}{8}\zeta_3 - \frac{7}{135}\pi^4 + \frac{1}{6}L\right)\ln x \\
&\qquad + \frac{7}{9}\ln^2 x\Bigg]x^2 + \mathcal{O}(x^3)\,,
\end{aligned}
$$

$$
\begin{aligned}
C^{HM}_{\gamma_\perp}(x) &= C^{HL}_{\gamma_\perp} + \left(\frac{349}{4050} - \frac{1}{54}\pi^2 - \frac{1}{15}\ln x\right)x^2 + \mathcal{O}(x^3)\,, \\
C^{LM}_{\gamma_\perp}(x) &= 2C^{LL}_{\gamma_\perp} + \left(\frac{77}{324}\pi^2 - \frac{11}{54}\pi^2\ln 2 - \frac{11}{108}\pi^2\ln x\right)x \\
&\quad + \left(-\frac{19}{54} + \frac{1}{54}\pi^2 + \frac{2}{27}\ln x - \frac{1}{9}\ln^2 x\right)x^2 + \mathcal{O}(x^3)\,, \\
C^{MM}_{\gamma_\perp}(x) &= C^{LL}_{\gamma_\perp} + \left(\frac{11}{405}\pi^2 - \frac{11}{108}\pi^2\ln x\right)x \\
&\quad + \left(-\frac{29}{108} + \frac{1}{54}\pi^2 + \frac{2}{27}\ln x - \frac{1}{9}\ln^2 x\right)x^2 + \mathcal{O}(x^3)\,,
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
C^{M}_{\gamma_\perp \not{p}}(x) &= C^{L}_{\gamma_\perp \not{p}} - \frac{5}{24}\pi^2 x + \left(\frac{7}{6} - \frac{1}{3}\ln x\right)x^2 + \mathcal{O}(x^3)\,, \\
C^{GM}_{\gamma_\perp \not{p}}(x) &= C^{GL}_{\gamma_\perp \not{p}} + \Bigg(-\frac{59221}{3888}\pi^2 + \frac{6295}{324}\pi^2\ln 2 - \frac{65}{648}\pi^3 + \frac{25}{144}\pi^2 L \\
&\qquad\qquad + \frac{541}{216}\pi^2\ln x\Bigg)x \\
&\quad + \Bigg[\frac{35653}{5184} + \frac{631}{432}\pi^2 - \frac{7}{108}\pi^2\ln 2 + \frac{313}{48}\zeta_3 - \frac{5}{6}\pi^2\zeta_3 - \frac{19}{2160}\pi^4 \\
&\qquad - \frac{5}{6}\zeta_5 - \frac{35}{36}L \\
&\qquad + \left(\frac{575}{432} + \frac{9}{4}\pi^2 - \frac{1}{8}\zeta_3 - \frac{43}{540}\pi^4 + \frac{5}{18}L\right)\ln x \\
&\qquad + \frac{1}{2}\ln^2 x\Bigg]x^2 + \mathcal{O}(x^3)\,, \\
C^{HM}_{\gamma_\perp \not{p}}(x) &= C^{HL}_{\gamma_\perp \not{p}} + \left(\frac{49}{4050} - \frac{1}{54}\pi^2 - \frac{1}{15}\ln x\right)x^2 + \mathcal{O}(x^3)\,, \\
C^{LM}_{\gamma_\perp \not{p}}(x) &= 2C^{LL}_{\gamma_\perp \not{p}} + \left(\frac{35}{108}\pi^2 - \frac{5}{18}\pi^2\ln 2 - \frac{5}{36}\pi^2\ln x\right)x \\
&\quad + \left(-\frac{23}{54} + \frac{1}{54}\pi^2 + \frac{2}{27}\ln x - \frac{1}{9}\ln^2 x\right)x^2 + \mathcal{O}(x^3)\,, \\
C^{MM}_{\gamma_\perp \not{p}}(x) &= C^{LL}_{\gamma_\perp \not{p}} + \left(\frac{1}{27}\pi^2 - \frac{5}{36}\pi^2\ln x\right)x \\
&\quad + \left(-\frac{37}{108} + \frac{1}{54}\pi^2 + \frac{2}{27}\ln x - \frac{1}{9}\ln^2 x\right)x^2 + \mathcal{O}(x^3)\,.
\end{aligned}
\tag{5.11}
$$

6 Meson matrix elements

We are now in the position to apply our results to the matrix elements between a B or B^* meson with momentum p and the vacuum. They are defined through

$$
\begin{aligned}
\langle 0|\left(\bar{q}\gamma_5^{\mathrm{AC}}Q\right)_\mu|B\rangle &= -im_B f_B^P(\mu)\,,\\
\langle 0|\bar{q}\gamma^\alpha\gamma_5^{\mathrm{AC}}Q|B\rangle &= if_B p^\alpha\,,\\
\langle 0|\bar{q}\gamma^\alpha Q|B^*\rangle &= im_{B^*}f_{B^*}e^\alpha\,,\\
\langle 0|\left(\bar{q}\sigma^{\alpha\beta}Q\right)_\mu|B^*\rangle &= f_{B^*}^T(\mu)(p^\alpha e^\beta - p^\beta e^\alpha)\,,
\end{aligned}
\tag{6.1}
$$

where e^α is the B^* polarization vector. The corresponding HQET matrix elements (at $m\to\infty$) in the v rest frame are

$$
\begin{aligned}
\langle 0|\left(\bar{q}\gamma_5^{\mathrm{AC}}Q_v\right)_\mu|B(\vec{k}\,)\rangle_{\mathrm{nr}} &= -iF(\mu)\,,\\
\langle 0|\left(\bar{q}\vec{\gamma}Q_v\right)_\mu|B^*(\vec{k}\,)\rangle_{\mathrm{nr}} &= iF(\mu)\vec{e},
\end{aligned}
\tag{6.2}
$$

where the single-meson states are normalized by the non-relativistic condition

$$
{}_{\mathrm{nr}}\langle B(\vec{k}\,')|B(\vec{k}\,)\rangle_{\mathrm{nr}} = (2\pi)^3\delta(\vec{k}\,'-\vec{k}\,)\,.
$$

We also remind the reader that $\bar{q}\Gamma\not{v}Q_v = \bar{q}\Gamma Q_v$, so that there are only two currents. These two matrix elements are characterized by a single hadronic parameter $F(\mu)$ due to the heavy-quark spin symmetry. From Eq. (2.9) we have [5]

$$
\frac{f_B^P(\mu)}{f_B} = \frac{m_B}{m(\mu)}\,.
\tag{6.3}
$$

Here $m_B = m+\bar{\Lambda}+\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/m)$ where $\bar{\Lambda}$ is the residual energy of the ground-state B meson in the limit $m\to\infty$. Neglecting $1/m$ corrections, we see that this equation coincides with (2.10). We have checked that our results agree with the known formulas for $m(\mu)/m$ at $m_c=0$ [23] and with m_c corrections taken into account [18].

Using the expressions of Section 5 we find the following results for the ratios of decay constants. Again, we present an expansion to the second order in x for the charm mass dependent terms. For the numerical evaluations we use an expansion to the eighth order in x and the values $\alpha_s^{(4)}(m_b) = 0.2163$ and $x = 0.3$.

$$
\begin{aligned}
\frac{f_{B^*}}{f_B} &= \frac{C_{\gamma_\perp}(m_b)}{C_{\not{v}}(m_b)} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_b}\right) = 1 - \frac{2}{3}\frac{\alpha_s^{(4)}(m_b)}{\pi} \\
&\quad + \Big[-7.749 - 0.028 n_h + 0.352 n_l \\
&\quad + \Big(0.352 - 1.097 x - 0.667 x^2 - 1.333 x^2 \ln x\Big) n_m\Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 \\
&\quad + \Big[-129.211 - 0.198 n_h + 14.294 n_l - 0.006 n_h^2 - 0.005 n_h n_l \\
&\quad - 0.331 n_l^2 + \Big(14.294 - 17.816 x + 11.697 x \ln x \\
&\qquad - 0.273 x^2 - 6.082 x^2 \ln x + 4.889 x^2 \ln^2 x\Big) n_m \\
&\quad + \Big(-0.005 - 0.040 x^2\Big) n_h n_m \\
&\quad + \Big(-0.661 + 0.692 x - 0.731 x \ln x \\
&\qquad + 0.879 x^2 + 0.296 x^2 \ln x - 0.444 x^2 \ln^2 x\Big) n_l n_m \\
&\quad + \Big(-0.331 + 0.195 x - 0.731 x \ln x \\
&\qquad + 1.213 x^2 + 0.296 x^2 \ln x - 0.444 x^2 \ln^2 x\Big) n_m^2\Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 \\
&\quad + \mathcal{O}\left(\alpha_s^4, \frac{\Lambda_{\rm QCD}}{m_b}\right) \\
&= 1 - \frac{2}{3}\frac{\alpha_s^{(4)}(m_b)}{\pi} - (6.370 + 0.189)\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 \\
&\quad - (77.549 + 6.575)\left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 + \mathcal{O}\left(\alpha_s^4, \frac{\Lambda_{\rm QCD}}{m_b}\right) \\
&= 1 - 0.046 - (0.030 + 0.001) - (0.025 + 0.002) \\
&= 0.899 - 0.003 = 0.896 + \mathcal{O}\Big(\alpha_s^4, \Lambda_{\rm QCD}/m_b\Big)\ . \qquad (6.4)
\end{aligned}
$$

In the second line from the bottom the corrections from tree level, first, second and third order in α_s are given separately. Also the contributions stemming from the finite charm mass are separated (the second number in the parentheses). In the first part of the last line, the m_c correction is also separated. Power corrections $\mathcal{O}(\Lambda_{\rm QCD}/m_b)$ are discussed in Refs. [37,8].

For the second ratio we obtain for $\mu = m_b$

$$
\frac{f_{B^*}^T(m_b)}{f_{B^*}} = \frac{C_{\not{v}\gamma_\perp}(m_b)}{C_{\gamma_\perp}(m_b)} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_b}\right)
$$

$$
\begin{aligned}
&= 1 + \Big[-4.690 - 0.041 n_h + 0.341 n_l \\
&\quad + \Big(0.341 - 0.548x + 0.333x^2\Big) n_m\Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 \\
&\quad + \Big[-70.923 - 0.666 n_h + 9.175 n_l - 0.026 n_h^2 - 0.016 n_h n_l \\
&\quad - 0.222 n_l^2 + \Big(9.175 - 7.859x + 6.031 x \ln x \\
&\qquad + 4.555x^2 + 3.256x^2 \ln x - 0.278 x^2 \ln^2 x\Big) n_m \\
&\quad + \Big(-0.016 - 0.074x^2\Big) n_h n_m \\
&\quad + \Big(-0.444 + 0.346x - 0.366 x \ln x - 0.074x^2\Big) n_l n_m \\
&\quad + \Big(-0.222 + 0.097x - 0.366 x \ln x - 0.074x^2\Big) n_m^2\Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 \\
&\quad + \mathcal{O}\left(\alpha_s^4, \frac{\Lambda_{\mathrm{QCD}}}{m_b}\right) \\
&= 1 - (3.367 + 0.142) \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 \\
&\quad - (38.530 + 3.973) \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 + \mathcal{O}\left(\alpha_s^4, \frac{\Lambda_{\mathrm{QCD}}}{m_b}\right) \\
&= 1 - (0.016 + 0.001) - (0.013 + 0.001) \\
&= 0.971 - 0.002 = 0.969 + \mathcal{O}\left(\alpha_s^4, \Lambda_{\mathrm{QCD}}/m_b\right) . \qquad (6.5)
\end{aligned}
$$

The coefficients of these perturbative series (at $m_c = 0$ and $\mu = m_b$) can be re-written via β_0':

$$
\begin{aligned}
\left(\frac{f_{B^*}}{f_B}\right)^{(2)} &= -0.53\beta_0' - 1.97 = -4.40 - 1.97 = -6.37 , \\
\left(\frac{f_{B^*}^T(m_b)}{f_{B^*}}\right)^{(2)} &= -0.51\beta_0' + 0.89 = -4.26 + 0.89 = -3.37 , \\
\left(\frac{f_{B^*}}{f_B}\right)^{(3)} &= -0.74\beta_0'^2 - 5.06\beta_0' + 16.33 \\
&= -51.67 - 42.21 + 16.33 = -77.55 , \\
\left(\frac{f_{B^*}^T(m_b)}{f_{B^*}}\right)^{(3)} &= -0.50\beta_0'^2 - 2.75\beta_0' + 19.07 \\
&= -34.69 - 22.91 + 19.07 = -38.53 . \qquad (6.6)
\end{aligned}
$$

Again, naive nonabelianization [5] works quite well for these ratios predicting the correct sign and order of magnitude.

Asymptotics of the perturbative coefficients for the matching coefficients at a large number of loops $l \gg 1$ have been investigated in Ref. [8] in a model-independent way. The results contain three unknown normalization constants $N_{0,1,2} \sim 1$. Let us in the following assume that the number of loops $l = 3$ is much larger than one and compare our results with Ref. [8]. The asymptotics of the perturbative coefficients for f_{B^*}/f_B contain N_0 and N_2 (see (5.6) in Ref. [8]); in the case of $m/\hat{m}$ it contains only N_0 (see (5.9) in Ref. [8]) [5]:

$$\left(\frac{f_{B^*}}{f_B}\right)^{(n+1)}_{L=-5/3} = -\frac{14}{27}\left\{1+\mathcal{O}\left(\frac{1}{n}\right)+\frac{2}{7}\left(\frac{50}{3}n\right)^{-9/25}\left[1+\mathcal{O}\left(\frac{1}{n}\right)\right]\frac{N_2}{N_0}\right\} \times \left(\frac{m}{\hat{m}}\right)^{(n+1)}_{L=-5/3} . \tag{6.7}$$

The coefficient of N_2/N_0 is about 0.08 at $n = 2$, and it seems reasonable to neglect this contribution. Neglecting also $1/n$ corrections, we obtain [8]

$$\left(\frac{f_{B^*}}{f_B}\right)^{(3)}_{L=-5/3} = -\frac{14}{27}\cdot 56.37 = -29.23\,. \tag{6.8}$$

Our exact result

$$\left(\frac{f_{B^*}}{f_B}\right)^{(3)}_{L=-5/3} = -37.787 \tag{6.9}$$

agrees with this prediction reasonably well, thus confirming the simple relation (5.14) in Ref. [8]. However, $1/n$ corrections are large and tend to break this agreement. It is natural to expect that $1/n^2$ (and higher) corrections are also substantial at $n = 2$.

7 Conclusion

We have calculated the N^{3}LO corrections to the matching coefficients of heavy–light currents in HQET. Our result takes into account effects due to the mass of the charm quark. Strictly speaking, our results should be used together with the N^{3}LO β-function [38,39] and anomalous dimensions of both the QCD currents and the HQET one. The four-loop anomalous dimensions are known for some of the QCD currents (for $\Gamma = \not{v}$, $\gamma_\perp$ the anomalous dimension is exactly zero; for $\Gamma = 1$ it is just the anomalous dimension of the $\overline{\text{MS}}$ mass [40,41] with a minus sign). However, the four-loop anomalous dimension of the HQET current is not known (this anomalous dimension does not depend on the Dirac

[5] Note that for convenience the choice $\mu = me^{-5/6}$ has been adopted in Ref. [8]; $\hat{m}$ is the renormalization-group invariant mass (the exact definition used here is given in the unnumbered formula after (3.8) in Ref. [8]).

structure Γ). [6] The effect of this unknown anomalous dimension cancels in ratios of B-meson decay constants, f_{B^*}/f_B and $f^T_{B^*}(\mu)/f_{B^*}$, discussed in Sect. 6.

All the previous experience shows that contributions from anomalous dimensions are numerically much smaller than from matching coefficients. Matching coefficients have renormalon singularities at the Borel parameter $u = 1/2$, this is the position closest to the origin out of all possible ones; this means that the factorial growth of their perturbative coefficients is fastest among all possible variants. On the other hand, anomalous dimensions have no renormalon singularities, and their perturbative series have finite convergence radii.

Only f_B has been measured experimentally [42]. Our results can be used for predicting the B^* decay constants. We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best. The effects due to the charm-quark mass are small and of the order of 10^{-3}.

The matching coefficients C_Γ can be used for extraction of B (and B^*) decay constants from lattice HQET simulations (see Ref. [9] for recent reviews), or from HQET sum rules.

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