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Charm and Bottom Quark Masses: an Update

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Using new four-loop results for the heavy quark vacuum polarization and new data for bottom quark production in electron-positron annihilation, an update on the determination of charm- and bottom-quark masses through sum rules has been performed. The previous result for the charm-quark mass, $m_c(3\text{ GeV}) = 0.986(13)$ GeV, based on the lowest moment, is supported by the new results from higher moments which lead to consistent values with comparable errors. The new value for the bottom quark, $m_b(10\text{ GeV}) = 3.610(16)$ GeV, corresponding to $m_b(m_b) = 4.163(16)$ GeV, makes use both of the new data and the new perturbative results and is consistent with the earlier determination.

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I. INTRODUCTION

The precise determination of charm and bottom quark masses has always been an important task both for theory and experiment. The most precise values have been obtained [1] from an analysis of the ITEP sum rules [2] (for reviews see Refs. [3–5]), combining data for the heavy quark production cross section in electron-positron collision with dispersion relations and a four-loop evaluation of the vacuum polarization induced by the heavy quark current. In this letter, we present an update of these results. We will include data recently published by the BABAR collaboration [6] and make use of new perturbative results which replace the estimates for the four-loop coefficients of higher moments used in the earlier publications.

II. ANALYTIC RESULTS

Our determination of the heavy quark masses follows closely Refs. [1, 7, 8]. It is based on the direct comparison of the theoretical and experimental evaluations of the contributions to the derivatives of the polarization function $\Pi_Q(q^2)$, the former evaluated in perturbative QCD, the latter through moments of the measured cross section for heavy quark production in electron-positron annihilation. Using dispersion relations, the moments of R_Q [28]

$$\mathcal{M}_n \equiv \int \frac{\mathrm{d}s}{s^{n+1}} R_Q(s)\,, \tag{1}$$

can be related to the derivatives of the vacuum polarization function at $q^2 = 0$

$$\mathcal{M}_n = \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_Q(q^2)\bigg|_{q^2=0}\,. \tag{2}$$

In its domain of analyticity $\Pi_Q(q^2)$ can be cast into the form

$$\Pi_Q(q^2) = Q_Q^2 \frac{3}{16\pi^2}\sum_{n\geq 0}\bar{C}_n z^n\,, \tag{3}$$

with $z = q^2/(4m_Q^2)$. Here $m_Q = m_Q(\mu)$ is the heavy quark mass with charge Q_Q in the $\overline{\mathrm{MS}}$ scheme at the scale μ. The coefficients $\bar{C}_n$ depend on α_s and on the heavy quark mass through logarithms of the form $l_{m_Q} = \ln(m_Q^2(\mu)/\mu^2)$. Equating theoretically calculated and experimentally measured moments, the heavy quark mass is given by

$$m_Q(\mu) = \frac{1}{2}\left(\frac{9Q_Q^2\bar{C}_n}{4\mathcal{M}_n^{\mathrm{exp}}}\right)^{1/(2n)}\,. \tag{4}$$

As a perturbative series the coefficients $\bar{C}_n$ can be written as

$$\begin{aligned}\bar{C}_n = {}&\bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_Q}\right)\\ &+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_Q} + \bar{C}_n^{(22)} l_{m_Q}^2\right)\\ &+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3\left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_Q} + \bar{C}_n^{(32)} l_{m_Q}^2\right.\\ &\left.+ \bar{C}_n^{(33)} l_{m_Q}^3\right) + \dots\,.\end{aligned} \tag{5}$$

The terms of order α_s^2 were evaluated up to $n = 8$ in Refs. [9–11] (and recently in Refs. [12, 13] even up to n=30). The four loop contributions to $\bar{C}_0$ and $\bar{C}_1$ were calculated in Refs. [14, 15]. For the higher moments the analysis of [1] was based on estimates for $\bar{C}_n^{(30)}$ with $n = 2,\ 3,\ 4$, which lead to an additional uncertainty in the mass determination. Recently the exact results for

n	1	2	3	4
charm	-5.6404	-3.4937	-2.8395	$-3.349(11)$
lower || upper limits	—	-6.0 || 7.0	-6.0 || 5.2	-6.0 || 3.1
bottom	-7.7624	-2.6438	-1.1745	$-1.386(10)$
lower || upper limits	—	-8.0 || 9.5	-8.0 || 8.3	-8.0 || 7.4

TABLE I: New results for the coefficients $\bar{C}_n^{(30)}$ in comparison with previous upper and lower limits as used in Ref. [1]. For less precise numerical results of $\bar{C}_n^{(30)}$ for $n = 3$ and $n = 4$ see Ref. [19].

the second [16] and third [17] moments were obtained. Combining these coefficients with additional information on the threshold and the high-energy behaviour and using the analyticity of $\Pi_Q(q^2)$ and Padé approximations, fairly precise numerical results were obtained [18] for the higher coefficients up to $n = 10$. (For an earlier analysis along similar lines see Ref. [19].) For the lowest four moments the four loop coefficients $\bar{C}_n^{(30)}$ are listed in Tab. I both for the charm and the bottom quark. All other coefficients relevant for $n = 1$ to 4 can be found in Tabs. 4 and 9 of [1]. It should be emphasized that these results are well within the estimates used in the analysis of [1], also shown in Tab. I. The impact of these new results on the quark mass determination will be studied below.

III. BOTTOM PRODUCTION CLOSE TO THRESHOLD

The determination of the bottom quark mass, as performed in [1, 7] relies heavily on the precise measurement of $R = \sigma(e^+e^- \to \text{hadrons})/\sigma_{\rm pt}$ (with $\sigma_{\rm pt} = \frac{4\pi\alpha^2}{3s}$), which enters the moments as defined above. Specifically, it is the contribution from the heavy quark current denoted as R_b with the light quark contribution subtracted. It is convenient to split the integration region into three pieces: The lowest region covering the narrow resonances, an intermediate "threshold" region between 10.62 GeV and 11.24 GeV, and the perturbative region above 11.24 GeV, where the measurement is replaced by the perturbative QCD prediction. The choice of 11.24 GeV corresponds to the upper end of the energy range covered by a CLEO measurement more than 20 years ago [20]. It also coincides approximately with the energy reach of a recent BABAR measurement [6]. In the analysis of [1], $\Upsilon(4S)$ with its mass $M_{\Upsilon(4S)} = 10.5794(12)$ GeV and width $\Gamma_{\Upsilon(4S)} = 20.5$ MeV has been considered together with the three lower, narrow resonances and thus the continuum part of the bottom-cross section was taken from 10.62 GeV upwards. Until recently the only measurement in the threshold region has been the one from the CLEO collaboration, which quotes a systematic error of about 6%. No radiative corrections had been applied. In Ref. [1] it has been argued, that a normalization factor 1/1.28 is necessary to reconcile these data with more recent and more precise CLEO results below the $\Upsilon(4S)$-resonance and with perturbative QCD at the high end. These "rescaled" data were the basis of the subsequent extraction of the bottom quark mass. However, in view of these uncertainties an overall systematic error of 10% was attributed to the contribution of the moments from this region. Thus, although this contribution to the moments is relatively small, its impact on the error was larger or equal than the one from the other two regions combined.

Recently a measurement of R_b in the energy region between 10.54 GeV and 11.20 GeV was performed by the BABAR collaboration with significantly improved statistics and with a correlated systematic error between 2.5% and 3% [6]. In principle this should allow an independent determination of the contribution to the moments with significantly reduced systematic error. However, no radiative corrections were applied to the published data and the radiative tails of the four lower Υ resonances were included in the quantity denoted R_b. In the following we describe the procedure used to obtain the contribution to the moments from these data.

In a first step we subtract the radiative tail of the $\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(3S)$ resonances, which is explicitly given in Ref. [6]. Subsequently we subtract the radiative tail of the $\Upsilon(4S)$ resonance. For the resonance shape we use a Breit-Wigner function with an electronic width of $\Gamma_{ee}(\Upsilon(4S)) = 0.272$ keV and an energy-independent total width of $\Gamma_{\rm tot}(\Upsilon(4S)) = 20.5$ MeV [21]. For the radiator function $G(z)$ we take the functional dependence as used in [22], based on the resummed NNLO result of [23]:

$$G(z) = \beta(1-z)^{\beta-1}\, {\rm e}^{\delta_{\rm yfs}}\, F\, \left(\delta_C^{V+S} + \delta_C^H\right) , \qquad (6)$$

with

$$\begin{aligned}
\beta =& \frac{2\alpha}{\pi}(L-1)\,, \qquad L = \ln\frac{s}{m_e^2}\,, \qquad F = \frac{{\rm e}^{-\beta\gamma_E}}{\Gamma(1+\beta)}\,, \\
\delta_{\rm yfs} =& \frac{\alpha}{\pi}\left(\frac{L}{2} - 1 + 2\zeta(2)\right) , \qquad (7)\\
\delta_c^{V+S} =& 1 + \frac{\alpha}{\pi}(L-1) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 L^2\,, \\
\delta_C^H =& -\frac{1-z^2}{2} + \frac{\alpha}{\pi}L\left[-\frac{1}{4}\left(1+3z^2\right)\ln z - 1 + z\right] .
\end{aligned}$$

The remainder $\hat{\sigma}$ corresponds to the continuum cross section distorted by initial-state radiation (ISR) and modified by vacuum polarization. It is related to σ, the cross section without ISR, through

$$\hat{\sigma}(s) = \int_{z_0}^{1} {\rm d}z\, G(z)\, \sigma(sz)\,, \qquad (8)$$

where the lower bound of the integration is given by $z_0 = (10.62{\rm GeV})^2/s$ corresponding to the point where

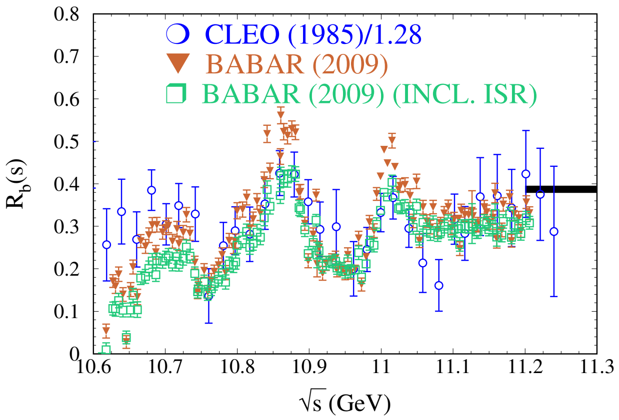

FIG. 1: Comparison of rescaled CLEO data for R_b with BABAR data before and after deconvolution. The black bar on the right corresponds to the theory prediction [24].

the continuum cross section (after subtraction of the $\Upsilon(4s)$ resonance) vanishes.

Given $\hat{\sigma}$, we can solve for σ in an iterative way as follows: Let us define $\delta G(z) \equiv G(z) - \delta(1-z)$ and evaluate a successive series of approximations,

$$\sigma_i = \sigma_0 - \int_{z_0}^{1} \mathrm{d}z \delta G(z) \sigma_{i-1}(sz) \,, \tag{9}$$

using as starting point $\sigma_0 = \hat{\sigma}$. The difference between σ_i and σ can be estimated by evaluating Eq. (8) with σ_i in place of σ. After five iterations the resulting function differs from $\hat{\sigma}$ by less than 0.5%.

Finally, the effect of the vacuum polarization must be taken into account and the result is normalised relative to the point cross section,

$$R_b = \sigma \frac{3s}{4\pi\alpha^2(s)} \,. \tag{10}$$

The integration region in Eq. (8) covers the energy interval between 10.62 GeV and 11.24 GeV, whence a constant value $(\alpha/\alpha(s))^2 = 0.93$ has been adopted.

In Fig. 1 we show the BABAR data [6] (after subtraction of the radiative tails of $\Upsilon(1S)$ to $\Upsilon(4S)$), together with R_b after deconvolution of ISR and correcting for the running of $\alpha(s)$. We also show the CLEO data [20] after the aforementioned rescaling.

It is now straightforward to evaluate the contribution to the moments. The result is listed in Tab. II and compared to our earlier analysis based on the CLEO result [20]. The error of this new result is dominated by the correlated systematic error of the BABAR measurement which amounts to about 3.5%. In addition we use a 2% error for the uncertainty from the details of the matching between the tail of $\Upsilon(4S)$ and the continuum around $\sqrt{s} = 10.62$ GeV, which we add in quadrature.

n	1	2	3	4
$\mathcal{M}_{n,\,\mathrm{old}}^{\mathrm{dat}} \times 10^{(2n+1)}$	0.296(32)	0.249(27)	0.209(22)	0.175(19)
$\mathcal{M}_{n,\,\mathrm{new}}^{\mathrm{dat}} \times 10^{(2n+1)}$	0.287(12)	0.240(10)	0.200(8)	0.168(7)
$\mathcal{M}_{n,\,\mathrm{new}}^{\mathrm{exp}} \times 10^{(2n+1)}$	4.592(31)	2.872(28)	2.362(26)	2.170(26)

TABLE II: Moments in $(\mathrm{GeV})^{-2n}$ for the bottom quark system from the threshold region $\langle 10.62$ GeV, 11.24 GeV$\rangle$ from Ref. [1] (old) and this letter (new). Also the new total experimental moments are given.

As is evident from Tab. II, the agreement between old and new result is remarkable giving additional confidence in the procedure used in Ref. [1]. The new experimental input and the new information on the coefficients $\bar{C}_n^{30}$ lead to a significant reduction of the error on m_b, as shown below.

IV. QUARK MASSES

In the absence of new data the analysis of m_c will be based on the moments listed in Tab. 6 of Ref. [1]. As emphasized earlier [1, 7] it is convenient to consider as primary quantity the running quark mass at scale 3 GeV. This is the natural scale for the sum rule (corresponding roughly to the charm threshold) and, as a consequence of the smaller strong coupling constant, the perturbative series exhibits a more stable behaviour.

If not stated otherwise, all input parameters and assumptions are identical to those of Ref. [1]. In particular we adopt $\alpha_s(M_Z) = 0.1189$. The new results and the corresponding errors are listed in Tab. III. Compared to [1], the shift induced by the analytic results for $\bar{C}_n^{(30)}$ amounts to 3, 4 and 8 MeV for n=2, 3 and 4 respectively. The results for all four moments are nicely consistent, and the three lowest moments exhibit comparable errors, ranging between 13 MeV and 17 MeV. Note, that the relative composition of the experimental input varies strongly from low to high moments: For $n = 1$ the contributions from narrow resonances and continuum are roughly comparable, for $n = 3$ the continuum contribution amounts to about 10%. Furthermore, the experimental contribution to the error decreases with increasing n from 9 MeV to 5 MeV, the μ-dependence, reflecting the theory uncertainly, increases from 2 MeV to 7 MeV. Despite the significant differences in the composition of the errors, the results are perfectly consistent. Since the result from $n = 1$ has the smallest dependence on the strong coupling and the smallest total error we take as our final value

$$m_c(3\ \mathrm{GeV}) = 986(13)\ \mathrm{MeV} \,, \tag{11}$$

and consider its consistency with $n = 2$, 3 and 4 as additional confirmation.

n	m_c(3 GeV)	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

TABLE III: Results for m_c(3 GeV) in MeV obtained from Eq. (4). The errors are from experiment, α_s, variation of μ and the gluon condensate.

n	m_b(10 GeV)	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

TABLE IV: Results for m_b(10 GeV) and $m_b(m_b)$ in MeV obtained from Eq. (4). The errors are from experiment, α_s and the variation of μ.

Transforming this to the scale-invariant mass $m_c(m_c)$ [25], including the four-loop coefficients of the renormalization group functions one finds [29] $m_c(m_c) = 1279(13)$ MeV. Let us recall at this point that a recent lattice determination, combining a lattice simulation for the data for the pseudoscalar correlator with the perturbative three- and four-loop result [11, 17, 26] has led to $m_c(3\text{ GeV}) = 986(10)$ MeV [27] in remarkable agreement with [1] and the present analysis.

The same approach is also applicable for the case of the bottom quark. Using the new moments with their significantly reduced experimental error (see Tab. II), one obtains the results for the bottom quark mass at the scale $\mu = 10\,\text{GeV}$ as listed in Tab. IV. In comparison with the previous determination a minute upwards shift of 1 MeV (resulting from an upward shift of +3 MeV from the new data and a downward shift of −2 MeV from the new theory input) and a reduction of both experimental and theory error is observed. The three results based on $n = 1$, 2 and 3 are of comparable precision. The relative size of the contribution from the continuum above 11.24 GeV which is modelled by perturbative QCD decreases for the higher moments $n = 2$ and 3. On the other hand the theory uncertainty, exemplified by the μ dependence is still acceptable. We therefore adopt the result from $n = 2$ (which is roughly between the $n = 1$ and $n = 3$ values and which exhibits the smallest error) as our final result

$$\begin{aligned} m_b(10\text{ GeV}) &= 3610(16)\text{ MeV}\,, \\ m_b(m_b) &= 4163(16)\text{ MeV}\,. \end{aligned} \tag{12}$$

These values are well consistent with the previous determination [1] $m_b(10\text{ GeV}) = 3609(25)$ MeV and $m_b(m_b) = 4164(25)$ MeV.

It is straightforward to evolve the new value for m_b to the normalization point at M_Z and $m_t(m_t) = 161.8$ GeV

$$\begin{aligned} m_b(M_Z) &= 2835 \pm 13 \pm 17\text{ MeV}\,, \\ m_b(161.8\text{ GeV}) &= 2703 \pm 12 \pm 19\text{ MeV}\,, \end{aligned} \tag{13}$$

where a matching to the $n_f = 6$ flavour theory has been performed in order to arrive at $m_b(161.8\text{ GeV})$. The first error originates from Eq. (12) the second from $\delta\alpha_s$.

For some of the applications it might be useful to explicitely exhibit the α_s dependence of our result, which is given by

$$\begin{aligned} m_c(3\text{ GeV}) &= \left(986 - \frac{\alpha_s - 0.1189}{0.002}\cdot 9 \pm 10\right)\text{ MeV}\,, \\ m_b(10\text{ GeV}) &= \left(3610 - \frac{\alpha_s - 0.1189}{0.002}\cdot 12 \pm 11\right)\text{ MeV}\,, \\ m_b(m_b) &= \left(4163 - \frac{\alpha_s - 0.1189}{0.002}\cdot 12 \pm 11\right)\text{ MeV}\,, \\ m_b(M_Z) &= \left(2835 - \frac{\alpha_s - 0.1189}{0.002}\cdot 27 \pm 8\right)\text{ MeV}\,, \\ m_b(161.8\text{ GeV}) &= \left(2703 - \frac{\alpha_s - 0.1189}{0.002}\cdot 28 \pm 8\right)\text{ MeV}\,, \end{aligned} \tag{14}$$

where $\alpha_s = \alpha_s(M_Z)$. When considering the ratio of charm and bottom quark masses, part of the α_s and of the μ dependence cancels

$$\frac{m_c(3\text{ GeV})}{m_b(10\text{ GeV})} = 0.2732 - \frac{\alpha_s - 0.1189}{0.002}\cdot 0.0014 \pm 0.0028\,, \tag{15}$$

which might be a useful input in ongoing analyses of bottom decays.

V. SUMMARY

Based on new four-loop results for the higher derivatives of the vacuum polarization function and new BABAR data for bottom quark production in the threshold region, a reanalysis of the charm- and bottom-quark mass determination has been performed. The new data, a posteriori, give additional support to the analysis of CLEO data presented in Ref. [20] and, furthermore, lead to a significant reduction of the experimental error. The new theory results for the higher moments lead to a further reduction of the theory uncertainty and, equally important, demonstrate the consistency between the analysis based on different moments. The final results, $m_c(3\text{GeV}) = 0.986(13)$ GeV and $m_b(10\text{GeV}) = 3.610(16)$ GeV are consistent with the earlier determination in Ref. [1] and, together with Ref. [27], constitute the most precise determination of charm- and bottom-quark masses to date.

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[1] J. H. Kühn, M. Steinhauser, C. Sturm. *Nucl. Phys.*, B778:192–215, 2007. hep-ph/0702103.
[2] V. A. Novikov, et al. *Phys. Rept.*, 41:1–133, 1978.
[3] L. J. Reinders, H. Rubinstein, S. Yazaki. *Phys. Rept.*, 127:1, 1985.
[4] P. Colangelo, A. Khodjamirian. 2000. hep-ph/0010175.
[5] B. L. Ioffe. *Prog. Part. Nucl. Phys.*, 56:232–277, 2006. hep-ph/0502148.
[6] B. Aubert, et al. *Phys. Rev. Lett.*, 102:012001, 2009. arXiv 0809.4120.
[7] J. H. Kühn, M. Steinhauser. *Nucl. Phys.*, B619:588–602, 2001. hep-ph/0109084.
[8] J. H. Kühn, M. Steinhauser. *JHEP*, 10:018, 2002. hep-ph/0209357.
[9] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser. *Phys. Lett.*, B371:93–98, 1996. hep-ph/9511430.
[10] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser. *Nucl. Phys.*, B482:213–240, 1996. hep-ph/9606230.
[11] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser. *Nucl. Phys.*, B505:40–64, 1997. hep-ph/9705254.
[12] R. Boughezal, M. Czakon, T. Schutzmeier. *Nucl. Phys. Proc. Suppl.*, 160:160–164, 2006. hep-ph/0607141.
[13] A. Maier, P. Maierhöfer, P. Marquard. *Nucl. Phys.*, B797:218–242, 2008. arXiv 0711.2636.
[14] K. G. Chetyrkin, J. H. Kühn, C. Sturm. *Eur. Phys. J.*, C48:107–110, 2006. hep-ph/0604234.
[15] R. Boughezal, M. Czakon, T. Schutzmeier. *Phys. Rev.*, D74:074006, 2006. hep-ph/0605023.
[16] A. Maier, P. Maierhöfer, P. Marquard. *Phys. Lett.*, B669:88–91, 2008. arXiv 0806.3405.
[17] A. Maier, P. Maierhöfer, P. Marquard, A. V. Smirnov. 2009. arXiv 0907.2117.
[18] Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard. 2009. arXiv 0907.2120.
[19] A. H. Hoang, V. Mateu, S. Mohammad Zebarjad. *Nucl. Phys.*, B813:349–369, 2009. arXiv 0807.4173.
[20] D. Besson, et al. *Phys. Rev. Lett.*, 54:381, 1985.
[21] C. Amsler et al. (Particle Data Group). *Phys. Lett,*, B667:1, 2008.
[22] K. G. Chetyrkin, J. H. Kühn, T. Teubner. *Phys. Rev.*, D56:3011–3018, 1997. hep-ph/9609411.
[23] S. Jadach, B. F. L. Ward. *Comput. Phys. Commun.*, 56:351–384, 1990.
[24] R. V. Harlander, M. Steinhauser. *Comput. Phys. Commun.*, 153:244–274, 2003. hep-ph/0212294.
[25] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser. *Comput. Phys. Commun.*, 133:43–65, 2000. hep-ph/0004189.
[26] C. Sturm. *JHEP*, 09:075, 2008. arXiv 0805.3358.
[27] I. Allison, et al. *Phys. Rev.*, D78:054513, 2008. arXiv 0805.2999.
[28] For the precise definition of R_Q, in particular the treatment of gluon splitting into $Q\bar{Q}$, the subtraction of singlet contributions, and the role of nonperturbative terms in the case of charm quarks we refer to Ref. [1].
[29] In Ref. [1] an imprecise routine has been used to compute the scale-invariant charm quark mass with the result $m_c(m_c) = 1286$ MeV for the central value. In this letter we solve the coupled system of differential equations numerically which has also been done in Ref. [27] where $m_c(m_c) = 1268$ MeV has been obtained using $\alpha_s(M_Z) = 0.1176$.