The SUSY-QCD β function to three loops

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Abstract

A number of $\overline{\mathrm{DR}}$ renormalization constants in softly broken SUSY-QCD are evaluated to three-loop level: the wave function renormalization constants for quarks, squarks, gluons, gluinos, ghosts, and ϵ-scalars, and the renormalization constants for the quark and gluino mass as well as for all cubic vertices. The latter allow us to derive the corresponding β functions through three loops, all of which we find to be identical to the expression for the gauge β function obtained by Jack, Jones, and North [1] (see also Ref. [2]). This explicitely demonstrates the consistency of DRED with SUSY and gauge invariance, an important pre-requisite for precision calculations in supersymmetric theories.

1 Introduction

Up to now, particle collider experiments could not provide clear evidence for physics beyond the Standard Model (BSM). The quest for extensions of the Standard Model comes either from non-accelerator observations, such as Dark Matter and Dark Energy, or it is based on purely theoretical considerations. For example, naturalness [3, 4, 5] requires some mechanism to stabilize the Higgs mass at the electro-weak scale (fine tuning problem); the intriguing evolution of the gauge couplings towards a common value at high energies needs an enlarged particle spectrum if unification is to occur at scales compatible with proton decay [6, 7]; any explanation for electro-weak symmetry breaking presumably requires an embedding of the Standard Model into a higher symmetry.

In the light of these considerations, supersymmetry (SUSY) is a strong candidate for an extension of the Standard Model [8]. Not only does it provide solutions for Dark Matter, it also solves the fine tuning problem by canceling the quadratic divergences in the Higgs self energies, it explains electro-weak symmetry breaking by a simple evolution of the parameters of the Higgs potential, and unification of the gauge couplings can be achieved for rather natural choices of the parameters.

This latter issue is a rather unique feature: it actually allows one to make quantitative predictions at energy scales that are several orders of magnitude larger than what can be achieved at current and probably also at any future particle collider. Crucial ingredients for such indirect measurements are precision data as well as precision calculations (see, e.g., Ref. [9]).

In order to study the unification of the gauge couplings in SUSY, one needs to include the effect of the supersymmetric particle spectrum into the running. In addition, dimensional regularization with minimal subtraction, called $\overline{\mathrm{MS}}$, is no longer a good renormalization scheme in the sense that it explicitely breaks supersymmetry. An alternative regularization, so-called Dimensional Reduction (DRED) was suggested by Siegel [10], however, inconsistencies of this method were pointed out only shortly afterwards [11]. Nevertheless, renormalization by combining DRED with minimal subtraction (the $\overline{\mathrm{DR}}$ scheme) has become the preferred scheme in higher order supersymmetric calculations [12]. A thorough formulation of DRED which isolates the source of the inconsistencies has been given in Ref. [13]. It was pointed out that, although a mathematical consistent formulation of DRED is possible, it could violate SUSY at higher orders of perturbation theory. It is the main goal of the current paper to demonstrate this consistency within SUSY-QCD up to the three-loop level.

Renormalization group functions, governing the energy dependence of masses and couplings, are among the simplest quantities to compute in perturbative quantum field theory. For example, the anomalous dimensions of the strong coupling constant in QCD (the QCD β function) and the quark masses are known at four-loop level in the $\overline{\mathrm{MS}}$ scheme [14, 15, 16, 17] (the conversion to the $\overline{\mathrm{DR}}$ scheme was done in Ref. [18]).

For supersymmetric gauge theories, one can devise a particular renormalization scheme, so-called NSVZ [19], where an all-orders relation between the gauge β function and the anomalous dimension of the chiral supermultiplet is valid. So, in the absence of the matter supermultiplet, *i.e.* for SUSY-Yang-Mills theory, the β function is known to all orders in the coupling constant. The problem with this scheme is that general conversion rules to schemes used in perturbative calculations such as $\overline{\mathrm{MS}}$ or $\overline{\mathrm{DR}}$ are not known. However, by explicit calculation of the abelian β function in the $\overline{\mathrm{DR}}$ scheme, comparison with the NSVZ result, and arguments based on its holomorphy, the authors of Ref. [1] found the two-loop conversion formula for the coupling also in the non-abelian case. The SUSY-QCD β function in the $\overline{\mathrm{DR}}$ scheme is thus known to three loops. The result was later confirmed by an explicit calculation using the background field method [2], which requires three-loop corrections to the gluon propagator.

Applying the same method based on holomorphy of the NSVZ scheme to softly broken SUSY gauge theory, the authors of Ref. [20] derived the renormalization group equation governing the running of the gaugino and squark masses, valid to all orders in the coupling constant. Using this result, an elegant relation between the gaugino and the gauge β functions was formulated in Ref. [21], both for the NSVZ and the $\overline{\mathrm{DR}}$ scheme.

One of the goals of the current paper is to add another confirmation of these results by calculating for the first time directly the relevant vertex diagrams through three-loop order. Dimensional reduction is implemented by explicitely introducing ϵ-scalars, the $(4-d)$-dimensional components of the gluon field.

In addition, we derive the gauge β function β_s from all possible three-point functions in SUSY-QCD that involve the strong coupling α_s. The fact that in each case we obtain the same expression provides an explicit check on the consistency of DRED with gauge invariance and supersymmetry. In particular, we calculate the renormalization of the quark-squark-gluino vertex which is not fixed by gauge invariance but by supersymmetry alone. Furthermore, we confirm the relation between β_s and the gluino mass anomalous dimension $\gamma_{\tilde{g}}$ by an explicit calculation of $\gamma_{\tilde{g}}$ to three loops. We also verify the three-loop formula for the quark anomalous dimension γ_q [22] that was derived in the superfield formalism. It is an essential ingredient for the derivation of the four-loop SUSY-QCD gauge β function in the approach of Ref. [22].

The ϵ-scalars occurring in DRED have Feynman rules that can be derived from the ones of the gluon by replacing d-dimensional objects with 2ϵ-dimensional ones. In a supersymmetric theory, the coupling constants for the ϵ-scalar and the gluon must be identical, which means that the β function for the ϵ-scalar couplings must be given by the one of the strong coupling constant α_s. We check also this relation through three loops, by evaluating three-point functions involving ϵ-scalars through three loops.

The remainder of this paper is organized as follows: In Sect. 2, we review the method of DRED and describe its implementation in our setup. Sect. 3 describes some details of the calculation for the Green's functions and the β functions, and also gives the main results. Concluding remarks are given in Sect. 4.

2 Dimensional Reduction

2.1 Notation and technical setup

We implement DRED by introducing the quasi-four, -d, and -2ϵ-dimensional spaces [23] (see also Ref. [13]) Q_4, Q_d, $Q_{2\epsilon}$, where $d = 4 - 2\epsilon$ and

$$Q_4 = Q_d \cup Q_{2\epsilon} \qquad \text{and} \qquad Q_d \cap Q_{2\epsilon} = 0 \,. \tag{1}$$

Also, we introduce quasi-$four$, -d and -2ϵ-dimensional sets of indices, implying the correspondence

$$\mu \leftrightarrow (\hat{\mu}, \bar{\mu}) \,, \quad \nu \leftrightarrow (\hat{\nu}, \bar{\nu}) \,, \quad \rho \leftrightarrow (\hat{\rho}, \bar{\rho}) \,, \quad \sigma \leftrightarrow (\hat{\sigma}, \bar{\sigma}) \,, \tag{2}$$

in the sense that, for example,

$$t^\mu = t^{\hat{\mu}} + t^{\bar{\mu}} \qquad \text{etc.} \tag{3}$$

for any vector t. I.e., $t^{\hat{\mu}}$ and $t^{\bar{\mu}}$ denote the projections of $t^\mu \in Q_4$ to the subspaces Q_d and $Q_{2\epsilon}$, respectively. For generic indices without reference to (quasi-)dimensionality we will use the letters $\alpha, \beta, \gamma, \delta$.

For the metric tensor, we have

$$\begin{gathered} g^{\hat{\mu}\bar{\mu}} = 0 \qquad \Rightarrow \qquad g^{\mu\nu} = g^{\hat{\mu}\hat{\nu}} + g^{\bar{\mu}\bar{\nu}} \,, \\ g^{\bar{\mu}\bar{\nu}} t_\mu = t^{\bar{\nu}} \,, \qquad g^{\hat{\mu}\hat{\nu}} t_\mu = t^{\hat{\nu}} \,, \qquad g^\mu_\mu = 4 \,, \qquad g^{\hat{\mu}}_{\hat{\mu}} = d \,, \qquad g^{\bar{\mu}}_{\bar{\mu}} = 2\epsilon \,. \end{gathered} \tag{4}$$

The usual commutation relations for four-dimensional Dirac matrices are carried forward to their d- and 2ϵ-dimensional projections:

$$\{\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}\} = 2g^{\hat{\mu}\hat{\nu}} \,, \qquad \{\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\} = 2g^{\bar{\mu}\bar{\nu}} \,, \qquad \{\gamma^{\hat{\mu}}, \gamma^{\bar{\nu}}\} = 0 \,. \tag{5}$$

These relations together with the constraint $\mathrm{Tr}\,\mathbb{1} = 4$ allow us to evaluate fermion traces (without γ_5) in the following way:

$$\mathrm{Tr}(\gamma^{\hat{\mu}_1}\cdots\gamma^{\hat{\mu}_n}\,\gamma^{\bar{\nu}_1}\cdots\gamma^{\bar{\nu}_n}) = \frac{1}{4}\mathrm{Tr}(\gamma^{\hat{\mu}_1}\cdots\gamma^{\hat{\mu}_n})\mathrm{Tr}(\gamma^{\bar{\nu}_1}\cdots\gamma^{\bar{\nu}_m})\,. \tag{6}$$

However, the quark-squark-gluino vertices involve γ_5, and the problems involving γ_5 in other than four dimensions are of course well known. We treat it as follows:

1. Use the relations

$$\{\gamma^{\hat{\mu}}, \gamma_5\} = \{\gamma^{\bar{\mu}}, \gamma_5\} = 0\,, \qquad (\gamma_5)^2 = 1\,, \tag{7}$$

   until there is at most one γ_5 per fermion trace. If all γ_5 matrices are eliminated in this way, one can continue with Eq. (6).

2. Fermion traces with a single γ_5 matrix are reduced to the form $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma_5)$ by using Eqs. (5) and (7), and then evaluated by applying the *formal* replacement

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma_5) = 4i\,\varepsilon^{\alpha\beta\gamma\delta}\,. \tag{8}$$

3. The intrinsically (*not* quasi-)four-dimensional Levi-Civita symbol $\varepsilon^{\alpha\beta\gamma\delta}$ in Eq. (8) with quasi-dimensional indices makes sense when combined with a second one of its kind. By applying appropriate projectors, our calculations are set up in such a way that this is always the case, so we can use

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha'\beta'\gamma'\delta'} = g^{[\alpha}_{[\alpha'}\,g^{\beta}_{\beta'}\,g^{\gamma}_{\gamma'}\,g^{\delta]}_{\delta']}\,, \tag{9}$$

   where the square brackets denote anti-symmetrization. Note that due to $g^{\hat{\mu}\bar{\mu}} = 0$, one can write, e.g.,

$$\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\bar{\sigma}}\varepsilon_{\hat{\mu}'\hat{\nu}'\hat{\rho}'\bar{\sigma}'} = g^{[\hat{\mu}}_{[\hat{\mu}'}\,g^{\hat{\nu}}_{\hat{\nu}'}\,g^{\hat{\rho}]}_{\hat{\rho}']}\,g^{\bar{\sigma}}_{\bar{\sigma}'}\,, \qquad \varepsilon^{\hat{\mu}\hat{\nu}\bar{\rho}\bar{\sigma}}\varepsilon_{\hat{\mu}'\hat{\nu}'\bar{\rho}'\bar{\sigma}'} = g^{[\hat{\mu}}_{[\hat{\mu}'}\,g^{\hat{\nu}]}_{\hat{\nu}']}\,g^{[\bar{\rho}}_{[\bar{\rho}'}\,g^{\bar{\sigma}]}_{\bar{\sigma}']}\,,$$
$$\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\bar{\sigma}}\varepsilon_{\hat{\mu}'\hat{\nu}'\bar{\sigma}'\bar{\tau}'} = 0\,, \qquad \text{etc.}$$

We made sure that the terms that need to be treated in this way generate no higher than simple poles in ϵ. Therefore, we can be sure that the above procedure is valid without introducing additional finite counterterms. Of course, the fact that we find the same β function from all vertices also supports our treatment of γ_5.

2.2 ϵ-scalars and corresponding Feynman rules

We work in $N = 1$ supersymmetric QCD with n_f quark flavors, corresponding to the strong sector of the MSSM. The Feynman rules are therefore given by the usual ones for QCD, involving quarks q, gluons g, and ghosts c, plus the Feynman rules for their supersymmetric partners: the squarks $\tilde{q}_L$, $\tilde{q}_R$, and the gluinos, $\tilde{g}$. In our implementation, instead of the partners of the left- and right-handed quarks, we will use the mass eigenstates

$$\begin{pmatrix}\tilde{q}_1\\ \tilde{q}_2\end{pmatrix} = \begin{pmatrix}\cos\theta_q & \sin\theta_q\\ -\sin\theta_q & \cos\theta_q\end{pmatrix}\begin{pmatrix}\tilde{q}_L\\ \tilde{q}_R\end{pmatrix}\,.$$

The mixing angle θ_q depends on the squark masses; however, all the quantities in this paper will be independent of any masses, and therefore, they must also be independent of the squark mixing angle. This serves as a welcome check on our calculations.

The tree-level Feynman rules for SUSY-QCD in DREG can be found in the literature; in this paper, we will use those given in Ref. [24]. DRED results are obtained by supplementing these Feynman rules by new ones for the so-called ϵ-scalars, as will be explained in what follows.

DRED is defined by assuming that all fields depend only on the coordinates $x^{\hat{\mu}}$ of a d-dimensional subspace of the usual four-dimensional Minkowski space. I.e., the 2ϵ-dimensional components $x^{\bar{\mu}}$ have no physical relevance. In particular, this means that derivatives ∂_μ and momenta p_μ can always be replaced by $\partial_{\hat{\mu}}$ and $p_{\hat{\mu}}$. The technical implementation is done by decomposing the quasi-four-dimensional vector fields into d- and 2ϵ-dimensional components:

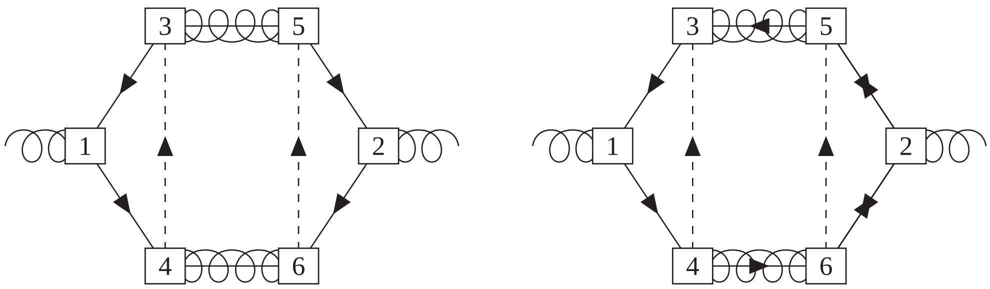
$$A_\mu = A_{\hat{\mu}} + A_{\bar{\mu}}\,. \tag{10}$$

For convenience, $A_{\hat{\mu}}$ will be simply called the "gluon field" in what follows, while A_μ will be explicitely referred to as "four-dimensional gluon field". $A_{\bar{\mu}}$ is the so-called ϵ-scalar. The corresponding Feynman rules for the ϵ-scalars are obtained from the d-dimensional SUSY-QCD ones by replacing the d-dimensional indices by 2ϵ-dimensional ones. In this way, one arrives at the following vertices: $q\bar{q}\epsilon$, $g\epsilon\epsilon$, $\tilde{g}\tilde{g}\epsilon$, $\epsilon\epsilon\tilde{q}\tilde{q}$, $gg\epsilon\epsilon$, $\epsilon\epsilon\epsilon\epsilon$, where ϵ denotes the ϵ-scalar. Their Feynman rules are given explicitely in Appendix A.

The bare coupling constants for the gluon and the ϵ-scalar are identical by construction. However, *a priori* it is not clear whether this holds also for the renormalized couplings. In principle, all ϵ-scalar couplings could be different without violating gauge invariance. In fact, it is well known that they differ in standard (i.e., non-SUSY) QCD (see, e.g., Refs. [25, 18, 26]). Even more: in order to renormalize the quartic ϵ-scalar vertex, one has to take into account all possible color structures for it, and attribute to each one a separate coupling constant. For SUSY-QCD, we have explicitly checked that at the one-loop order only the β function associated with the usual color structure of the four-gluon interaction, *i.e.* $f_{abe}f_{cde}$[1], does not vanish and it equals the one-loop gauge β function. So, through one-loop, one can identify the coupling constant of the corresponding ϵ-scalar quartic interaction with the strong coupling constant and set to zero the other three couplings. This order of accuracy is sufficient for the results discussed in this paper, as the ϵ-scalar quartic interactions contribute to the anomalous dimensions starting from the two-loop order.

In order for the renormalized Lagrangian to obey SUSY, the decomposition of Eq. (10) should hold also at higher orders. Therefore, the renormalized gluon and ϵ-scalar coupling constants must be equal, i.e., their β functions must be the same. We will explicitely confirm this for the $q\bar{q}\epsilon$, the $\tilde{g}\tilde{g}\epsilon$, and the $g\epsilon\epsilon$ vertex through three loops, thus confirming consistency of DRED with SUSY at next-to-next-to-next-to-leading order of perturbation theory.

[1] f_{abe} denotes the structure constants of the gauge group.



(a) (b)

Figure 1: (a) Feynman diagram involving Majorana fermions in a non-trivial way. Solid lines are quarks, dashed lines are squarks, slashed springy lines are gluinos, and the external lines are gluons. The arrows on the lines denote the charge flow. Note that the fermion line involves quarks with opposite charge flow. (b) The same diagram after applying the prescription proposed in Ref. [27, 28]. The arrows on the gluino lines denote the external flow (see text); lines with two "clashing" arrows are "flipped" quarks which have external flow opposite to the charge flow.

```
|1|g
|2|g
|3,1|fQ,fq
|1,4|fQ,fq
|5,2|fQ,fq
|2,6|fQ,fq
|4,3|Sq1,sq1
|5,3|fgm,fgm
|6,4|fgm,fgm
|6,5|Sq1,sq1
```

```
|1|g
|2|g
|3,1|fQ,fq
|1,4|fQ,fq
|2,5|fQx,fqx
|6,2|fQx,fqx
|4,3|Sq1,sq1
|5,3|fgmx,fgm
|4,6|fgmx,fgm
|6,5|Sq1,sq1
```

Figure 2: The diagrams of Fig. 1 in topological notation.

2.3 Majorana fermions

Another technical issue is given by the Majorana character of the gluinos. A well-defined and practical prescription how to deal with Majoranas in higher order perturbative calculations was given in Ref. [27, 28]. Unfortunately, using this method in combination with the Feynman diagram generator `qgraf` [29] is not straightforward. We implement it by parsing the output of `qgraf` with a `PERL` program, giving each chain of fermion lines a well-defined direction. If the chain contains a Dirac fermion f whose ordinary fermion direction ("charge flow") is opposite to the one chosen by the `PERL` program ("external flow"), we replace this particle with a "flipped fermion" $\check{f}$ which has opposite charge flow. Majoranas have no definite charge flow and are simply interpreted as Dirac particles with charge flow equal to the external flow.

An example is shown in Fig. 1. In topological notation, diagram (a) is represented by the left column of Fig. 2, where each line denotes a propagator in obvious notation. After application of the prescription of Ref. [27, 28], the diagram in Fig. 1 (b) results, or, again in topological notation, the right column of Fig. 2. One can see that the quarks (`fQ,fq`) connecting vertices 2 with 5 and 6 with 2 have been replaced by "flipped" quarks (`fQx,fqx`), and the Majorana notation for the gluinos (`fgm,fgm`) has been changed to the Dirac notation, where particle and anti-particle are distinguishable (`fgmx,fgm`).

The result looks like regular `qgraf` output with only Dirac fermions. The flipped fermions have Feynman rules that are easily derived from the original fermions using Ref. [27, 28]. They are given explicitely in Appendix A. Finally, the overall sign is determined in the usual way by counting the number of closed fermion loops.

Note that the same approach has already been applied successfully to earlier calculations [30].

For the remainder of the calculation, we use the following setup: the diagrams are brought to `MINCER` notation [31] with the help of the `C++` program `q2e` [32]. The actual calculation is done in `FORM` [33]: After applying appropriate projectors in Lorentz, Dirac and color space and evaluating the fermion traces, we use `MINCER` to evaluate the Feynman integrals in terms of a Laurent series in ϵ.

The representation of the four-gluon vertex through an auxiliary, non-propagating particle allows us to factorize the color factor from the rest of the diagram and to evaluate it separately. For that purpose, we use the program `color.frm` [34] which expresses the result in terms of color invariants.

3 Evaluation of the renormalization constants

We closely follow the method of Refs. [35, 36], i.e., we calculate connected, amputated Green's functions and derive the coupling constant renormalization from Slavnov-Taylor identities. For example, if we compute the N-point Green's function with external fields $\phi_1, \cdots, \phi_n$ and denote its coupling constant by g, we obtain

$$Z_g = \frac{Z_{\phi_1 \cdots \phi_N}}{\sqrt{Z_{\phi_1} \cdots Z_{\phi_N}}}\,, \tag{11}$$

where the Z_{ϕ_i} are the wave function renormalization constants for the ϕ_i, $Z_{\phi_1 \cdots \phi_N}$ is the corresponding vertex renormalization constant, and Z_g the charge renormalization.

Since we work in a minimal subtraction scheme, the renormalization constants (and thus also the β function) are independent of any mass scale. We remind the reader that due to this fact, decoupling of heavy particles has to be done "by hand" in the $\overline{\rm DR}$ scheme, by introduction of decoupling constants for the renormalized parameters of the model. The decoupling constants in the MSSM for the strong coupling constant and the quark mass have been calculated to two-loop order in Refs. [9, 37].

The anomalous dimension for the coupling g can be obtained from the simple pole of Z_g in the usual way via

$$0 \equiv \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g^{\mathrm{B}} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\mu^{\epsilon} Z_g g\right)\,, \tag{12}$$

where $g^{\rm B}$ is the bare coupling, and μ is the (arbitrary) renormalization scale. Z_g does not explicitely depend on μ, and if g is the only coupling constant in the theory, then

$$\mu\frac{\rm d}{{\rm d}\mu}g = \frac{-\epsilon g}{1+g\frac{\rm d}{{\rm d}g}Z_g}\,. \tag{13}$$

Applying DRED to a non-SUSY theory like standard QCD, Z_g in general depends also on the ϵ-scalar couplings, and Eq. (13) assumes a slightly more complicated form (see, e.g., Ref. [25]). The β function is defined through

$$\beta(\alpha_s) = \mu^2\frac{\rm d}{{\rm d}\mu^2}\frac{\alpha_s(\mu^2)}{\pi}\,,\quad \text{where}\quad \alpha_s = \frac{g^2}{4\pi}\,. \tag{14}$$

In analogy to Eq. (14) we define the anomalous dimension for fermion (quark and gluino) masses

$$\gamma_i = \frac{\mu^2}{m_i}\frac{{\rm d}m_i}{{\rm d}\mu^2} = -\pi\beta_s\frac{{\rm d}\ln Z_{m,i}}{{\rm d}\alpha_s}\,,\quad i=q\,,\tilde{g}\,, \tag{15}$$

where $Z_{m,i}$ is the mass renormalization constant.

Since we work in a SUSY theory, we expect the renormalization constants for the ϵ-scalar couplings to be identical to the ones of the strong coupling g_s, provided DRED is consistent with SUSY. When evaluating the strong coupling renormalization Z_s at n-loop, at most the $(n-1)$-loop expression for the ϵ-scalar coupling renormalization Z_e is needed. Therefore, the calculation of Z_s at n-loop order checks the identity $Z_s = Z_e$ through $(n-1)$-loop order: if $Z_s \neq Z_e$ at $(n-1)$-loop order, in most cases the n-loop expression for Z_s would not be local. However, in some cases it can also happen that simply the $1/\epsilon$ pole is wrong.

In our calculation it is $n=3$, so we check the equality to two-loop order. In addition to that, however, we have explicitly evaluated a number of ϵ-scalar couplings to three-loop order (see Table 1) and also find equality to the strong coupling constant. In this way, consistency of DRED with SUSY is confirmed through three-loop order.

For the calculation of the renormalization constants within the minimal subtraction scheme, one is free to choose any masses and external momenta, as long as infra-red divergences are avoided. We set all masses to zero, as well as one of the two independent external momenta in the three-point functions. We therefore arrive at three-loop integrals with one non-vanishing external momentum q which can be calculated with the help of `MINCER`. As a check, we have also calculated some of the three-point functions with non-zero particle masses m by expanding them in the limit $m^2/q^2 \ll 1$ with the help of asymptotic expansions [38]. In the final expression the limit $m \to 0$ can be taken and the result coincides with the one obtain with the massless set-up. Possible infra-red singularities would manifest in $\ln m^2/q^2$ terms which are absent in our calculations.

Traces with a single γ_5 do not contribute to any of the two-point functions that we calculated. They do contribute for some of the three-point functions though, in particular the $q\tilde{q}\tilde{g}$, the $\tilde{g}\tilde{g}\epsilon$, and the $q\bar{q}\epsilon$ vertex. An example diagram for the latter vertex is shown in Fig. 3. Such diagrams contribute (among others) a color factor $d_R^{abcd}d_A^{abcd}$ (for the notation, see Refs. [34, 26]), but they cancel against the same factors from other sources in the final result for the renormalization constants and the β function.

# loops	1	2	3
$c\bar{c}$	1	14	423
$q\bar{q}$	4	86	3583
$\epsilon\epsilon$	7	100	3902
$\tilde{g}\tilde{g}$	6	130	5577
$\tilde{q}\tilde{q}$	8	157	6760
gg	11	171	6954

# loops	0	1	2	3
$c\bar{c}g$	1	2	77	3920
$q\bar{q}\epsilon$	1	6	319	21669
$q\bar{q}g$	1	8	439	30078
$\tilde{g}\tilde{g}\epsilon$	1	8	445	31815
$q\tilde{q}\tilde{g}$	1	6	430	36868
$\epsilon\epsilon g$	1	14	618	38741
$\tilde{g}\tilde{g}g$	1	12	657	46314
$\tilde{q}\tilde{q}g$	1	12	674	52205
ggg	1	26	1105	70705

Table 1: The number of diagrams contributing to the Green's functions evaluated in this work. Left table: two-point functions; right table: three-point functions. The first column indicates the external legs of the Green's function, the other columns show the number of diagrams at the individual loop orders (0–3). The number depends on the implementation of the quartic vertices. In our calculation, we split the $gggg$, $gg\epsilon\epsilon$, and the $\epsilon\epsilon\epsilon\epsilon$ each into two cubic vertices by introducing non-propagating auxiliary particles.

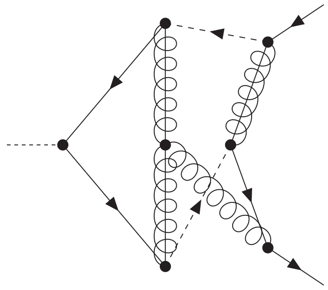


Figure 3: Sample diagram for the three-loop $q\bar{q}\epsilon$ vertex where a non-vanishing trace with a single γ_5 matrix occurs. The internal line styles are as in Fig. 1.

The results for the renormalization constants can be obtained in electronic form from Ref. [39]. It is straightforward to obtain from them the three-loop SUSY-QCD β function which reads

$$
\begin{aligned}
&\beta(\alpha_s) = -\sum_{n\geq 0}\left(\frac{\alpha_s}{\pi}\right)^{n+2}\beta_n\,,\\
\beta_0 &= \frac{3}{4}C_A - \frac{1}{2}T_f\,,\\
\beta_1 &= \frac{3}{8}C_A^2 - T_f\left(\frac{1}{2}C_F + \frac{1}{4}C_A\right),\\
\beta_2 &= \frac{21}{64}C_A^3 + T_f\left(\frac{1}{4}C_F^2 - \frac{13}{16}C_AC_F - \frac{5}{16}C_A^2\right) + T_f^2\left(\frac{3}{8}C_F + \frac{1}{16}C_A\right),
\end{aligned} \tag{16}
$$

where $C_F = (n_c^2-1)/(2n_c)$, $C_A = n_c$ are the Casimir operators of SU(n_c), and $2T_f = n_f$ is the number of quark flavors (which is equal to the number of squark flavors). The result in Eq. (16) is in full agreement with Ref. [1, 2].

The case $T_f = 0$ (SUSY–Yang-Mills) has been treated before, both in the superfield formalism [22] and the diagrammatic approach [18]. Full agreement has been found between the two methods up to four-loop order.

The anomalous dimension for the quark mass is independent of any SUSY-breaking parameters and is given to three-loop level by

$$
\begin{aligned}
&\gamma_q(\alpha_s) = -\sum_{n\geq 0}\left(\frac{\alpha_s}{\pi}\right)^{n+1}\gamma_n^q\,,\\
\gamma_0^q &= \frac{1}{2}C_F\,,\\
\gamma_1^q &= -\frac{1}{4}C_F^2 + \frac{3}{8}C_AC_F - \frac{1}{4}T_fC_F\,,\\
\gamma_2^q &= \frac{1}{4}C_F^3 - \frac{3}{16}C_AC_F^2 + \frac{3}{16}C_A^2C_F + T_f\left[\left(-\frac{1}{2}+\frac{3}{4}\zeta(3)\right)C_F^2 + \left(\frac{1}{16}-\frac{3}{4}\zeta(3)\right)C_AC_F\right]\\
&\quad -\frac{1}{8}T_f^2C_F\,,
\end{aligned} \tag{17}
$$

where ζ is Riemann's zeta function with $\zeta(3) = 1.20206\ldots$. The result of Eq. (17) is in agreement with Ref. [22].

In Ref. [21] it has been shown that the anomalous dimension of the gluon mass defined through

$$
\gamma_{\tilde{g}}(\alpha_s) = -\sum_{n\geq 0}\left(\frac{\alpha_s}{\pi}\right)^{n+1}\gamma_n^{\tilde{g}}\,, \tag{18}
$$

can be related to the gauge β function of Eq. (16) via

$$
\gamma_n^{\tilde{g}} = (n+1)\beta_n\,. \tag{19}
$$

We have explicitly verified that this relation holds through three loops for the case of vanishing SUSY-breaking parameters except the gluino mass.

4 Conclusions

The main result of this paper is a confirmation of the consistency of DRED with SUSY at three-loop order.

We have shown that the same result for the SUSY-QCD β function is obtained from all three-particle vertices, even the ones involving ϵ-scalars which are used for the implementation of DRED. Furthermore, we verify a predicted relation between the β function and the gluino mass anomalous dimension to three-loop order and confirm the result for the quark mass anomalous dimension present in the literature.

Along with this paper we provide the results for all two- and three-point renormalization constants [39] which should be useful for other calculations within the MSSM.

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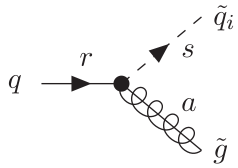
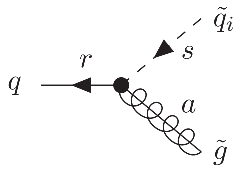
A Feynman Rules

In this appendix we give the Feynman rules involving ϵ-scalars and flipped fermions. The notation is the same as in Appendix A.2 of Ref. [24], except that all Lorentz indices occuring there are understood as d-dimensional ones. Note that in that paper, the sign of the $gg\tilde{q}\tilde{q}$ vertex should be reversed, and there is a typo for one of the $q\tilde{q}\tilde{g}$ vertices which should read

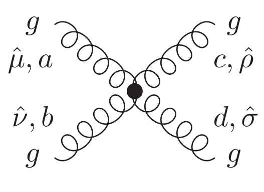


$$ig_s T^a_{sr}\sqrt{2}(\mathcal{R}^q_{i1}P_L - \mathcal{R}^q_{i2}P_R)$$



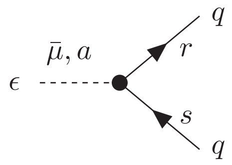
$$ig_s T^a_{rs}\sqrt{2}(\mathcal{R}^q_{i1}P_R - \mathcal{R}^q_{i2}P_L)$$

Also, the four-gluon vertex did not contribute in Ref. [24], so for completeness, let us quote it here:

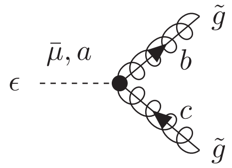


$$\begin{aligned}
&-ig_s^2 f^{abe} f^{cde}\left[g^{\hat{\mu}\hat{\rho}}g^{\hat{\nu}\hat{\sigma}} - g^{\hat{\mu}\hat{\sigma}}g^{\hat{\nu}\hat{\rho}}\right] \\
&- ig_s^2 f^{ace} f^{bde}\left[g^{\hat{\mu}\hat{\nu}}g^{\hat{\rho}\hat{\sigma}} - g^{\hat{\mu}\hat{\sigma}}g^{\hat{\nu}\hat{\rho}}\right] \\
&- ig_s^2 f^{ade} f^{bce}\left[g^{\hat{\mu}\hat{\nu}}g^{\hat{\rho}\hat{\sigma}} - g^{\hat{\mu}\hat{\rho}}g^{\hat{\nu}\hat{\sigma}}\right]
\end{aligned}$$

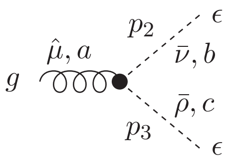
The following are the tri-linear vertices involving ϵ-scalars (all momenta incoming):



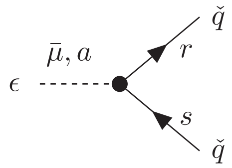
$$ig_s T^a_{rs} \gamma^{\bar{\mu}}$$



$$g_s f^{abc} \gamma^{\bar{\mu}}$$

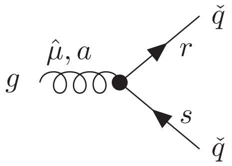


$$g_s f^{abc} (p_2 - p_3)^{\hat{\mu}} g^{\bar{\nu}\bar{\rho}}$$

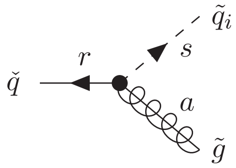


$$-ig_s T^a_{sr} \gamma^{\bar{\mu}}$$

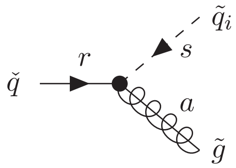
As introduced in the main text, $\check{q}$ denotes a "flipped" quark whose "external flow" (indicated by the arrow on the line) is opposite to its charge flow. For clarity, the external flow has also been indicated on the gluino lines in the $\epsilon\tilde{g}\tilde{g}$ vertex. The other Feynman rules involving flipped quarks are given as follows:



$$-ig_s T^a_{sr} \gamma^{\hat{\mu}}$$

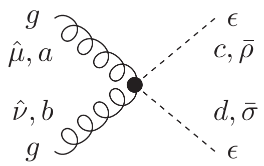


$$ig_s T^a_{sr} \sqrt{2} (\mathcal{R}^q_{i1} P_L - \mathcal{R}^q_{i2} P_R)$$

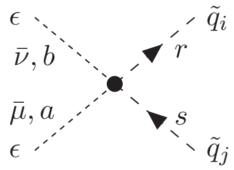


$$ig_s T^a_{rs} \sqrt{2} (\mathcal{R}^q_{i1} P_R - \mathcal{R}^q_{i2} P_L)$$

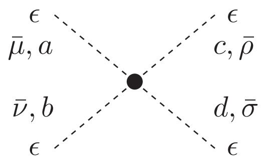
The quartic vertices involving ϵ-scalars are



$$-ig_s^2\, g^{\hat{\mu}\hat{\nu}} g^{\bar{\rho}\bar{\sigma}} \left[f^{ace} f^{bde} + f^{ade} f^{bce} \right]$$



$$ig_s^2 \left\{ T^a, T^b \right\}_{rs} g^{\bar{\mu}\bar{\nu}} \delta_{ij} = ig_s^2 \left(\tfrac{1}{3} \delta_{ab} \delta_{rs} + d_{abc} T^c_{rs} \right) g^{\bar{\mu}\bar{\nu}} \delta_{ij}$$



$$\begin{aligned} &-ig_s^2 f^{abe} f^{cde} \left[g^{\bar{\mu}\bar{\rho}} g^{\bar{\nu}\bar{\sigma}} - g^{\bar{\mu}\bar{\sigma}} g^{\bar{\nu}\bar{\rho}} \right] \\ &-ig_s^2 f^{ace} f^{bde} \left[g^{\bar{\mu}\bar{\nu}} g^{\bar{\rho}\bar{\sigma}} - g^{\bar{\mu}\bar{\sigma}} g^{\bar{\nu}\bar{\rho}} \right] \\ &-ig_s^2 f^{ade} f^{bce} \left[g^{\bar{\mu}\bar{\nu}} g^{\bar{\rho}\bar{\sigma}} - g^{\bar{\mu}\bar{\rho}} g^{\bar{\nu}\bar{\sigma}} \right] \end{aligned}$$

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