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Nine-Propagator Master Integrals for Massless Three-Loop Form Factors

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Abstract

We complete the calculation of master integrals for massless three-loop form factors by computing the previously-unknown three diagrams with nine propagators in dimensional regularisation. Each of the integrals yields a six-fold Mellin-Barnes representation which we use to compute the coefficients of the Laurent expansion in ϵ. Using Riemann ζ functions of up to weight six, we give fully analytic results for one integral; for a second, analytic results for all but the finite term; for the third, analytic results for all but the last two coefficients in the Laurent expansion. The remaining coefficients are given numerically to sufficiently high accuracy for phenomenological applications.

1 Introduction

The quark form factor $\gamma^* \to q\bar{q}$ and gluon form factor $H \to gg$ (effective coupling) are the simplest processes containing infrared divergences at higher orders in massless quantum field theory, and therefore are of particular interest in many aspects. They have, for instance, been used to predict the infrared pole structure of multi-leg amplitudes at a given order [1–4]. The form factors can also be exploited to extract resummation coefficients [5, 6], and they enter the purely virtual corrections to a number of collider reactions (Drell-Yan process, Higgs production and decay, DIS).

Besides phenomenological applications, a major motivation for obtaining analytic results at three-loop order and beyond is finding and understanding structures in massless gauge theories that generalize to an arbitrary number of loops. Much progress has been achieved recently in the prediction of all-order singularity structures in QCD [7, 8], in conjectures about the all-orders behaviour of maximally supersymmetric Yang-Mills theories [9–17] and in investigations of the finiteness of N=8 supergravity [18–23]. It has also been shown, at two loops [24, 25] and recently even for the matter contributions at three loops [26], that the soft anomalous dimension matrix in any massless gauge theory is proportional to the one-loop matrix.

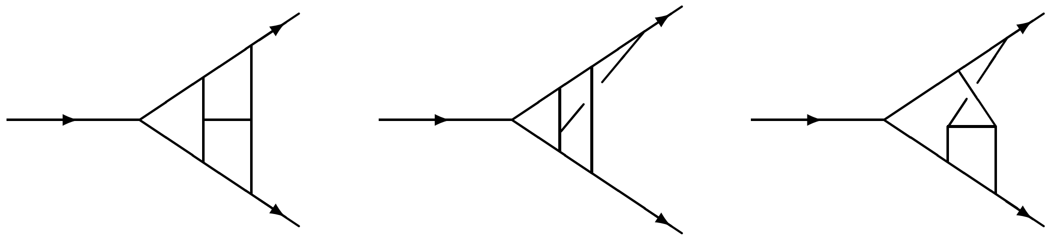
The two-loop corrections to the massless-quark [27] and gluon [28, 29] form factors were computed in dimensional regularisation with $D = 4 - 2\epsilon$ to order ϵ^0 and subsequently extended to all orders in ϵ in ref. [30]. Two-loop corrections to order ϵ^0 are also available for massive quarks [31]. The three-loop form factors to order ϵ^{-1} (and ϵ^0 for contributions involving fermion loops in the quark form factor) were computed in refs. [6, 32]; see also ref. [33].

In order to calculate the quark and gluon form factors at higher orders in perturbation theory, the amplitudes are reduced to a small set of master integrals by means of algebraic reduction procedures [34–38]. At the three-loop level, the master integrals for massless form factors were identified in ref. [39], and results for certain subsets are available in the literature [34, 39–42]. Among the three-loop master integrals, the genuine three-loop vertex functions are the most challenging ones from a computational point of view. They correspond to two-particle cuts of the master integrals for massless four-loop off-shell propagator integrals [43], which have been used in the calculation of the scalar R-ratio [44]. In fact, such a correspondence, via two-particle cuts, between these two families of master integrals follows from the general result of ref. [45]. The derivation of the three-loop vertex integrals is of comparable complexity to massless four-loop propagator integrals.

Working in dimensional regularisation and expanding the master integrals in a Laurent series in ϵ, the finite part of the three-loop form factors requires the extraction of all coefficients through (polylogarithmic) weight six[‡], i.e. coefficients containing terms up to π^6 or ζ_3^2.

Those genuine three-loop vertex functions which contain one-loop or two-loop propagator insertions were computed in ref. [39]. The three-loop master integrals which are sufficient in order to obtain the fermion loop contributions within a Feynman diagrammatic approach were computed in ref. [40]. The purpose of this Letter is to give the results of the remaining three diagrams which have nine propagators each. We present analytic results for all but three coefficients in the ϵ expansion, along with accurate numerical values for the remaining ones.

[‡]We prefer to use the term "weight" instead of "transcendentality", because from a mathematical point of view there is no proof that ζ_3 is a transcendental number.



$A_{9,1}$ $A_{9,2}$ $A_{9,4}$

Figure 1: Three-loop master integrals with nine massless propagators. The incoming momentum is $q = p_1 + p_2$. Outgoing momenta are taken to be on-shell and massless, $p_1^2 = p_2^2 = 0$.

2 Computational Methods and Results

In this section, we list the results we obtained for the three-loop master integrals with nine propagators. They are depicted in Fig. 1 and labeled as in refs. [39,40]. All other diagrams with up to eight propagators were already given in the same references. We will work in dimensional regularisation with $D = 4 - 2\epsilon$.

2.1 Diagram $A_{9,1}$

The first diagram to be considered is $A_{9,1}$ which can be written as follows,

$$\begin{aligned} A_{9,1} &= \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \\ &\times\frac{1}{k^2\,(k+p_1)^2\,(k+l)^2\,(k-r)^2\,(l+r)^2\,(l+p_2)^2\,l^2\,(r+p_1)^2\,(r-p_2)^2}\,. \end{aligned} \quad (1)$$

Here and in the following we tacitly assume that all propagators contain an infinitesimal $+i\eta$. The integral in eq. (1) can be written in terms of the following six-fold Mellin-Barnes (MB) representation [46–49]:

$$\begin{aligned} A_{9,1} &= i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-3-3\,\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)}\int\limits_{c_1-i\infty}^{c_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{c_2-i\infty}^{c_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{c_3-i\infty}^{c_3+i\infty}\frac{dw_3}{2\pi i}\int\limits_{c_4-i\infty}^{c_4+i\infty}\frac{dw_4}{2\pi i}\int\limits_{c_5-i\infty}^{c_5+i\infty}\frac{dw_5}{2\pi i}\int\limits_{c_6-i\infty}^{c_6+i\infty}\frac{dw_6}{2\pi i} \\ &\times\frac{\Gamma(-w_2)\,\Gamma(2+w_1+w_2)\,\Gamma(-w_3)\,\Gamma(w_3-w_2-w_4)\,\Gamma(-w_4)\,\Gamma(-w_5)\,\Gamma(1+w_3+w_5)}{\Gamma(-w_2-w_4)\,\Gamma(2+w_3+w_5)\,\Gamma(2+w_5+w_6)} \\ &\times\frac{\Gamma(w_4+w_5-w_1)\,\Gamma(-w_6)\,\Gamma(1+w_5+w_6)\,\Gamma(-2-3\epsilon-w_3-w_5)}{\Gamma(-1-4\epsilon-w_5)\,\Gamma(-1-3\epsilon-w_2)\,\Gamma(3+\epsilon+w_1+w_2)\,\Gamma(3+3\epsilon+w_5)} \\ &\times\Gamma(-1-2\epsilon-w_2-w_4)\,\Gamma(-2-3\epsilon-w_5-w_6)\,\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(w_1-w_5-\epsilon) \\ &\times\Gamma(3+2\epsilon+w_2+w_4+w_5+w_6)\,\Gamma(3+3\epsilon+w_3+w_5)\,\Gamma(3+3\epsilon+w_5+w_6) \\ &\times\Gamma(-1-\epsilon-w_1)\,\Gamma(-1-\epsilon-w_2)\;, \end{aligned} \quad (2)$$

$$\text{where} \qquad q^2 = (p_1+p_2)^2 \qquad \text{and} \qquad S_\Gamma = \frac{1}{(4\pi)^{D/2}\,\Gamma(1-\epsilon)}\,. \quad (3)$$

The representation (2) was obtained from eq. (1) by first introducing Feynman parameters; then, integrating over the momenta, loop by loop, and finally introducing MB parameters to decompose sums into products where appropriate. As usual, the contour integrals in the complex plane are along curves which separate left poles of Γ functions from right ones, where "left poles" are poles stemming from a $\Gamma(\ldots + w)$ dependence, while "right poles" stem from a $\Gamma(\ldots - w)$ dependence [49]. The most convenient choice for these contours are straight lines parallel to the imaginary axis, that is with constant real parts along the curves. According to refs. [47, 48], these real parts, together with the parameter ϵ, must be chosen in such a way as to have positive arguments in all occurring Γ functions in order to separate left and right poles in the desired way. In certain situations, such admissible straight contours and an appropriate starting value of ϵ do not exist. This can be cured via the introduction of an auxiliary analytic regularisation (see, e.g., refs. [47, 48]).

The regularisation of the MB integral as described above, as well as the analytic continuation to $\epsilon = 0$, was done with the `MB` package [50] and, alternatively, with the `MBresolve` package [51], which is based on the strategy formulated in ref. [46]. (Within this latter strategy, straight lines for the contours along the imaginary axis are not required at the beginning.) These packages were also used for numerical cross checks. Moreover, we have also derived sevenfold MB representations in two different ways: using the `AMBRE` package [52] and starting from the MB representation of ref. [10] derived for general powers of the propagators for the tennis court diagram. The numerical evaluation based on these two MB representations was again performed with `MB`. In addition we performed numerical checks with the sector decomposition methods of [53, 54] and the `FIESTA` [55] package. (See ref. [56] for another implementation of sector decomposition.)

As analytic techniques we apply Barnes's lemmas and the theorem of residues to the multiple Mellin-Barnes integrals, and insert integral representations of hypergeometric functions as well as ψ functions and Euler's B function where appropriate. We also make use of the `HPL` [57–60], `HypExp` [61, 62], and `barnesroutines` [63] packages, as well as an in-house implementation of the nested sums algorithm [64, 65]. The final result for $A_{9,1}$ reads,

$$
\begin{aligned}
A_{9,1} \;=\; & i\, S_\Gamma^3 \left[-\, q^2 - i\, \eta \right]^{-3-3\,\epsilon} \\
& \times \Big[-\frac{1}{18\epsilon^5} + \frac{1}{2\epsilon^4} + \Big(-\frac{53}{18} - \frac{4\pi^2}{27} \Big) \frac{1}{\epsilon^3} + \Big(\frac{29}{2} + \frac{22\pi^2}{27} - 2\zeta_3 \Big) \frac{1}{\epsilon^2} \\
& \quad + \Big(-\frac{129}{2} - \frac{8\pi^2}{3} + \frac{158}{9}\zeta_3 - \frac{20\pi^4}{81} \Big) \frac{1}{\epsilon} \\
& \quad + \Big(\frac{537}{2} + 6\pi^2 - \frac{578}{9}\zeta_3 + \frac{322\pi^4}{405} - \frac{14}{3}\pi^2\zeta_3 - \frac{238}{3}\zeta_5 \Big) \\
& \quad + \Big(-\frac{2133}{2} - 4\pi^2 + 158\zeta_3 - \frac{302\pi^4}{135} - \frac{26}{3}\pi^2\zeta_3 + \frac{826}{3}\zeta_5 - \frac{2398\pi^6}{5103} - \frac{466}{3}\zeta_3^2 \Big) \epsilon \\
& \quad + \mathcal{O}(\epsilon^2) \Big] \, . \qquad (4)
\end{aligned}
$$

We emphasize at this point that all terms in eq. (4) have been derived by purely analytic steps.

We also derived this result by evaluating first the following integral with a numerator:

$$A_{9,1}^{(n)} = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \times\frac{r^2}{k^2\;(k+p_1)^2\;(k+l)^2\;(k-r)^2\;(l+r)^2\;(l+p_2)^2\;l^2\;(r+p_1)^2\;(r-p_2)^2}\,. \quad (5)$$

Starting from the above mentioned MB representation in ref. [10] for the tennis court diagram and setting one of the indices to minus one (the index corresponding to the numerator r^2), we obtained the following seven-dimensional MB representation:

$$\begin{aligned}A_{9,1}^{(n)} &= i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-2-3\,\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)}\int\limits_{c_1-i\infty}^{c_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{c_2-i\infty}^{c_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{c_3-i\infty}^{c_3+i\infty}\frac{dw_3}{2\pi i}\int\limits_{c_4-i\infty}^{c_4+i\infty}\frac{dw_4}{2\pi i}\int\limits_{c_5-i\infty}^{c_5+i\infty}\frac{dw_5}{2\pi i}\int\limits_{c_6-i\infty}^{c_6+i\infty}\frac{dw_6}{2\pi i}\int\limits_{c_7-i\infty}^{c_7+i\infty}\frac{dw_7}{2\pi i}\\
&\times\frac{\Gamma(-w_2)\Gamma(1+w_1+w_2+w_3)\Gamma(-w_3)\Gamma(-w_4)\Gamma(1+w_1+w_4)\Gamma(-1-\epsilon-w_1-w_3)}{\Gamma(1-w_2)\Gamma(1-w_3)\Gamma(1-2\epsilon+w_1+w_2+w_3)}\\
&\times\frac{\Gamma(-1-\epsilon-w_1-w_2-w_4)\,\Gamma(2+\epsilon+w_1+w_2+w_3+w_4)\,\Gamma(-\epsilon+w_1+w_3-w_5)}{\Gamma(-4\epsilon-w_5)\,\Gamma(1-w_6)\,\Gamma(1-w_4-w_7)}\\
&\times\Gamma(2+3\epsilon+w_5)\,\Gamma(-1-3\epsilon-w_5-w_6)\,\Gamma(-w_6)\,\Gamma(1+w_5+w_6)\\
&\times\Gamma(-1-3\epsilon-w_4-w_5-w_7)\,\Gamma(-\epsilon+w_1+w_2-w_5-w_6-w_7)\,\Gamma(-w_5)\,\Gamma(-w_7)\\
&\times\Gamma(-w_1+w_5+w_7)\,\Gamma(1+\epsilon-w_1-w_2-w_3+w_5+w_6+w_7)\;. \qquad (6)\end{aligned}$$

It turns out that at each order in the ϵ expansion of $A_{9,1}^{(n)}$, the coefficients have *homogeneous* weight. This property turns out to be very helpful when one uses the so-called PSLQ algorithm [66]. Postulating that a given numerical result can be represented as a linear combination of certain constants (typically ζ values and powers of π) accompanied by rational coefficients, this algorithm solves for the latter. Starting from eq. (6) and using the `MB` and `MBresolve` packages, we applied the Barnes lemmas whenever possible to MB integrals which appeared after expanding in ϵ. At this point, at worst two-dimensional MB integrals were left. We calculated these integrals to an accuracy of 25 digits, which was sufficient to obtain very stable PSLQ results. For the term of highest weight we used the assumption that it is a linear combination, with rational coefficients, of π^6 and ζ_3^2. This lead to the following result:

$$\begin{aligned}A_{9,1}^{(n)} &= i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-2-3\,\epsilon}\\
&\times\Big[-\frac{1}{36\epsilon^6}-\frac{\pi^2}{18\epsilon^4}-\frac{14\zeta_3}{9\epsilon^3}-\frac{47\pi^4}{405\epsilon^2}\\
&\quad+\left(-\frac{85}{27}\pi^2\zeta_3-20\zeta_5\right)\frac{1}{\epsilon}+\left(-\frac{1160\pi^6}{5103}-\frac{137}{3}\zeta_3^2\right)+\mathcal{O}(\epsilon)\Big]\;. \qquad (7)\end{aligned}$$

Subsequently, we derived an analytic relation between $A_{9,1}$ and $A_{9,1}^{(n)}$ by means of a Laporta reduction [35] using, independently, the `AIR` [37] and `FIRE` [38] packages. The result reads[§]

$$
\begin{aligned}
A_{9,1}^{(n)} \;=\;& \frac{8(2D-7)(2D-5)(3D-10)(3D-8)(D-3)(1311D^2-11764D+26396)}{9(D-4)^5(3D-14)(5D-22)\,(q^2)^4}\,A_4 \\
&+\frac{80(2D-7)(3D-14)(3D-8)(D-3)^2}{3(D-4)^4(5D-22)\,(q^2)^3}\,A_{5,1} \\
&-\frac{64(2D-7)(D-3)^3(69D^2-580D+1220)}{9(D-4)^4(3D-14)(5D-22)\,(q^2)^3}\,A_{5,2} \\
&+\frac{8(3D-14)(3D-10)(3D-8)(D-3)^2}{(D-4)^4(5D-22)\,(q^2)^3}\,A_{5,1}^{(M)} \\
&-\frac{32(2D-7)(D-3)^3(45D-202)}{3(D-4)^4(5D-22)\,(q^2)^3}\,A_{5,2}^{(M)} \\
&+\frac{64(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)\,(q^2)^2}\,A_{6,1}-\frac{20(2D-7)(5D-18)}{9(D-4)^2\,(q^2)^2}\,A_{6,2} \\
&+\frac{8(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)\,(q^2)^2}\,A_{6,3} \\
&-\frac{2(3D-14)}{(5D-22)q^2}\,A_{7,3}-\frac{(3D-14)^2q^2}{2(D-4)(5D-22)}\,A_{9,1}\;. \qquad (8)
\end{aligned}
$$

The integrals A_4, $A_{5,1}^{(M)}$, and $A_{5,2}^{(M)}$ are listed in the Appendix, while all other integrals with fewer than nine propagators are given in refs. [39, 40]. For convenience, all the corresponding graphs are shown in the Appendix, Fig. 2. We have checked that eqs. (4),(7) satisfy this relation.

2.2 Diagram $A_{9,2}$

The next diagram we consider is $A_{9,2}$. It reads

$$
\begin{aligned}
A_{9,2} \;=\;& \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \qquad (9)\\
&\times\frac{1}{k^2\,(k+p_1)^2\,(k-l+p_1)^2\,(k-r-l)^2\,(l+r)^2\,(l+p_2)^2\,l^2\,(r+p_1)^2\,(r-p_2)^2}\;.
\end{aligned}
$$

Like $A_{9,1}$, it can be written in terms of a six-fold Mellin-Barnes representation.

$$
\begin{aligned}
A_{9,2} \;=\;& i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-3-3\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)}\int\limits_{c_1-i\infty}^{c_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{c_2-i\infty}^{c_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{c_3-i\infty}^{c_3+i\infty}\frac{dw_3}{2\pi i}\int\limits_{c_4-i\infty}^{c_4+i\infty}\frac{dw_4}{2\pi i}\int\limits_{c_5-i\infty}^{c_5+i\infty}\frac{dw_5}{2\pi i}\int\limits_{c_6-i\infty}^{c_6+i\infty}\frac{dw_6}{2\pi i} \\
&\times\frac{\Gamma(-w_1)\,\Gamma(2+w_1+w_2)\,\Gamma(-w_5)\,\Gamma(-w_2+w_3+w_4+1)\,\Gamma(w_5-w_2)\,\Gamma(w_5-w_4)}{\Gamma(1-w_4+w_5)\,\Gamma(-w_1-w_3+w_6)\,\Gamma(1-w_2+w_5+w_6)\,\Gamma(2-\epsilon+w_1+w_3+w_4)}
\end{aligned}
$$

[§]We also thank Beat Tödtli for correspondence on this point [68].

$$\times \frac{\Gamma(-w_6)\,\Gamma(w_6+1)\,\Gamma(w_6-w_3)\,\Gamma(1+w_5+w_6)\,\Gamma(-2-2\epsilon-w_1-w_3)}{\Gamma(2-2\epsilon+w_1+w_3+w_4)\,\Gamma(-1-w_1-3\epsilon)\,\Gamma(1-\epsilon+w_4-w_5)\,\Gamma(3+\epsilon+w_1+w_2)}$$

$$\begin{aligned}&\times\Gamma(-1-\epsilon-w_2)\,\Gamma(1-\epsilon+w_1+w_3)\,\Gamma(w_2-w_4-\epsilon)\,\Gamma(1+w_4-\epsilon)\,\Gamma(-1-\epsilon-w_1)\\ &\times\Gamma(1-\epsilon+w_1+w_3+w_4-w_5-w_6)\,\Gamma(\epsilon-w_1-w_3-w_4+w_5+w_6)\\ &\times\Gamma(w_4-w_5-\epsilon)\,\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(3+2\epsilon+w_1+w_3+w_4)\ .\end{aligned} \tag{10}$$

The techniques we apply are the same as before. The final result for $A_{9,2}$ reads

$$\begin{aligned}A_{9,2} &= i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-3-3\,\epsilon}\\ &\times\Big[\frac{2}{9\epsilon^6}+\frac{5}{6\epsilon^5}+\Big(-\frac{20}{9}-\frac{7\pi^2}{27}\Big)\frac{1}{\epsilon^4}+\Big(\frac{50}{9}-\frac{17\pi^2}{27}-\frac{91}{9}\zeta_3\Big)\frac{1}{\epsilon^3}\\ &+\Big(-\frac{110}{9}+\frac{4\pi^2}{3}-\frac{166}{9}\zeta_3-\frac{373\pi^4}{1080}\Big)\frac{1}{\epsilon^2}\\ &+\Big(\frac{170}{9}-\frac{16\pi^2}{9}+\frac{494}{9}\zeta_3-\frac{187\pi^4}{540}+\frac{179}{27}\pi^2\zeta_3-167\zeta_5\Big)\frac{1}{\epsilon}\\ &+(-670.0785\pm0.0326)+\mathcal{O}(\epsilon)\Big]\ .\end{aligned} \tag{11}$$

The number for the finite term was obtained with `MB` [50]. Again, all pole terms in eq. (11) have been derived by purely analytic steps. As in the previous case we performed an independent analytic calculation of an integral with a numerator, which again turns out to have homogeneous weight[¶] at each order in the ϵ expansion:

$$A_{9,2}^{(n)} = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \tag{12}$$

$$\times\frac{(l-p_1)^2}{k^2\ (k+p_1)^2\ (k-l+p_1)^2\ (k-r-l)^2\ (l+r)^2\ (l+p_2)^2\ l^2\ (r+p_1)^2\ (r-p_2)^2}\ .$$

Again, we relate $A_{9,2}$ to $A_{9,2}^{(n)}$ by means of a Laporta reduction [37, 38]. The result reads

$$\begin{aligned}A_{9,2}^{(n)} &= -\frac{16(2D-7)(2D-5)(3D-10)(3D-8)(D-3)(70D-303)}{3(D-4)^4(2D-9)(5D-22)\,(q^2)^4}\,A_4\\ &-\frac{64(2D-7)(3D-8)(D-3)^2(19D-84)}{3(D-4)^3(2D-9)(5D-22)\,(q^2)^3}\,A_{5,1}-\frac{64(2D-7)(D-3)^3}{3(D-4)^3(2D-9)\,(q^2)^3}\,A_{5,2}\\ &-\frac{16(3D-10)(3D-8)(D-3)^2}{(D-4)^2(2D-9)(5D-22)\,(q^2)^3}\,A_{5,1}^{(M)}+\frac{64(2D-7)(3D-10)(D-3)^3}{3(D-4)^3(2D-9)(5D-22)\,(q^2)^3}\,A_{5,2}^{(M)}\\ &-\frac{8(2D-7)(5D-18)(16D-71)}{3(D-4)(2D-9)(5D-22)\,(q^2)^2}\,A_{6,2}+\frac{10(D-3)}{(2D-9)q^2}\,A_{7,1}\\ &-\frac{8(D-4)^2}{(2D-9)(5D-22)q^2}\,A_{7,3}+\frac{8(3D-13)(3D-11)}{(2D-9)(5D-22)q^2}\,[A_{7,4}+A_{7,5}]\\ &-\frac{(3D-14)}{(5D-22)}\,\Big[A_8-q^2\,A_{9,2}\Big]\ .\end{aligned} \tag{13}$$

[¶]We thank Lance Dixon for the suggestion that this particular numerator generates homogeneous weights.

We observed homogeneity of weights for $A^{(n)}_{9,2}$ in all coefficients where we have an analytic result.

To evaluate $A^{(n)}_{9,2}$ we started again with eq. (10) and repeatedly applied the Barnes lemmas as much as possible. The resulting integrals were treated numerically. Afterwards, the numbers were plugged into the right-hand side of eq. (13) and an expansion in ϵ was performed. Assuming homogeneous weight helped to minimize the number of possible constants in $A^{(n)}_{9,2}$. There is only one constant each at weight 0,2,3 and 4 (1, π^2, ζ_3, π^4) and two constants each at weight 5 ($\pi^2\zeta_3$ and ζ_5) and 6 (ζ_3^2 and π^6). Using PSLQ, we reproduced the result in eq. (11) up to order $1/\epsilon^2$. We note that the MB integrals contributing to the finite part of $A_{9,2}$ have dimensionality as high as five, and therefore prevent us from achieving an accuracy which is sufficient for a successful application of the PSLQ algorithm.

For the integral with the numerator, the result reads

$$
\begin{aligned}
A^{(n)}_{9,2} \;=\;& i\,S_\Gamma^3\left[-\,q^2-i\,\eta\right]^{-2-3\,\epsilon} \\
&\times\Big[-\frac{2}{9\epsilon^6}+\frac{7\pi^2}{27\epsilon^4}+\frac{91\zeta_3}{9\epsilon^3}+\frac{373\pi^4}{1080\epsilon^2} \\
&\quad+\left(-\frac{179}{27}\pi^2\zeta_3+167\zeta_5\right)\frac{1}{\epsilon}+(395.3405\pm 0.0326)+\mathcal{O}(\epsilon)\Big]\,. \qquad (14)
\end{aligned}
$$

2.3 Diagram $A_{9,4}$

The last diagram we consider is $A_{9,4}$. It reads

$$
\begin{aligned}
A_{9,4} \;=\;& \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \qquad (15)\\
&\times\frac{1}{k^2\;(k+p_1)^2\;(k-r)^2\;(k-r-l)^2\;(l+r)^2\;(l+p_2)^2\;l^2\;(r+p_1)^2\;(r-p_2)^2}\,.
\end{aligned}
$$

Like the previous integrals, it can be written in terms of a six-fold Mellin-Barnes representation,

$$
\begin{aligned}
A_{9,4} \;=\;& i\,S_\Gamma^3\left[-\,q^2-i\,\eta\right]^{-3-3\,\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)\Gamma(-1-4\,\epsilon)} \\
&\times\int\limits_{c_1-i\,\infty}^{c_1+i\,\infty}\frac{dw_1}{2\pi i}\int\limits_{c_2-i\,\infty}^{c_2+i\,\infty}\frac{dw_2}{2\pi i}\int\limits_{c_3-i\,\infty}^{c_3+i\,\infty}\frac{dw_3}{2\pi i}\int\limits_{c_4-i\,\infty}^{c_4+i\,\infty}\frac{dw_4}{2\pi i}\int\limits_{c_5-i\,\infty}^{c_5+i\,\infty}\frac{dw_5}{2\pi i}\int\limits_{c_6-i\,\infty}^{c_6+i\,\infty}\frac{dw_6}{2\pi i} \\
&\times\frac{\Gamma(-w_1)\,\Gamma(1+w_1+w_2)\,\Gamma(-w_3)\,\Gamma(1-w_1+w_3)\,\Gamma(w_3-w_2)\,\Gamma(1+w_4)\,\Gamma(1+w_5)}{\Gamma(1-w_1)\,\Gamma(w_1+w_2-w_3-w_4+w_5-2\epsilon)\,\Gamma(1-2\epsilon+w_1+w_2)} \\
&\times\frac{\Gamma(-w_5)\,\Gamma(w_4-w_5+1)\,\Gamma(w_5-w_4)\,\Gamma(-w_6)\,\Gamma(1+w_3+w_4+w_6-w_5)}{\Gamma(2-w_1+w_3+w_4)\,\Gamma(1-w_2+w_3+w_4-w_5)\,\Gamma(2+w_3+w_4+w_6)} \\
&\times\Gamma(-2-3\epsilon-w_4)\,\Gamma(w_1+w_2-w_3-2\epsilon)\,\Gamma(-w_1-\epsilon)\,\Gamma(w_1-\epsilon)\,\Gamma(-1-\epsilon-w_2) \\
&\times\Gamma(-2-3\epsilon-w_3-w_4+w_5-w_6)\,\Gamma(1+\epsilon-w_1-w_2+w_3+w_4) \\
&\times\Gamma(1+w_2-\epsilon)\,\Gamma(2+\epsilon+w_1+w_2+w_6)\,\Gamma(3+3\epsilon+w_3+w_4+w_6)\;\;. \qquad (16)
\end{aligned}
$$

The techniques we apply are the same as above. The final result for $A_{9,4}$ reads,

$$\begin{aligned}
A_{9,4} &= i\, S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-3-3\,\epsilon} \\
&\times \Big[\frac{1}{9\epsilon^6} + \frac{8}{9\epsilon^5} + \Big(-1 - \frac{10\pi^2}{27}\Big)\frac{1}{\epsilon^4} + \Big(-\frac{14}{9} - \frac{47\pi^2}{27} - 12\zeta_3\Big)\frac{1}{\epsilon^3} \\
&\quad + \Big(17 + \frac{71\pi^2}{27} - \frac{200}{3}\zeta_3 - \frac{47\pi^4}{810}\Big)\frac{1}{\epsilon^2} \\
&\quad + \Big(117.3999538 \pm 0.0000032\Big)\frac{1}{\epsilon} \\
&\quad + (1948.167043 \pm 0.000025) + \mathcal{O}(\epsilon)\Big] \,. \qquad (17)
\end{aligned}$$

The two numbers were again obtained with `MB` [50]. All higher pole terms in eq. (17) have again been derived by purely analytic steps. In the case of $A_{9,4}$ a homogeneous-weight master integral also exists, with a numerator:

$$\begin{aligned}
A_{9,4}^{(n)} &= \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D} \qquad (18) \\
&\times \frac{r^2}{k^2\,(k+p_1)^2\,(k-r)^2\,(k-r-l)^2\,(l+r)^2\,(l+p_2)^2\,l^2\,(r+p_1)^2\,(r-p_2)^2} \,.
\end{aligned}$$

For $A_{9,4}$, we also used the same alternative method of calculation as in the case of $A_{9,2}$. This procedure was again based on a relation between $A_{9,4}$ to $A_{9,4}^{(n)}$, which follows from a Laporta reduction [37, 38]. The relation reads

$$\begin{aligned}
A_{9,4}^{(n)} &= -\frac{16(2D-7)(2D-5)(3D-10)(3D-8)(D-3)(348D^2-3037D+6618)}{9(D-4)^4(2D-9)(3D-14)(5D-22)\,(q^2)^4}\,A_4 \\
&+\frac{32(2D-7)(3D-8)(D-3)^2(31D-138)}{3(D-4)^3(2D-9)(5D-22)\,(q^2)^3}\,A_{5,1} \\
&+\frac{128(2D-7)(D-3)^3(195D^2-1726D+3816)}{9(D-4)^3(2D-9)(3D-14)(5D-22)\,(q^2)^3}\,A_{5,2} \\
&-\frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)\,(q^2)^2}\,A_{6,1} + \frac{8(2D-7)(5D-18)(28D-123)}{9(D-4)(2D-9)(5D-22)\,(q^2)^2}\,A_{6,2} \\
&-\frac{8(2D-7)(3D-10)(D-3)(7D-30)}{(D-4)^2(2D-9)(5D-22)\,(q^2)^2}\,A_{6,3} + \frac{20(D-3)}{(2D-9)q^2}\,A_{7,2} \\
&-\frac{8(3D-13)(3D-11)}{(2D-9)(5D-22)q^2}\,[A_{7,4}+A_{7,5}] - \frac{(3D-14)^2q^2}{2(2D-9)(5D-22)}\,A_{9,4} \,. \qquad (19)
\end{aligned}$$

For the integral with numerator, the result reads

$$\begin{aligned}
A_{9,4}^{(n)} &= i\, S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-2-3\,\epsilon} \\
&\times \Big[\frac{1}{9\epsilon^6} - \frac{10\pi^2}{27\epsilon^4} - \frac{12\zeta_3}{\epsilon^3} - \frac{47\pi^4}{810\epsilon^2} \\
&\quad + (206.7612077 \pm 0.0000032)\,\frac{1}{\epsilon} + (1237.300592 \pm 0.000035) + \mathcal{O}(\epsilon)\Big] \,. \qquad (20)
\end{aligned}$$

As was the case for $A_{9,2}$, the remaining Mellin-Barnes integrals are of too high dimension to allow for a stable PSLQ fit to determine the remaining coefficients analytically. However, we are confident that a dedicated effort to determine the remaining coefficients analytically will eventually be successful.

3 Conclusions and Outlook

In this Letter we have evaluated the three nine-propagator master integrals needed for computing the quark and gluon form factors to three-loop order. Each of the three integrals can be expressed in terms of a six-fold Mellin-Barnes representation from which we determine all coefficients through weight six in the Riemann ζ function. One integral is given fully analytically up to order ϵ. For the second one we give an analytic representation for all pole parts, and the third one is given analytically except for the coefficients of $1/\epsilon$ and ϵ^0 in the Laurent expansion in ϵ. The remaining Laurent coefficients are given numerically to an accuracy which is sufficient for all phenomenological applications.

Note added: Our results for the coefficients of the three master integrals $A_{9,1}$, $A_{9,2}$ and $A_{9,4}$ partially overlap with those of ref. [67] where these integrals were evaluated in an indirect way. Agreement has been found for all common coefficients: terms up to ϵ^0 for $A_{9,1}$, up to $1/\epsilon$ for $A_{9,2}$ and up to $1/\epsilon^2$ for $A_{9,4}$. To avoid confusion, we would like to point out that the convention for the overall prefactor used in ref. [67] differs from our one. For other recent progress on the three-loop quark form factor see ref. [68].

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A Additional integrals

In this appendix we collect some additional integrals which appear in the Laporta reduction of the nine-propagator integrals with numerator. Three of the integrals are two-point functions of the MINCER type [42]. The fourth one is the eight-propagator butterfly graph. The diagrams are depicted in Fig. 2, together with diagrams which have been calculated in Ref. [40]. The results for the two-point functions are

$$A_4 = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D}\,\frac{1}{(k+q)^2\;(k+l)^2\;(l+r)^2\;r^2}$$
$$= -i\,S_\Gamma^3\,\Big[-q^2-i\,\eta\Big]^{2-3\,\epsilon}\,\frac{\Gamma^7(1-\epsilon)\Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)}\,, \tag{21}$$

$$A_{5,1}^{(M)} = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D}\,\frac{1}{k^2\,(k+q)^2\;(l+q)^2\;(l+r)^2\;r^2}$$
$$= i\,S_\Gamma^3\,\Big[-q^2-i\,\eta\Big]^{1-3\,\epsilon}\,\frac{\Gamma^8(1-\epsilon)\Gamma(\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(2-2\epsilon)\Gamma(3-3\epsilon)}\,, \tag{22}$$

$$A_{5,2}^{(M)} = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D}\,\frac{1}{(r+q)^2\;k^2\;l^2\,(k+r)^2\;(l+r)^2}$$
$$= i\,S_\Gamma^3\,\Big[-q^2-i\,\eta\Big]^{1-3\,\epsilon}\,\frac{\Gamma^8(1-\epsilon)\Gamma^2(\epsilon)\Gamma(-1+3\epsilon)\Gamma(2-3\epsilon)}{\Gamma^2(2-2\epsilon)\Gamma(2\epsilon)\Gamma(3-4\epsilon)}\,. \tag{23}$$

The last integral which we give is the eight-propagator butterfly graph $A_{8,B}$, depicted in Fig. 2. It is not a master integral, but it is useful in quite a number of calculations. It is obtained from $A_{9,1}$ by shrinking the horizontal propagator to a point. From the Laporta reduction we obtain

$$A_{8,B} = \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D}$$
$$\times\frac{1}{k^2\;(k+p_1)^2\;(k-r)^2\;(l+r)^2\;(l+p_2)^2\;l^2\;(r+p_1)^2\;(r-p_2)^2}$$
$$= -\frac{128(2D-7)(D-3)^3}{(D-4)^3(3D-14)\,(q^2)^3}\,A_{5,2}+\frac{32(2D-7)(D-3)^2}{3(D-4)^2(3D-14)\,(q^2)^2}\,A_{6,1}$$
$$+\frac{48(2D-7)(2D-5)(3D-10)(3D-8)(D-3)}{(D-4)^4(3D-14)\,(q^2)^4}\,A_4\,. \tag{24}$$

Hence the integral can be written entirely in terms of Γ functions. Its expansion reads

$$A_{8,B} = i\,S_\Gamma^3\,\Big[-q^2-i\,\eta\Big]^{-2-3\,\epsilon}$$
$$\times\Big[\frac{1}{9\epsilon^5}-\frac{1}{3\epsilon^4}+\Big(1-\frac{2\pi^2}{9}\Big)\frac{1}{\epsilon^3}+\Big(-3+\frac{2\pi^2}{3}-\frac{52}{9}\zeta_3\Big)\frac{1}{\epsilon^2}$$
$$+\Big(9-2\pi^2+\frac{52}{3}\zeta_3-\frac{7\pi^4}{90}\Big)\frac{1}{\epsilon}$$
$$+\Big(-27+6\pi^2-52\zeta_3+\frac{7\pi^4}{30}+\frac{68}{9}\pi^2\zeta_3-\frac{140}{3}\zeta_5\Big)$$
$$+\Big(81-18\pi^2+156\zeta_3-\frac{7\pi^4}{10}-\frac{68}{3}\pi^2\zeta_3+140\zeta_5+\frac{2473\pi^6}{34020}+\frac{1136}{9}\zeta_3^2\Big)\epsilon$$
$$+\mathcal{O}(\epsilon^2)\Big]\,. \tag{25}$$

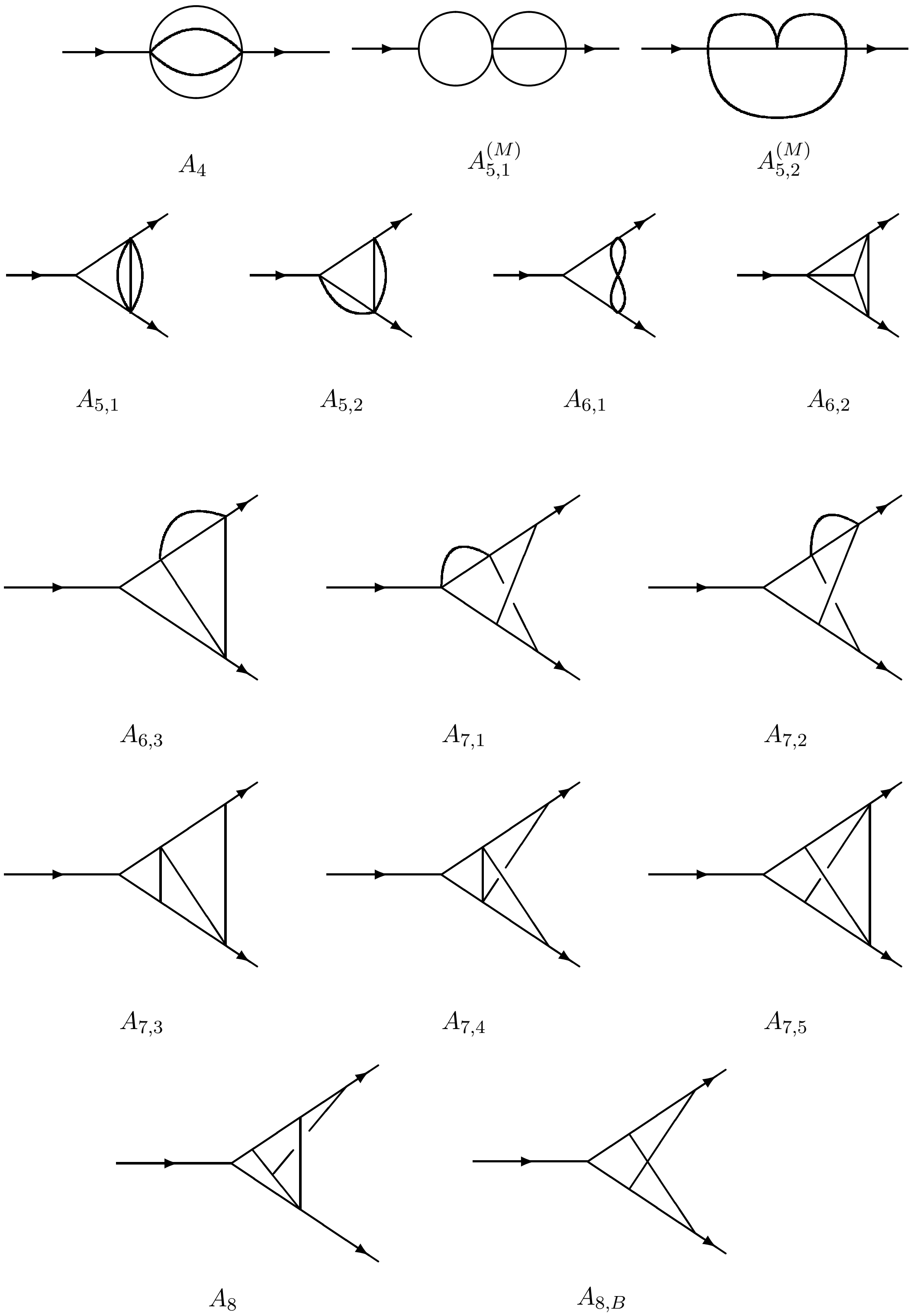

Figure 2: Graphs with fewer than nine propagators. The incoming momentum is $q = p_1 + p_2$, $p_1^2 = p_2^2 = 0$.

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