Quark and gluon form factors to three loops

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We compute the form factors of the photon-quark-anti-quark vertex and the effective vertex of a Higgs boson and two gluons to three-loop order within massless perturbative Quantum Chromodynamics. These results provide building blocks for many third-order cross sections. Furthermore, this is the first calculation of complete three-loop vertex corrections.

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In the recent years various next-to-next-to-leading order (NNLO) calculations to physical observables have been completed. Among them are the total threshold cross section for top quark pair production in electron positron annihilation [1], the Higgs boson production in gluon fusion [2–4], the rare decay rate of the B meson into a meson containing a strange quark and a photon [5, 6] and the three-jet cross section at lepton colliders [7, 8]. There exist also a few results at next-to-next-to-next-to-leading order (NNNLO), like the total hadronic cross section in electron positron annihilation [9], the hadronic τ lepton [9] and Higgs boson decay [10]. It is common to all NNNLO results that the calculation can be reduced to two-point functions and that only one mass scale is involved in the computation.

In this Letter we provide the first NNNLO calculation of a three-point function within Quantum Chromodynamics (QCD). To be precise, we consider gauge invariant building blocks for NNNLO cross sections, namely the virtual third-order corrections for the hadronic Higgs boson production and the process $e^+e^- \to 2$ jets. The results are conveniently expressed in terms of form factors of the photon-quark and the effective gluon-Higgs boson vertex originating from integrating out the heavy top-quark loops. Denoting the corresponding vertex functions by Γ_q^μ and $\Gamma_g^{\mu\nu}$, respectively, the scalar form factors are obtained via

$$F_q(q^2) \;=\; -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\,(\not{q}_2\,\Gamma^\mu\not{q}_1\,\gamma_\mu)\;,$$
$$F_g(q^2) \;=\; \frac{(q_1\cdot q_2\;g_{\mu\nu} - q_{1,\mu}\,q_{2,\nu} - q_{1,\nu}\,q_{2,\mu})}{2(1-\epsilon)}\Gamma_g^{\mu\nu}\;, \quad (1)$$

where $d = 4-2\epsilon$ is the space-time dimension, $q = q_1 + q_2$ and q_1 (q_2) is the incoming (anti-)quark momentum in the case of F_q, and F_g depends on the gluon momenta q_1 and q_2 with polarization vectors $\varepsilon^\mu(q_1)$ and $\varepsilon^\nu(q_2)$. Some sample Feynman diagrams contributing to F_q and F_g are shown in Fig. 1. Starting from three-loop order a new class of diagrams occurs, the so-called singlet diagrams, where the external photon is not connected to the fermion line involving the final-state quarks (see Fig. 1 (b)). Since at three-loop order there are no counterterm contributions to the singlet diagrams and furthermore there is no corresponding real emission contribution the sum of all diagrams has to be finite. This constitutes an important check on the correctness of our result.

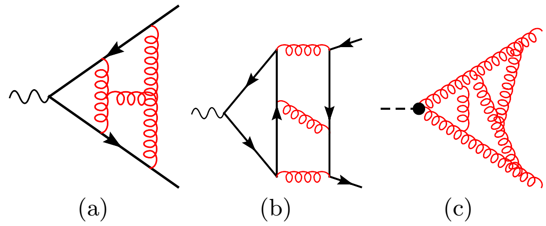

FIG. 1: Sample Feynman diagrams contributing to the F_q ((a) and (b)) and F_g (c) at three-loop order. Straight and curly lines denote quarks and gluons, respectively.

In the recent years the evaluation of the three-loop form factor has attracted much attention. After the pioneering work more than 20 years ago [11–13] where the quark form factor has been computed to two-loop order the corresponding quantity for the Higgs-gluon coupling has been evaluated by Harlander in Ref. [14] (see also [15]). The latter constitutes a building block for the NNLO predictions of the Higgs boson production in gluon fusion at the Fermilab Tevatron and CERN Large Hadron Collider [2–4]. More recently, in Ref. [16] the two-loop results have been reconsidered and more terms in the ϵ-expansion have been added in order to match the three-loop accuracy. Furthermore, in Refs. [17, 18] almost all master integrals necessary for the three-loop calculation have been evaluated. However, the most complicated master integrals are still unknown.

First steps towards three-loop results for the form factors have been undertaken in the Refs. [19, 20] where the pole parts of F_q [19] and F_g [20] have been extracted from the behaviour of the three-loop coefficient function for inclusive deep-inelastic scattering [21]. Furthermore, in Ref. [20] also the finite part of the fermionic contribution to F_q could be evaluated. With our calculation we were able to confirm these results but also add the finite contributions which are necessary for the physical observables.

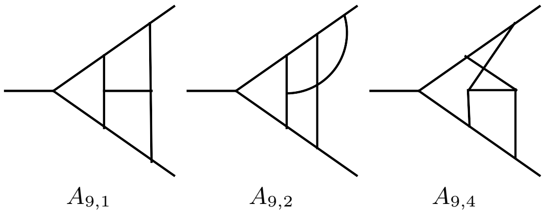

FIG. 2: Three most complicated master integrals entering the result for the three-loop form factor. The notation is adopted from Ref. [17, 18].

For the evaluation of the Feynman integrals we developed two independent set-ups which have in common that a reduction of all occurring integrals to so-called master integrals is performed in d space-time dimensions. Afterwards the (ϵ-expanded) master integrals are inserted.

Following Refs. [22–24] one considers integral representations of the coefficient functions of the individual master integrals in the limit of large space-time dimension d, evaluates several expansion terms and reconstructs in this way the complete rational dependence on d. The most CPU-consuming step, the large d expansion, has been performed by a program written in `ParFORM` [25, 26], the parallel version of the computer algebra program `FORM` [27]. For the singlet contribution, which involves the most complicated integrals, also a second approach has been employed. After generating the Feynman diagrams with the help of `QGRAF` [28] they are further processed with `q2e` and `exp` [29, 30] where a mapping to the underlying family of the diagrams is achieved. In a next step the reduction of the integrals is performed with the program package `FIRE` [31] which implements a combination of the Laporta algorithm [32] and a generalization [33] of the Buchberger algorithm (see, e.g., Ref. [34]) to construct Gröbner bases.

Our results are expressed in terms of 22 master integrals. Eight master integrals are either two-point functions or products of one- and two-loop integrals and are thus well-known since many years (see, e.g., Ref. [16, 35–37]). The results for eleven three-point master integrals can be found in Ref. [17, 18], however, the three most complicated integrals, which are shown in Fig. 2, are not yet known in the literature. Our calculation fixes, by comparing with Ref. [19], the divergent parts of $A_{9,2}$ and $A_{9,4}$ and the finite part of $A_{9,1}$ and leaves only three coefficients of the ϵ expansion undetermined. The results read (assuming massless propagators of the form $1/(k^2+i0)$ and pulling out a factor $(i\pi^{d/2}e^{-\gamma_E\epsilon})^3(-q^2-i0)^{-3-3\epsilon}$)

$$
\begin{aligned}
A_{9,1} &= \frac{1}{18\epsilon^5} - \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{53}{18} + \frac{29\zeta(2)}{36}\right) + \frac{1}{\epsilon^2}\left(-\frac{29}{2}\right. \\
&\quad \left. - \frac{149\zeta(2)}{36} + \frac{35\zeta(3)}{18}\right) + \frac{1}{\epsilon}\left(\frac{129}{2} + \frac{139\zeta(2)}{12}\right. \\
&\quad \left. - \frac{307\zeta(3)}{18} + \frac{5473\zeta(4)}{288}\right) - \frac{537}{2} - \frac{57\zeta(2)}{4} \\
&\quad + \frac{1103\zeta(3)}{18} - \frac{15625\zeta(4)}{288} + \frac{871\zeta(2)\zeta(3)}{36} \\
&\quad + \frac{793\zeta(5)}{10} + \epsilon\, X_{9,1} + \mathcal{O}\left(\epsilon^2\right) ,
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
A_{9,2} &= -\frac{2}{9\epsilon^6} - \frac{5}{6\epsilon^5} + \frac{1}{\epsilon^4}\left(\frac{20}{9} + \frac{17\zeta(2)}{9}\right) + \frac{1}{\epsilon^3}\left(-\frac{50}{9}\right. \\
&\quad \left. + \frac{181\zeta(2)}{36} + \frac{31\zeta(3)}{3}\right) + \frac{1}{\epsilon^2}\left(\frac{110}{9} - \frac{34\zeta(2)}{3}\right. \\
&\quad \left. + \frac{347\zeta(3)}{18} + \frac{595\zeta(4)}{24}\right) + \frac{1}{\epsilon}\left(-\frac{170}{9} + 19\zeta(2)\right. \\
&\quad - \frac{514\zeta(3)}{9} + \frac{489\zeta(4)}{32} - \frac{341\zeta(2)\zeta(3)}{6} \\
&\quad \left. + \frac{2507\zeta(5)}{15}\right) + X_{9,2} + \mathcal{O}\left(\epsilon\right) ,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
A_{9,4} &= -\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \frac{1}{\epsilon^4}\left(1 + \frac{43\zeta(2)}{18}\right) + \frac{1}{\epsilon^3}\left(\frac{14}{9}\right. \\
&\quad \left. + \frac{106\zeta(2)}{9} + \frac{109\zeta(3)}{9}\right) + \frac{1}{\epsilon^2}\left(-17\right. \\
&\quad \left. - \frac{311\zeta(2)}{18} + \frac{608\zeta(3)}{9} - \frac{481\zeta(4)}{144}\right) + \frac{1}{\epsilon}\left(84\right. \\
&\quad + \frac{11\zeta(2)}{3} - \frac{949\zeta(3)}{9} + \frac{425\zeta(4)}{6} + \frac{3463\zeta(5)}{45} \\
&\quad \left. - \frac{2975\zeta(2)\zeta(3)}{18}\right) + X_{9,4} + \mathcal{O}\left(\epsilon\right) .
\end{aligned} \tag{4}
$$

We obtained a numerical result for the coefficient $X_{9,1}$ using the Mellin-Barnes (MB) method [38–40], starting from the general MB representation for the tennis court diagram of Ref. [41], and applying the corresponding packages [42, 43]. To evaluate numerically $X_{9,2}$ and $X_{9,4}$ we used the program `FIESTA` [44] which is a convenient and efficient implementation of the sector decomposition algorithm. Our results read

$$
X_{9,1} \approx 1429(1)\,, X_{9,2} \approx 528.0(4)\,, X_{9,4} \approx -2085(5)\,, \tag{5}
$$

where the accuracy is sufficient for all foreseeable physical applications. Finally, let us mention that we evaluate the colour factors with the help of the program `color` [45].

In the following we want to present explicit results for F_q and F_g. We parameterize the results in terms of the bare coupling which allows us to factorize all occurring logarithms of the form $\ln(Q^2/\mu^2)$ where $Q^2 = -q^2 > 0$. Furthermore, we cast the results in the form ($x = q, g$)

$$F_x \;=\; 1 + \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \left(\frac{\mu^2}{Q^2}\right)^{n\epsilon} F_x^{(n)}\,, \qquad (6)$$

and split $F_q^{(3)}$ into the singlet, fermionic and remaining gluonic part

$$F_q^{(3)} \;=\; F_q^{(3),g} + F_q^{(3),n_f} + \sum_{q'} Q_{q'} F_q^{(3),sing}\,, \qquad (7)$$

where n_f stands for the number of active quarks. The results for $F_q^{(1)}$ and $F_q^{(2)}$ (expanded in ϵ sufficient for the three-loop calculation) can be found in Eqs. (3.5) and (3.6) of Ref. [19] and $F_g^{(1)}$, $F_g^{(2)}$ and $F_q^{(3),n_f}$ are given in Eqs. (7), (8) and (6) of Ref. [20], respectively. The pole parts of $F_q^{(3),g}$ and $F_g^{(3)}$ are listed in Eqs. (3.7) of Ref. [19] and (9) of Ref. [20], respectively. Our expressions agree with all these results which constitutes a strong cross check since in Refs. [19, 20] a completely different approach has been chosen to evaluate the Feynman integrals. In particular, no reduction to master integrals has been performed. In this Letter new results for $F_q^{(3),g}$, $F_q^{(3),sing}$ and $F_g^{(3)}$ are presented. Since the pole parts are already available in the literature we display only the corresponding finite parts which read in the case of a $SU(N_c)$ colour group

$$\begin{aligned}
F_q^{(3),g+n_f}\Big|_{\rm fin} \;=\;& C_F^3\left(\frac{26871}{8} - \frac{95137\zeta(2)}{60} + \frac{5569\zeta(3)}{5} + \frac{95375\zeta(4)}{48} + \frac{30883\zeta(2)\zeta(3)}{15} - \frac{16642\zeta(5)}{5} + \frac{2669(\zeta(3))^2}{3}\right.\\
&\left. + \frac{1961387\zeta(6)}{2880} - \frac{24X_{9,1}}{5} + \frac{24X_{9,2}}{5} + \frac{6X_{9,4}}{5}\right) + C_A C_F^2\left(\frac{20003431}{29160} + \frac{4239679\zeta(2)}{1620} - \frac{121753\zeta(3)}{30}\right.\\
& - \frac{11155817\zeta(4)}{4320} - \frac{92554\zeta(2)\zeta(3)}{45} + \frac{610462\zeta(5)}{225} - \frac{36743(\zeta(3))^2}{30} - \frac{1118529\zeta(6)}{640} + \frac{24X_{9,1}}{5}\\
&\left. - \frac{16X_{9,2}}{5} - \frac{9X_{9,4}}{5}\right) + C_A^2 C_F\left(-\frac{88822328}{32805} - \frac{3486997\zeta(2)}{2916} + \frac{3062512\zeta(3)}{1215} + \frac{4042277\zeta(4)}{4320}\right.\\
&\left. + \frac{5233\zeta(2)\zeta(3)}{12} - \frac{202279\zeta(5)}{450} + \frac{63043(\zeta(3))^2}{180} + \frac{4741699\zeta(6)}{11520} - X_{9,1} + \frac{2X_{9,2}}{5} + \frac{3X_{9,4}}{5}\right)\\
& + C_F^2 n_f T\left(-\frac{2732173}{1458} - \frac{45235\zeta(2)}{81} + \frac{102010\zeta(3)}{81} + \frac{40745\zeta(4)}{216} - \frac{686\zeta(3)\zeta(2)}{9} + \frac{556\zeta(5)}{45}\right)\\
& + C_A C_F n_f T\left(\frac{17120104}{6561} + \frac{442961\zeta(2)}{729} - \frac{90148\zeta(3)}{81} - \frac{5465\zeta(4)}{27} + \frac{736\zeta(3)\zeta(2)}{9} - \frac{416\zeta(5)}{3}\right)\\
& + C_F n_f^2 T^2\left(-\frac{2710864}{6561} - \frac{248\zeta(2)}{3} + \frac{12784\zeta(3)}{243} - \frac{166\zeta(4)}{27}\right)\,, \qquad (8)\\
F_q^{(3),sing}\Big|_{\rm fin} \;=\;& d^{abc}d^{abc}\left(\frac{2}{3} + \frac{5\zeta(2)}{3} + \frac{7\zeta(3)}{9} - \frac{\zeta(4)}{6} - \frac{40\zeta(5)}{9}\right)\,, \qquad (9)\\
F_g^{(3)}\Big|_{\rm fin} \;=\;& C_A^3\left(\frac{14423912}{6561} + \frac{384479\zeta(2)}{2916} - \frac{370649\zeta(3)}{486} + \frac{280069\zeta(4)}{864} + \frac{1821\zeta(2)\zeta(3)}{4} - \frac{66421\zeta(5)}{90}\right.\\
&\left. + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right) + C_A^2 n_f T\left(-\frac{10021313}{6561} - \frac{75736\zeta(2)}{729} - \frac{1508\zeta(3)}{27}\right.\\
&\left. + \frac{437\zeta(4)}{12} - \frac{878\zeta(3)\zeta(2)}{9} + \frac{6476\zeta(5)}{45}\right) + C_F C_A n_f T\left(-\frac{155629}{243} - \frac{82\zeta(2)}{3} + \frac{23584\zeta(3)}{81} - 16\zeta(4)\right.\\
&\left. + 96\zeta(3)\zeta(2) + \frac{64\zeta(5)}{9}\right) + C_F^2 n_f T\left(\frac{608}{9} + \frac{592\zeta(3)}{3} - 320\zeta(5)\right) + C_F n_f^2 T^2\left(\frac{42248}{81} - \frac{64\zeta(2)}{3}\right.\\
&\left. - \frac{2816\zeta(3)}{9} - \frac{224\zeta(4)}{3}\right) + C_A n_f^2 T^2\left(\frac{2958218}{6561} + \frac{304\zeta(2)}{27} + \frac{47296\zeta(3)}{243} + \frac{1594\zeta(4)}{27}\right)\,, \qquad (10)
\end{aligned}$$

where $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$, $T = 1/2$ and $d^{abc}d^{abc} = (N_c^2-1)(N_c^2-4)/N_c$. Inserting numerical values leads to $F_q^{(3),g+n_f}|_{\rm fin} \approx -13656.8 + 3062.1 n_f - 164.2 n_f^2 \pm 2.2\delta_{9,1} \pm 0.4\delta_{9,2} \pm 2.2\delta_{9,4}$, $F_q^{(3),sing}|_{\rm fin} \approx -5.944$, and $F_g^{(3)}|_{\rm fin} \approx 26102.7 - 8298.8 n_f + 585.3 n_f^2 \pm 27.0\delta_{9,1} \pm 21.6\delta_{9,2}$, where $\delta_{9,i} = 1$ corresponds to the one sigma uncertainty given in Eq. (5).

It is interesting to specify our result to a supersymmetric Yang-Mills theory containing a bosonic and fermionic degree of freedom in the same colour representation. This is achieved by setting $C_A = C_F = 2T$ and $n_f = 1$ which leads to

$$F_q^{(3),g+n_f}\Big|_{\rm fin} = C_A^3\Bigg(\frac{389216}{243} - \frac{155935\zeta(2)}{972} - \frac{54703\zeta(3)}{162} + \frac{23897\zeta(4)}{72} + \frac{15875\zeta(2)\zeta(3)}{36} - \frac{11279\zeta(5)}{10} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\Bigg), \tag{11}$$

$$F_g^{(3)}\Big|_{\rm fin} = C_A^3\Bigg(\frac{676219}{486} + \frac{61937\zeta(2)}{972} - \frac{93295\zeta(3)}{162} + \frac{95171\zeta(4)}{288} + \frac{16361\zeta(2)\zeta(3)}{36} - \frac{1645\zeta(5)}{2} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\Bigg). \tag{12}$$

Although we do not know three coefficients analytically, we believe that the growth of the transcendentality level continues when going to the next order in ϵ so that all the results are at most of transcendentality six as was predicted in Refs. [41, 46]. It is interesting to note that these terms agree between the two form factors.

To summarize, in this Letter we compute the form factors of the photon-quark and effective Higgs boson-gluon vertex to three-loop order within massless QCD. Our results constitute important building blocks for a number of physical applications. Among them are the two-jet cross section in e^+e^- collisions, the Higgs boson production in gluon fusion and the lepton pair production in proton collisions via the Drell-Yan mechanism. Let us stress that our result represents the first complete evaluation of three-loop QCD corrections to a three-point function. Our results for the coefficients of the three master integrals $A_{9,1}$, $A_{9,2}$ and $A_{9,4}$ partially overlap with those of Ref. [47] where these integrals were evaluated in a direct way. Agreement has been found for all common coefficients.

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